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A NOTE ON CODIVISIBLE AND WEAKLY CODIVISIBLE MODULES

(Dedicated to Prof. G. Azumaya for his sixtieth birthday)

By

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1. Recently in his paper [6] M. Sato has given, among other things, the following conjecture: If every right R-module has a codivisible cover with respect to a radical t of mod-R, then $T(t)$ is pseudo-hereditary. However we can provide an example which shows that the conjecture is not true.

We shall give some condition which is equivalent to that $T(t)$ is pseudo-hereditary. Finally some characterizations of codivisible and weakly codivisible right R-modules are also given. One of which can be seen as a generalization of Azumaya [1, Theorem 4] and using this we can give more simple proofs of results due to Sato.

2. Let R be a ring with identity and mod-R the category of all unitary right R-modules. A subfunctor t of the identity functor of mod-R is called a *preradical* of mod-R. It is called *idempotent* if $t(t(M))=t(M)$ and a *radical* if $t(M/t(M))=0$ for all right R-modules M.

To each preradical t of mod-R we can associate two classes of right R-modules, namely

$$T(t)=\{M \mid t(M)=M\} \quad \text{and} \quad F(t)=\{M \mid t(M)=0\}.$$

$T(t)$ is closed under factor modules and direct sums, while $F(t)$ is closed under submodules and direct products. $T(t)$ is called the *pretorsion class* and $F(t)$ the *pretorsionfree class*. Right R-modules in $T(t)$ are called *t-torsion* and those in $F(t)$ are *t-torsionfree*.

For a given preradical t of mod-R, a right R-module C is called *codivisible*, following Bland [3], if every row exact diagram of right R-modules

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$$\begin{array}{ccccc} & & C & & \\ & & \downarrow & & \\ A & \xrightarrow[f]{} & B & \longrightarrow & 0 \end{array}$$

where $Ker(f)$ is t-torsionfree can be completed to a commutative diagram. For a right R-module M, a codivisible module C together with an epimorphism $g: C \to M$ such that $Ker(g)$ is t-torsionfree and is small in C is called a *codivisible cover* of M.

Let t again be a preradical of mod-R. Then the pretorsion class $T(t)$ is called *pseudo-hereditary*, following Teply [7], if every submodule of $t(R)$ is again t-torsion. In case t is a radical, as was pointed out by [5, p. 480], $T(t)$ is pseudo-hereditary if and only if, for any right R-module M, every submodule of $M \cdot t(R)$ is t-torsion.

An *epiradical* is a preradical t of mod-R for which every epimorphism $M \to M''$ induces an epimorphism $t(M) \to t(M'')$ by restriction. Then a preradical t of mod-R is an epiradical if and only if $t(M) = M \cdot t(R)$ for every right R-module M. Following Beachy [2] we call an idempotent epiradical a *cotorsion radical.*

From this definition we have at once

LEMMA 1. *Let t be a cotorsion radical of mod-R. Then t is left exact if and only if $T(t)$ is pseudo-hereditary.*

Assume now that R is a right perfect ring. Then, as was shown by Bland [3, Theorem 3.2], for any idempotent radical t of mod-R, every right R-module M has a codivisible cover. Indeed first take a projective cover

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

of M. Then

$$0 \longrightarrow K/t(K) \longrightarrow P/t(K) \longrightarrow M \longrightarrow 0$$

is a desired codivisible cover of M. Hence, according to Lemma 1, to show that the conjecture cited above is not true it is enough to give a right perfect ring R and a non-left exact cotorsion radical t of mod-R. The following example is one given by [4, p. 570].

