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SOME GEOMETRICAL ASPECTS OF RIEMANNIAN MANIFOLDS WITH A POLE

(dedicated to Professor Isamu Mogi for his 60'th birthday)

By

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The aim of this paper is to describe some geometrical aspects of Riemannian manifolds with a pole. A point o of a Riemannian manifold is called a *pole*, if the exponential map exp is a diffeomorphism at o. Simply connected complete Riemannian manifolds of nonpositive curvature (the Euclidean space, the hyperbolic space and a simply connected symmetric space of noncompact type, etc.) and a paraboloid of revolution are typical examples of Riemannian manifolds with a pole.

We give in ***1*** a sufficient condition on the existence of a pole in terms of curvature. Hessian comparison theorem, conformal changes of a metric and a generalization of Cartan's fixed point theorem are discussed in ***2*** ([6], [1] [2]). And we argue in ***3*** the order of a holomorphic function on a Kähler manifold with a pole ([7]).

1. As an easy consequence, a Riemannian manifold with a pole is diffeomorphic to the Euclidean space. On the contrary, any complete Riemannian manifold diffeomorphic to the Euclidean space does not necessarily have a pole.

The following proposition gives a sufficient condition on the existence of a pole.

PROPOSITION 1. *Let M be a connected complete Riemannian manifold and N be a complete surface with a pole p. Assume that M has a point o such that the sectional curvature $K(\Pi_{\gamma}t))\leqq$ Gaussian curvature of N at a point with distance t from p for all $t>0$, every normal geodesic γ issuing from o and every plane $\Pi(t)$ containing $\dot{\gamma}(t)$. Then* $\exp_0$ *is of maximal rank. If, moreover, M is simply connected, then o is a pole.*

PROOF. It is sufficient to show that o has no conjugate point on each geo-

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desic issuing from o ([4]). Let γ be a normal geodesic from o and J a Jacobi field along γ such that $J(0)=0$ and $\nabla_t J\neq 0$ at $t=0$. Suppose that $J(t_0)=0$ for $t_0>0$. Without loss of generality we may assume that $J(t)\neq 0$ for $0<t<t_0$ and that J is perpendicular to γ. Since J satisfies the Jacobi's equation, $\langle\nabla_t^2 J, J\rangle = -\langle R(J,\dot\gamma)\dot\gamma, J\rangle = -K(J\wedge\dot\gamma)\|J\|^2$. On the other hand, by Schwarz' inequality $\langle\nabla_t^2 J, J\rangle = 1/2\, d^2/dt^2(\|J\|^2) - \|\nabla_t J\|^2 \leqq 1/2\, d^2/dt^2(\|J\|^2) - (d/dt\, \|J\|)^2 = \|J\|(d^2/dt^2\, \|J\|)$ for $0<t<t_0$. Then we have

$$\frac{d^2}{dt^2}\|J\| + K(J(t)\wedge\dot\gamma(t))\|J\| \geqq 0\,. \tag{1}$$

Note that $|d/dt\,\|J\||$ is bounded for $t\to+0$, since $|d/dt\,\|J\|| \leqq \|\nabla_t J\|$ for $0<t<t_0$.

Let γ' be a normal geodesic issuing from p in N. Since p is a pole, each nontrivial Jacobi field J' along γ' such that $J'(0)=0$ and $J'\perp\gamma'$ has no zero point for $t>0$. Since N is two dimensional, $J'(t)=h(t)E(t)$ where E is a parallel unit field along γ' and $h(t)$ is a smooth function such that $h(0)=0$ and $h(t)>0$ for $t>0$. By Jacobi's equation, we have

$$\frac{d^2}{dt^2}h(t) + K'(J'(t)\wedge\dot\gamma'(t))h(t) = 0\,. \tag{2}$$

On the other hand, by the curvature condition together with the lemma below, it follows that h has a zero for $0<t<t_0$. Thus we have a contradiction.

