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A NOTE ON SOME STRONG WHITNEY-REVERSIBLE PROPERTIES

By

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1. All spaces considered in this paper are assumed to be metric. A continuum means a compact connected space and a map means a continuous function. The letter X will always denote a continuum. Let $C(X)$ denote the hyperspace of all non-empty subcontinua of X with the Hausdorff metric (see [7]). Whitney [10] proved that for every continuum X there exists a map $\mu: C(X)\rightarrow[0, +\infty)$ satisfying

(1) if $A, B\in C(X)$ and $A\subsetneqq B$, then $\mu(A)\lneqq\mu(B)$ and

(2) $\mu(\{x\})=0$ for every $x\in X$.

We shall call any map from $C(X)$ to $[0, +\infty)$ satsfying the above conditions (1) and (2) a *Whitney map for* $C(X)$.

Nadler [7] introduced the concept of a strong Whitney-revesible property. Let P be a topological property. We say that P is a *strong Whitney-reversible property* provided whenever X is a continuum such that $\mu^{-1}(t)$ has the property P for some Whitney map μ for $C(X)$ and every $0<t\leqq\mu(X)$, then so does X. Moreover he has shown that some topological properties are strong Whitney-reversible properties. For example hereditary indecomposability and local connectedness are such properties.

We refer readers to see [1] and [7] for the shape theory and the hyperspace theory respectively if necesary.

2. We shall show that some topological properties are strong Whitney-reversible properties.

THEOREM 1. *Let* μ *be a Whitney map for* $C(X)$. *If there is a sequence* $\{t_n\}$ $n\geqq 1$ *in* $(0, \mu(X)]$ *such that* $t_n\rightarrow 0$ *as* $n\rightarrow+\infty$ *and* $\mu^{-1}(t_n)$ *is an FAR for each* $n=1, 2, 3, \cdots$, *then* X *is also an FAR.*

Hence the property of being an FAR is a strong Whitney-reversible property.

PROOF. Let M be an arbitrary ANR and $f: X\rightarrow M$ be an arbitrary map. Since M is an ANR and we can identify X with $\mu^{-1}(0)=\{\{x\}\mid x\in X\}$, there are

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an open neighborhood U of X and a map $\tilde{f}: U \to M$ such that $\tilde{f}|X=f$. Then there is an integer $n \geqq 1$ such that $\mu^{-1}([0, t_n]) \subset U$. Since $\mu^{-1}(t_n)$ is an FAR, $\tilde{f}|\mu^{-1}(t_n) \simeq 0$, where 0 is a constant map. Hence there exists a map $g: \mu^{-1}([t_n, \mu(X)]) \to M$ such that $g|\mu^{-1}(t_n)=\tilde{f}|\mu^{-1}(t_n)$. Now we can define a map $h: C(X) \to M$ as the following formula;

$$h|\mu^{-1}([0, t_n])=\tilde{f}|\mu^{-1}([0, t_n]) \quad \text{and}$$
$$h|\mu^{-1}([t_n, \mu(X)])=g.$$

Since $C(X)$ is an FAR (see [3]), $h \simeq 0$. Hence $f=h|X \simeq 0$. Therefore X is an FAR.

REMARK 1. By the example of Petrus [8] the converse of Theorem 1 is false.

REMARK 2. By the proof of Theorem 1 the property of being acyclic is a strong Whitney-reversible property. But it is not Whitney property (see [5]) by the same example of Petrus [8].

THEOREM 2. *Let μ be a Whitney map for $C(X)$. Let $\mathfrak{P}$ be a class of compact connected polyhedra. If there is a sequence $\{t_n\}$ $n \geqq 1$ in $(0, \mu(X)]$ such that $t_n \to 0$ as $n \to +\infty$ and $\mu^{-1}(t_n)$ is an hereditarily indecomposable $\mathfrak{P}$-like continuum (see [6]) for each $n=1, 2, 3, \cdots$, then X is also an hereditarily indecomposable $\mathfrak{P}$-like continuum.*

Hence the property of being an hereditarily indecomposable $\mathfrak{P}$-like continuum is a strong Whitney-reversible property.

