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A GEOMETRIC MEANING OF THE RANK OF HERMITIAN SYMMETRIC SPACES

Dedicated to Professor I. Mogi on his 60th birthday

By

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§1. Introduction.

Let M be a Kaehler manifold and denote by H the holomorphic sectional curvature of M. We say that H is *δ-pinched* if there exists a positive constant c such that

$$\delta c \leqq H \leqq c .$$

In this paper, we shall prove the following

THEOREM. *Let M be a compact irreducible Hermitian symmetric space of rank r. Then the holomorphic sectional curvature of M is $\frac{1}{r}$-pinched.*

Although it is possible to verify the result for each Hermitian symmetric space one by one by using the curvature tensors given by E. Calabi and E. Vesentini [1], we shall given here a systematic proof.

§2. Preliminaries.

We begin by constructing a compact Hermitian symmetric space. For details, see e.g. [3].

Let $\tilde{\mathfrak{g}}$ be a complex simple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\tilde{\mathfrak{g}}$. The dual space of the complex vector space $\mathfrak{h}$ is denoted by $\mathfrak{h}^*$. An element α of $\mathfrak{h}^*$ is called a *root* of $(\tilde{\mathfrak{g}}, \mathfrak{h})$ if there exists a non-zero vector X_α in $\tilde{\mathfrak{g}}$ such that

$$[H, X_\alpha]=\alpha(H)X_\alpha \qquad \text{for} \quad H\in\mathfrak{h}.$$

We denote by Δ the set of all non-zero roots of $(\tilde{\mathfrak{g}}, \mathfrak{h})$ and put $\mathfrak{g}_\alpha=\boldsymbol{C}X_\alpha$. Then we have a direct sum decomposition:

$$\tilde{\mathfrak{g}}=\mathfrak{h}+\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha .$$

Since the Killing form K of $\tilde{\mathfrak{g}}$ is non-degenerate on $\mathfrak{h}\times\mathfrak{h}$, for each $\xi\in\mathfrak{h}^*$ we can

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define $H_\xi \in \mathfrak{h}$ by

$$K(H, H_\xi)=\xi(H) \quad \text{for} \quad H\in\mathfrak{h}.$$

Put $\mathfrak{h}_0=\sum_{\alpha\in\Delta} \boldsymbol{R}H_\alpha$. Then the dual space $\mathfrak{h}_0^*$ of $\mathfrak{h}_0$ can be considered as a real subspace of $\mathfrak{h}^*$. Define an inner product (,) on $\mathfrak{h}_0^*$ by

$$(\xi, \eta)=K(H_\xi, H_\eta) \quad \text{for} \quad \xi, \eta\in\mathfrak{h}_0^*.$$

For each $\alpha\in\Delta$ we choose a basis E_α of $\mathfrak{g}_\alpha$ so that $\{H_{\alpha_j}\ (j=1, \cdots, 1), E_\alpha\ (\alpha\in\Delta)\}$ forms Weyl's canonical basis of $\tilde{\mathfrak{g}}$. Then we have $[E_\alpha, E_{-\alpha}]=H_\alpha$, and a Lie algebra $\mathfrak{g}$ defined as follows is a compact real form of $\tilde{\mathfrak{g}}$:

$$\mathfrak{g}=\sum_{\alpha\in\Delta}\boldsymbol{R}\sqrt{-1}H_\alpha+\sum_{\alpha\in\Delta}\boldsymbol{R}(E_\alpha+E_{-\alpha})+\sum_{\alpha\in\Delta}\boldsymbol{R}\sqrt{-1}(E_\alpha-E_{-\alpha}).$$

We denote by $\{\alpha_1, \cdots, \alpha_l\}$ the fundamental root system of $\tilde{\mathfrak{g}}$ with respect to a linear ordering in $\mathfrak{h}_0^*$ (so that $\dim_C\mathfrak{h}=l$).