LEMMA (*Sturm's Comparison Theorem* [3]). *Let u_i be C^2-functions defined on $[0, a]$, $i=1, 2$, which satisfy*

$$\begin{gathered}\frac{d^2}{dt^2}u_1(t) + A_1(t)u_1(t) \geqq 0\\ \frac{d^2}{dt^2}u_2(t) + A_2(t)u_2(t) = 0\,,\\ u_1(0)=u_2(0)=0\,,\quad \dot u_1(0)>0 \quad \text{and} \quad \dot u_2(0)>0\,,\end{gathered} \tag{3}$$

where A_i are C^0-functions on $[0, a]$. Assume that $A_1(t)\leqq A_2(t)$ for $0\leqq t\leqq a$ and u_2 never vanishes on $(0, a]$. Then u_1 also never vanishes on $(0, a]$.

PROOF OF LEMMA. Note that $u_2(t)>0$ for $t>0$ from the initial condition. Suppose that $u_1(t_0)=0$ for some $t_0\in(0, a]$. Without loss of generality we may assume that $u_1(t)>0$ for $0<t<t_0$. From (3), we have, for $0<t<t_0$,

$$0<\int_0^t \{u_2(\ddot u_1 + A_1 u_1) - u_1(\ddot u_2 + A_2 u_2)\}\, dt$$

$$=(u_2\dot{u}_1-u_1\dot{u}_2)\big|_0^t-\int_0^t(A_2-A_1)u_1u_2dt$$
$$<u_2(t)\dot{u}_1(t)-u_1(t)\dot{u}_2(t),$$

hence $\dot{u}_2(t)/u_2(t)<\dot{u}_1(t)/u_1(t)$. Then we have, for sufficiently small positive number c, $\log\{u_2(t)/u_2(c)\}=\int_c^t\{\dot{u}_2(t)/u_2(t)\}dt\leqq\int_c^t\{\dot{u}_1(t)/u_1(t)\}dt=\log\{u_1(t)/u_1(c)\}$ for $c<t<t_0$. Since $u_i(c)>0$, $i=1, 2$, $u_2(t_0)=\lim_{t\to t_0-0}u_2(t)<\lim_{t\to t_0-0}\{u_2(c)/u_1(c)\}u_1(t)=0$. This leads a contradiction.

2. Let M be a Riemannian manifold with a pole o. The distance function $\rho(\cdot)=d(\cdot, o)$ has singularity only at o. By comparing the radial curvatures, Siu and Yau [6] and also Greene and Wu [1] showed the comparison theorem on Hessian of the distance functions. By *radial curvature* $K(t)$ for a normal geodesic $\gamma:[0, \infty)\to M$, $\gamma(0)=o$, we mean the sectional curvature of a plane which contains the tangent vector $\dot{\gamma}$ at $\gamma(t)$. *Hessian* of a smooth function f is defined by $\mathrm{Hess}(f)(X, Y)=X(\tilde{Y}f)-(\nabla_X\tilde{Y})f$, where $\tilde{Y}$ is a local extension of Y.

By using Schwarz' inequality again, we have a description of the comparison theorem in a free manner on any dimensional condition.

PROPOSITION 2 (*Hessian Comparison Theorem*). *Let (M, o) and (N, p) be Riemannian manifolds with poles o and p respectively. Assume that for all $t>0$, the radial curvatures satisfy $K_M(t)\leqq K_N(t)$ for each normal geodesics γ and σ issuing from the poles. Then*

$$Hess_M(\rho_M)(X, X)\geqq Hess_N(\rho_N)(Y, Y),$$

where X and Y are unit vectors at $\gamma(t)$ and $\sigma(t)$ such that $X\perp\gamma(t)$ and $Y\perp\sigma(t)$, $t>0$, respectively.

Note that if f is an increasing smooth function on $(0, \infty)$, then

$$Hess_M(f\circ\rho_M)(X, X)\geqq Hess_N(f\circ\rho_N)(Y, Y),$$

since $Hess(f\circ\rho)=f'\cdot Hess(\rho)+f''d\rho\otimes d\rho$.

