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ON ORTHOGONALITY IN S. I. P. SPACES

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Abstract: In this paper we study the orthogonality in an s. i. p. space and prove a theorem of Giles under weaker conditions.

Let X be a complex (real) vector space. A complex (real) function $[\,,\,]$ on $X\times X$ is called a semi-inner-product (s. i. p.) on X if it satisfies the following properties:

(i) $[x+y, z]=[x, z]+[y, z]$

(ii) $[\lambda x, y]=\lambda[x, y]$

(iii) $[x, x]>0$ for $x\neq 0$

(iv) $|[x, y]|^2\leqq[x, x]\cdot[y, y]$

for all x, y, z in X and for all scalars λ. A vector space X with an s. i. p. is called a semi-inner-product space (s. i. p. space).

In what follows we shall consider complex s. i. p. spaces. An s. i. p. space X is normed linear space with the norm given by $\|x\|=[x, x]^{1/2}$ for x in X ([4], Theorem 2, p. 31) and the topology on an s. i. p. space is the one induced by this norm. It is also known ([4], p. 31) that every normed linear space can be made into an s. i. p. space (in general in infinitely many different ways). An s. i. p. space is said to have the homogeneity property when the s. i. p. satisfies

(v) $[x, \lambda y]=\bar{\lambda}[x, y]$ for all x, y in X and for all complex numbers λ. Giles [2] showed that every normed linear space can be represented as an s. i. p. space with the homogeneity property.

An s. i. p. space X is said to be continuous if for every x, y in X

$$Re[y, x+\lambda y]\rightarrow Re[y, x] \quad \text{for all real } \lambda\rightarrow 0.$$

We say that in an s. i. p. space X, x is orthogonal to y if $[y, x]=0$.

If M is a subset of an s. i. p. space X, let

$$M^{\perp}=\{x\in X: [m, x]=0 \quad \text{for all } m\in M\}.$$

In this paper we study the properties of $M^{\perp}$; in particular we show that $M^{\perp}$ is always closed and establish a decomposition theorem for s. i. p. spaces.

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The Riesz representation theorem for linear functionals on an s. i. p. space is also proved under conditions weaker than those used by Giles [2], Husain and Malviya [3].

Let M be a subset of an s. i. p. space X. Then it follows from the definition of orthogonality that

$$M^{\perp} \cap M = \{0\}.$$

Our definition of orthogonality is equivalent to that of James (see Giles [2]) in the case of normed linear spaces if the s. i. p. satisfies the continuity condition as the following lemma shows:

LEMMA 1. *Let X be a continuous s. i. p. space and let $x, y \in X$. The following are equivalent*:

(a) $[y, x]=0$.

(b) $\|x+\lambda y\| \geqq \|x\|$ for all complex numbers λ.

This is Theorem 2 of Giles ([2], p. 438)

PROPOSITION 2. *If X is a continuous s. i. p. space and $x, y \in X$, then $[y, x]$ $=0$ implies $[y, \alpha x]=0$ for all complex numbers α. In particular*

$$\alpha M^{\perp} \subset M^{\perp}$$

for each complex number α.

PROOF. Suppose X is a continuous s. i. p. space and $[y, x]=0$ with $x, y \in X$. We shall show that $[y, \alpha x]=0$ for all complex α. It is trivial when $\alpha=0$. We thus assume $\alpha \neq 0$. Now using Lemma 1, we see that

$$\|\alpha x+\lambda y\|=|\alpha|\,\|x+\frac{\lambda}{\alpha} y\| \geqq |\alpha|\,\|x\|=\|\alpha x\|$$

for all complex λ and the condition follows from Lemma 1.

PROPOSITION 3. *Let M be a subset of a continuous s. i. p. space X. Then $M^{\perp}$ is closed in X.*

PROOF. Let $\{x_n\}$ be a sequence of elements in $M^{\perp}$ such that $x_n \rightarrow x \in X$. It is enough to show that $x \in M^{\perp}$. Let $\varepsilon>0$ be given. Since $x_n \rightarrow x$, we can find n_0 such that

$$(1) \qquad \|x_n-x\|<\varepsilon \quad \text{for all } n \geqq n_0$$

This implies

(2) $$\|x\|-\varepsilon<\|x_n\|<\|x\|+\varepsilon \quad \text{for all } n\geqq n_0.$$

Let $m\in M$. Then by Lemma 1, for each n,

(3) $$\|x_n+\mu m\|\geqq\|x_n\| \quad \text{for all complex } \mu.$$

Now,

(4) $$\begin{aligned}\|x+\mu m\|&=\|x-x_n+x_n+\mu m\|\\ &\geqq\|x_n+\mu m\|-\|x-x_n\|\end{aligned}$$

Let $n>n_0$. Then using (1), (2) and (3) we get from (4)

$$\|x+\mu m\|\geqq\|x\|-2\varepsilon\,.$$

Since ε is arbitrary we conclude that $\|x+\mu m\|\geqq\|x\|$ for all complex μ. Hence by Lemma 1, $[m, x]=0$. This being true for each m in M, our assertion follows.