Now we fix a simple root α_i $(i=1, \cdots, l)$. For simplicity, we put $A_\alpha=E_\alpha+E_{-\alpha}$ and $B_\alpha=\sqrt{-1}(E_\alpha-E_{-\alpha})$. We define a subset Δ_i of Δ, a subalgebra $\mathfrak{k}_i$ of $\mathfrak{g}$ and a subspace $\mathfrak{m}_i$ of $\mathfrak{g}$ by

$$\Delta_i=\{\alpha=\sum_j m_j\alpha_j\,;\ m_i\geqq 1\},$$

$$\mathfrak{k}_i=\sum_{\alpha\in\Delta}\boldsymbol{R}\sqrt{-1}H_\alpha+\sum_{\alpha\in\Delta^+-\Delta_i}(\boldsymbol{R}A_\alpha+\boldsymbol{R}B_\alpha),$$

$$\mathfrak{m}_i=\sum_{\alpha\in\Delta_i}(\boldsymbol{R}A_\alpha+\boldsymbol{R}B_\alpha),$$

where Δ^+ denotes the set of all positive roots.

Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and K_i the connected Lie subgroup of G with algebra $\mathfrak{k}_i$. Let π denote the natural projection of G onto a compact homogeneous space $M_i=G/K_i$ and put $o=\pi(K_i)$. Then we can identify the vector space $\mathfrak{m}_i$ with the tangent space $T_0(M_i)$ of M_i at o. It is easily seen that there exists a unique G-invariant Riemannian metric g on M_i such that $g=-K|\mathfrak{m}_i\times\mathfrak{m}_i$ at o. It is known that a compact Riemannian homogeneous space M_i obtained as above from a pair $(\tilde{\mathfrak{g}}, \alpha_i)$ of a complex simple Lie algebra $\tilde{\mathfrak{g}}$ and a simple root α_i becomes a Hermitian symmetric space if and only if the coefficient m_i of α_i in every $\alpha\in\Delta_i$ is equal to 1 and the center $\mathfrak{z}(\mathfrak{k}_i)$ of $\mathfrak{k}_i$ is 1-dimensional, and that every compact irreducible Hermitian symmetric space can be obtained in this way.

Hereafter we assume that M_i is a Hermitian symmetric space. Then it is known that there exists an element Z_0 in $\mathfrak{z}(\mathfrak{k}_i)$ such that the complex structure of M_i at o is given by $I=\mathrm{ad}\,Z_0|\mathfrak{m}_i$ and $IA_\alpha=B_\alpha$, $IB_\alpha=-A_\alpha$ for $\alpha\in\Delta_i$. Since $Z_0\in\mathfrak{z}(\mathfrak{k}_i)$, we have

(1) $$I\circ Ad(k)=Ad(k)\circ I \qquad \text{for} \quad k\in K_i.$$

Let θ^{α}, $\theta^{-\alpha}$ be the dual forms of E_{α}, $E_{-\alpha}$. Then we have at o

(2) $$g=2\sum_{\alpha\in\Delta_i}\theta^{\alpha}\theta^{-\alpha},$$

since $K(E_{\alpha}, E_{-\alpha})=-1$. The norm of $X\in\mathfrak{m}_i$ is denoted by $|X|$.

§ 3. Proof of Theorem.

First we state a fundamental lemma without proof.

LEMMA (E. Cartan). *Let $\mathfrak{a}$ and $\mathfrak{a}'$ be two maximal abelian subspaces of $\mathfrak{m}_i$. Then*

(i) *there exists an element k in K_i such that $Ad(k)\mathfrak{a}=\mathfrak{a}'$, and*

(ii) $\mathfrak{m}_i=\bigcup_{k\in K_i} Ad(k)\mathfrak{a}$.