RROOF. By [7] X is hereditarily indecomposable. Hence it is sufficient to show that X is $\mathfrak{P}$-like. Without loss of generality we may assume that the sequence $\{t_n\}$ $n \geqq 1$ is decreasing. Now for each $n=1, 2, 3, \cdots$ we define a function $\eta_n: X \to \mu^{-1}(t_n)$ such that $x \in \eta_n(x) \in \mu^{-1}(t_n)$ for every $x \in X$. Since X is hereditarily indecomposable, for each $n=1, 2, 3, \cdots$, η is well-defined and continuous (see [2]). Similarily for each $n=1, 2, 3, \cdots$ we can define a map $p_n: \mu^{-1}(t_{n+1}) \to \mu^{-1}(t_n)$ such that $A \subset p_n(A)$ for each $A \in \mu^{-1}(t_{n+1})$. Then $\{\mu^{-1}(t_n), p_n\}$ is an inverse sequence of $\mathfrak{P}$-like continua and onto bonding maps. Moreover we hold that $p_n \eta_{n+1} = \eta_n$ for each $n=1, 2, 3, \cdots$. Then it is clear that X is homeomorphic to the invese limit $\varprojlim \{\mu^{-1}(t_n), p_n\}$. Therefore X is $\mathfrak{P}$-like.

In particular the convese of the result of Krasinkiewicz (4.2. [4]) is hold.

COROLLARY 1. *Let μ be a Whitney map for $C(X)$. If there exists a sequence*

$\{t_n\}$ $n\geqq 1$ in $(0, \mu(X)]$ such that $t_n\to 0$ as $n\to+\infty$ and $\mu^{-1}(t_n)$ is an hereditarily indecomposable tree-like continuum for each $n=1, 2, 3, \cdots$, then X is also an hereditarily indecomposable tree-like continuum.

The next lemma is usefull for our results.

LEMMA (Krasinkiewicz and Nadler [5]). *Let μ be a Whitney map for $C(X)$. If X contains an n-odd ($n\geqq 3$), there exists $t_0>0$ such that $\mu^{-1}(t_0)$ contains an $(n-1)$-disk.*

THEOREM 3. *Let μ be a Whitney map for $C(X)$. If* $\dim \mu^{-1}(t)\leqq n<+\infty$ *for every $t\in(0, \mu(X)]$ and one of the following conditions is satisfied, then* $\dim X\leqq n$:

(1) $\dim X<+\infty$,

(2) *$\mu^{-1}(t)$ is locally connected for every $t\in(0, \mu(X)]$,*

(3) *$\mu^{-1}(t)$ is hereditarily indecomposable for every $t\in(0, \mu(X)]$.*

PROOF. First we shall show the case (1). The following inequality is clearly hold.

$$\dim C(X)\leqq 1+\max\{\dim \mu^{-1}(t)\mid t\in[0, \mu(X)]\}<+\infty.$$

Then by the result of Rogers [9] $\dim X\leqq\dim \mu^{-1}(t)$ for some $t\in(0, \mu(X)]$. Hence $\dim X\leqq n$.

Next we shall the case (2). Then X is locally connected by [7]. If $\dim X\geqq 2$, for every $m\geqq 3$ X contains an $(m+1)$-odd. But by Lemma this fact contradicts the assumption. Hence $\dim X=1$.

In the case (3) by the same way of the proof of Theorem 2 we can show that $\dim X\leqq n$.

COROLLARY 2. *Let μ be a Whitney map for $C(X)$. If $\mu^{-1}(t)$ is locally connected and* $\dim \mu^{-1}(t)\leqq n<+\infty$ *for every $t\in(0, \mu(X)]$, then X is a finite graph. In particular if* $\dim \mu^{-1}(t)=1$ *for every $t\in(0, \mu(X)]$, X is an arc or a circle.*

PROOF. By the proof of Theorem 3 X is one-dimensional and locally connected. If X has infinitely many ramification points or a point with an infinite order, for every $m>1$ X contains $(m+1)$-odd. Then by Lemma $\dim \mu^{-1}(t)\gneqq n$ for some $t\in(0, \mu(X)]$. This contradicts our assumption. Hence X has at most finitely many ramification points and the order of each point of X is finite. Therefore X is a finite graph.

The following corollary is an easy consequence of Theorem 1 and Corollary 2.

COROLLARY 3. *Let μ be a Whitney map for $C(X)$. If $\mu^{-1}(t)$ is locally connected,* $\dim \mu^{-1}(t) \leqq n < +\infty$ *and an FAR for every $t \in (0, \mu(X)]$, X is a tree. In particular if* $\dim \mu^{-1}(t)=1$ *for every $t \in (0, \mu(X)]$, X is an arc.*

REMARK 3. Corollary 1 also can be proved by Theorem 1, Theorem 3 and the fact that hereditary indecomposability is a strong Whitney-reversible property.

REMARK 4. The author does not know whether the conditions of Theorem 3 are essential. But it seems not to be essential.

Related to Theorem 1 the following problem is open.

PROBLEM. *Is the property of being an FANR or a movable continuum a strong Whitney-reversible property?*

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