EXAMPLE 2. Let K be a field and R the ring of all 2×2 lower triangular matrices over K. R is a right perfect ring. Let

$$I = \begin{pmatrix} 0 & 0 \\ K & K \end{pmatrix}.$$

Then I is an idempotent two-sided ideal of R. Hence I determines a 3-fold torsion theory (T_1, T_2, T_3) for mod-R by

$$T_1=\{M \mid MI=M\},$$

$$T_2=\{M \mid MI=0\}$$

and

$$T_3=\{M \mid l_M(I)=0\}$$

where $l_M(I)$ denotes the left annihilator of I in M and the idempotent radical t_1 corresponding to the torsion theory (T_1, T_2) is a cotorsion radical of mod-R. This t_1 is not left exact because, for

$$N=\begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix} \quad \text{and} \quad M=R,$$

we have $t_1(N)=0$ and $t_1(M)\cap N\neq 0$.

3. In his Theorem 17, Sato presupposed that if A_R is codivisible, then $A/t(B)$ is also codivisible for any submodule B of A. As an equivalent condition to this, we have

PROPOSITION 3. *Let t be a radical of mod-R. Then the following conditions are equivalent:*

(1) *If a right R-module A is codivisible, then $A/t(B)$ is also codivisible for any submodule B of A.*

(2) *If a right R-module A is codivisible, then $t(B)$ is t-torsion for any submodule B of A.*

PROOF. (1) ⇒ (2) follows from [6, Proposition 3(3)] and (2) ⇒ (1) is evident.

For example, in mod-Z, we consider the radical $t=t_\eta$ where $\eta=\{Z/(p)\}$ and p is a prime (see [6: § 5, Example]). Since each nonzero submodule of Z is not t-torsion, if B is a submodule of Z and $t(B)\neq 0$, then $Z/t(B)$ can not be codivisible.

In contrast with the case of weakly codivisible modules, a homomorphic image A/B of a codivisible module A, where B is a submodule of $t(A)$, need not be codivisible (cf. [6, Lemma 11(1)]). Concerning this we have, by a similar way as above

PROPOSITION 4. *Let t be a radical of mod-R. Then the following conditions are equivalent:*

(1) *If a right R-module A is codivisible and B is a submodule of* $t(A)$, *then* A/B *is also codivisible.*

(2) *If a right R-module A is codivisible, then any submodule of* $t(A)$ *is t-torsion.*

Note that clearly each condition of Proposition 4 is stronger than that of Proposition 3, but the converse is not true in general. To see this, using [6, Theorem 13] and Lemma 1, it is enough to give a cotorsion radical of mod-R which is not left exact. The radical t_1 of Example 2 was such a non-left exact cotorsion radical.

4. Let t be a preradical of mod-R. Sato [6] has called a right R-module M *weakly codivisible* if every row exact diagram of right R-modules

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow & & \\ A & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

where A is t-torsionfree can be completed to a commutative diagram. The following characterization of weakly codivisible modules is often useful. This also can be seen as a generalization of Azumaya [1, Theorem 4].

THEOREM 5. *Let t be a radical of mod-R. Then the following conditions are equivalent for a right R-module M*:

(1) *M is weakly codivisible.*

(2) *Every row exact diagram of right R-modules*

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow & & \\ A & \underset{f}{\longrightarrow} & B & \longrightarrow & 0 \end{array}$$

where $t(A)\cap Ker(f)=0$ *can be completed to a commutative diagram.*

(3) *Every exact sequence* $N \xrightarrow{f} M \longrightarrow 0$ *of right R-modules such that* $t(N)\cap Ker(f)=0$ *splits.*

PROOF. (1) $\Rightarrow$ (2). Assume (1) and let $0\to K\to A \xrightarrow{f} B\to 0$ be an exact sequence with $t(A)\cap K=0$ and $g: M\to B$ a homomorphism. Then

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ A/t(A) & \xrightarrow[\bar{f}]{} & B/f(t(A)) \end{array}$$

is a pullback diagram for $\bar{f}$ and π, where π and π' are canonical maps and $\bar{f}$ is the map induced by f. Since $A/t(A)$ is t-torsionfree, there exists a homomorphism $h: M\to A/t(A)$ such that $\bar{f}h=\pi g$. Hence we can find a (unique) homomorphism $k: M\to A$ such that $fk=g$. (2) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (1). Assume (3). Using [6, Lemma 11(3)], there exists a weakly codivisible right R-module N and an epimorphism $f: N\to M$ such that $t(N)\cap Ker(f)=0$. Hence by assumption M is a direct summand of N and thus it is also weakly codivisible. This completes the proof.