The rank r of M_i as a symmetric space is, by definition, the common dimension of maximal abelian subspaces of $\mathfrak{m}_i$. By a theorem of Harish-Chandra ([2], Lemma 8), there exist r roots $\delta_1, \cdots, \delta_r$ in Δ_i such that none of $\delta_i\pm\delta_j$ belong to Δ, which are called strongly orthogonal roots. Thus the space $\mathfrak{a}_0$ spanned by $A_{\delta_1}, \cdots, A_{\delta_r}$ over $\boldsymbol{R}$ is a maximal abelian subspace of $\mathfrak{m}_i$. We denote by R the curvature tensor of (M_i, g). Then we have the following formula due to E. Cartan:

$$R(X, Y)Z=-[[X, Y], Z] \qquad \text{for} \quad X, Y, Z\in\mathfrak{m}_i.$$

Put $S=\{X\in\mathfrak{m}_i;\ |X|=1\}$. Then, for $X\in S$, the holomorphic sectional curvature $H(X)$ of the plane section spanned by X and IX is given by

(3) $$\begin{aligned} H(X)&=g(R(X, IX)IX, X)\\ &=-g([[X, IX], IX], X)\\ &=|[X, IX]|^2. \end{aligned}$$

We assert that the range of the function H on S coincides with that of H on $S\cap\mathfrak{a}_0$. In fact, Lemma implies that, for every $H\in S$, there exists an element k in K_i such that $Ad(k)X\in S\cap\mathfrak{a}_0$. Therefore from (1) and (3) we have

$$\begin{aligned} H(Ad(k)X)&=|[Ad(k)X, IAd(k)X]|^2\\ &=|[Ad(k)X, Ad(k)IX]|^2\\ &=|Ad(k)[X, IX]|^2\\ &=|[X, IX]|^2 \end{aligned}$$

$$=H(X),$$

which proves our assertion.

Let $X=\sum_{j=1}^{r} x_j A_{\delta_j} \in S \cap \mathfrak{a}_0$. Then by (2) we have

$$1=|X|^2=\sum_{j,k=1}^{r} x_j x_k g(E_{\delta_j}+E_{-\delta_j}, E_{\delta_k}+E_{-\delta_k})$$

$$=2\sum_{j=1}^{r} x_j^2,$$

and

$$[X, IX]=\left[\sum_{j=1}^{r} x_j A_{\delta_j}, \sum_{k=1}^{r} x_k B_{\delta_k}\right]$$

$$=\sum x_j^2 [A_{\delta_j}, B_{\delta_j}]$$

$$=\sum x_j^2 [E_{\delta j}+E_{-\delta_j}, \sqrt{-1}(E_{\delta_j}-E_{-\delta_j})]$$

$$=-2\sqrt{-1}\sum x_j^2 [E_{\delta_j}, E_{-\delta_j}]$$

$$=-2\sqrt{-1}\sum x_j^2 H_{\delta_j}.$$

Hence

$$|[X, IX]|^2=4|\sum x_j^2 H_{\delta_j}|^2$$

$$=4\sum x_j^4(\delta_j, \delta_j).$$

But by a theorem of C. C. Moore ([3], p. 362) we have $(\delta_1, \delta_1)=\cdots=(\delta_r, \delta_r)$. Thus the range of H is given by

$$4r\left(\frac{1}{2r}\right)^2(\delta_1, \delta_1)\leqq H\leqq 4\left(\frac{1}{2}\right)^2(\delta_1, \delta_1),$$

since $\sum x_j^2=\frac{1}{2}$. Therefore our theorem is proved.

§4. Remark.

Let (M_λ, g_λ) be a compact irreducible Hermitian symmetric space of rank r_λ and H_λ the holomorphic sectional curvature of (M_λ, g_λ), $\lambda=1, \cdots, n$. Assume that $\max H_1=\cdots=\max H_n$. Then a compact Hermitian symmetric space $(M_1\times \cdots \times M_n, g_1\times \cdots \times g_n)$ of rank $r_1+\cdots+r_n$ is $\frac{1}{r_1+\cdots+r_n}$-pinched

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