A normed linear space is strictly convex if each point of the unit sphere is an extreme point of the unit ball. A condition for strict convexity is given by

LEMMA 4. *An s. i. p. space X is strictly convex if and only if whenever $[x, y]=\|x\|\|y\|$, $x, y\neq 0$, then $y=\mu x$ for some $\mu>0$.*

This is ([1], Theorem 5.1, p. 381).

LEMMA 5. *If N is a proper closed subspace of a strictly convex reflexive Banach space X and if $y\notin N$, then there exists a unique vector $x_0\in N$ such that*

$$\|y-x_0\|=inf\{\|y-x\|: x\in N\}.$$

This is contained in ([5], Corollary 3.4 of Theorem 3.2)

PROPOSITION 6. *Let M be a proper closed linear subspace of a continuous s. i. p. space X which is strictly convex, complete and reflexive. Then every element x in X can be represented in the form $x=y+z$, where $y\in M$ and $z\in M^{\perp}$.*

PROOF. If $x\in M$, then $x=x+0$ which is the required representation. If $x\notin M$, by Lemma 5, there exists a unique vector $y\in M$ such that

$$\|x-y\|=inf\{\|x-m\|: m\in M\}.$$

Letting $z=x-y$, we have

$$\|z\|\leqq\|z+m\| \quad \text{for all } m\in M,$$

which implies $z\in M^{\perp}$ by Lemma 1. Further $x=y+z$. To prove the uniqueness of the representation, if possible, let us suppose that $x=y'+z'$ with $y'\in M$ and

$z' \in M^{\perp}$. Then $y'+z'=y+z$, which gives $z'-z=y-y' \in M$. Since $z \in M^{\perp}$, we have $[z'-z, z]=0$, which gives

$$\|z\|^2=[z, z]=[z', z] \leqq \|z'\| \cdot \|z\| \tag{5}$$

Similarly, since $z' \in M^{\perp}$,

$$\|z'\|^2=[z', z']=[z, z'] \leqq \|z\| \cdot \|z'\|. \tag{6}$$

From (5) and (6) we see that

$$\|z'\|=\|z\| \tag{7}$$

and

$$[z', z]=\|z'\| \cdot \|z\| \tag{8}$$

Since the space is strictly convex, it follows from (8) by Lemma 4 that there exists a positive number c such that $z=cz'$. This together (7) implies $c=1$. Hence $z=z'$ and $y=y'$.

The Riesz representation theorem for an s. i. p. space can now be stated as follows.

PROPOSITION 7. *In a continuous s.i.p. space X which is complete in its norm, strictly convex and reflexive, to every continuous linear functional f on X there exists a unique y in X such that $f(x)=[x, y]$ for all x in X.*

Propositions 6 and 7 were proved by Husain and Malviya [3] for a continuous s. i. p. space X which is complete in its norm and satisfying the inequality

$$\|x+y\|^2+\mu^2\|x-y\|^2 \leqq 2\|x\|^2+2\|y\|^2 \tag{9}$$

for $x, y \in X$ and $0<\mu<1$. Giles [2] established proposition 7 when the continuous s. i. p. space is complete and uniformly convex and has the homogeneity property. As every normed linear space satisfying the inequality (9) is uniformly convex, the condition of Husain and Malviya is stronger than uniform convexity used by Giles. Since every uniformly convex space is strictly convex and reflexive, we have much weaker conditions in proposition 7. We have also omitted the homogeneity property.

We omit the proof of the proposition 7 as it follows along the same lines as that given by Giles. Notice that in every s. i. p. space, the relation

$$[x, \alpha x]=\bar{\alpha}[x, x] \quad \text{for all complex } \alpha \text{ is valid.} \tag{10}$$

By virtue of (10), proposition 2 and Lemma 5 which guarantees the existence of a non-zero vector orthogonal to the null space of a non-zero functional on X (as in the proof of proposition 6) Giles, proof can be suitably modified to give proposition 7.

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