Using this theorem we can give more simple proofs of results due to Sato. As an example of the simplification, we give a proof of the uniqueness of weakly codivisible covers. For the definition of weakly codivisible covers and the proof of their uniqueness cf. [6, § 2] and [6, Lemma 6(3)].

Let M be a right R-module and let $0\to K\to C\xrightarrow{f} M\to 0$ and $0\to K'\to C'\xrightarrow{f'} M\to 0$ be weakly codivisible covers of M. By Theorem 5, there exists a homomorphism $g: C\to C'$ such that $f'g=f$. Since K' and K are small in C' and C respectively, this g is an epimorphism and $Ker(g)$ is small in C. Moreover $Ker(g)\cap t(C)=0$. Hence again by Theorem 5, the sequence $0\to Ker(g)\to C\xrightarrow{g} C'\to 0$ splits and thus $C\cong^{g} C'$.

Corresponding to Theorem 5, the following theorem characterizes codivisible modules.

THEOREM 6. *Let t be a radical of mod-R. Then the following conditions are equivalent for a right R-module M:*

(1) *M is codivisible.*

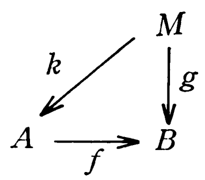
(2) *Every row exact diagram of right R-modules*

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow & & \\ A & \xrightarrow[f]{} & B & \longrightarrow & 0 \end{array}$$

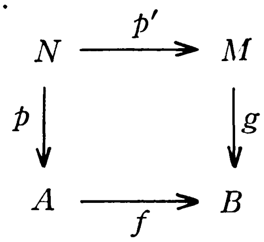
where $t(A)\cap Ker(f)$ is t-torsionfree can be completed to a commutative diagram.

(3) *Every exact sequence* $N\xrightarrow{f}M\to 0$ *of right R-modules such that* $t(N)\cap Ker(f)$ *is t-torsionfree splits.*

PROOF. (1) $\Rightarrow$ (2). Assume that M is codivisible. Let $0\to K\to A\xrightarrow{f}B\to 0$ be an exact sequence of right R-modules with $t(K)$ t-torsionfree and $g: M\to B$ a homomorphism. Then since $0\to K/t(K)\to A/t(K)\xrightarrow{\bar{f}}B\to 0$ is exact, where $\bar{f}$ is the map induced by f, and $K/t(K)$ is t-torsionfree, there exists a homomorphism $h: M\to A/t(K)$ such that $\bar{f}h=g$. Furthermore $0\to t(K)\to A\xrightarrow{\pi}A/t(K)\to 0$ is exact, where π is canonical, and $t(K)$ is t-torsionfree by assumption. Hence there exists a homomorphism $k: M\to A$ such that $\pi k=h$. Therefore we have a commutative diagram



as was desired. (2) $\Rightarrow$ (3) is evident. (3) $\Rightarrow$ (1). Assume (3) and let $A\xrightarrow{f}B\to 0$ be an exact sequence of right R-modules with $Ker(f)$ t-torsionfree and $g: M\to B$ a homomorphism. Consider a pullback diagram for f and g:



Then p' is an epimorphism and $Ker(p')\cong Ker(f)$, and so $N\xrightarrow{p'}M\to 0$ splits by assumption. There exists a homomorphism $h: M\to N$ such that $p'h=1_M$, the identity map of M, and therefore we have $f(ph)=g$. This completes the proof of the theorem.

Note that, as is seen from the proof of the theorem, in (2) $t(A)\cap Ker(f)$ can be replaced by $t(Ker(f))$ and furthermore in (3) we can also replace $t(N)\cap Ker(f)$ by $t(Ker(f))$ or $Ker(f)$.

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