PROOF. We shall prove this by following [6]. Since o is a pole, there is a global vector field $\tilde{X}$ on M such that (1) $\tilde{X}(o)=0$, (2) $\tilde{X}(\gamma(t))=X$, (3) $[\tilde{X}, \partial/\partial\rho]=0$ and (4) $\tilde{X}$ is a Jacobi field along $\gamma|_{[0,t]}$ perpendicular to γ. Then we have

$$Hess_M(\rho_M)(X, X)=\int_0^t\{\|\nabla_{\partial/\partial\rho}\tilde{X}(s)\|^2-K_M(\tilde{X}(s)\wedge\dot{\gamma}(s))\|\tilde{X}(s)\|^2\}ds$$
$$=I_0^t(\tilde{X}).$$

There is also a global field $\tilde{Y}$ satisfying the similar condition and $Hess_N(\rho_N)(Y, Y) = I_0^t(\tilde{Y})$. Let $Z(s)$ be a vector field along σ defined by $Z(s)=\|X(s)\|E(s)$, where E is a unit parallel field along σ such that $E(t)=Y$. Then $\|Z\|=\|X\|$, $Z(0)=0$ and $Z(t)=Y$. By Schwarz' inequality, we have $\|\nabla_{\partial/\partial\rho_N}Z\|\leqq\|\nabla_{\partial/\partial\rho_M}X\|$. From the curvature condition, $K_M(X(s)\wedge\dot{\gamma}(s))\|X(s)\|^2\leqq K_N(Z(s)\wedge\dot{\sigma}(s))\|Z(s)\|^2$, hence

$$Hess_M(\rho_M)(X, X)=I_0^t(\tilde{X})\geqq I_0^t(\tilde{Z})\,.$$

From the property of the quadratic form I_0^t, we have $I_0^t(\tilde{Z})\geqq I_0^t(\tilde{Y})=Hess_N(\rho_N)(Y, Y)$.

A C^2-function f is called *convex* (*strictly convex*) if and only if $Hess(f)\geqq 0$ (>0). Note that f is convex (strictly convex) if and only if $(f\circ\gamma)''\geqq 0$ $((f\circ\gamma)''>0)$ for every geodesic γ. The Hessian comparison theorem gives an estimation on the (strictly) convexity of a radial function. A function f on M is called *radial* if and only if f is a composition of ρ_M and a function defined on $\boldsymbol{R}^+$.

COROLLARY 3. *Let (M, o) and (N, p) be as in Proposition 2. If the curvature assumption in the proposition is satisfied and there is an increasing function $f: \boldsymbol{R}^+\to\boldsymbol{R}$, $f'>0$ such that $f\circ\rho_N$ is (strictly) convex, then $f\circ\rho_M$ is also (strictly) convex.*

The Hessian of a radial function of a manifold with a pole is not necessarily positive definte. The above corollary gives an estimation of the convexity. By construction of a surface of revolution with Gaussian curvature $K(s)$, the following theorem is obtained [1]: Suppose $\int_0^\infty s\,\bar{K}(s)ds<1$, where $\bar{K}(s)=\max\{0$, radial curvature at x with $\rho(x)=s\}$. Then $(\mu/t)(g-d\rho\otimes d\rho)(X, X)\leqq Hess(\rho)(X, X)$ at x with $\rho(x)=t$, $t>0$ for a positive constant μ such that $1-\int_0^\infty s\,\bar{K}(s)ds\leqq\mu\leqq 1$.

Since $Hess(\rho^2)=2\rho\cdot Hess(\rho)+2d\rho\otimes d\rho$, we have a crucial estimation for the strictly convexity of ρ^2.

Consider a paraboloid of revolution, $2z=x^2+y^2$. Then the origin is a pole. The Gaussian curvature $K(p)$ at $p=(x, y, z)$ and $\rho(p)$ are written as $K(p)= 1/\{(1+|p|^2)^2\}$ and $\rho(p)=1/2\{|p|\sqrt{1+|p|^2}+\log(|p|+\sqrt{1+|p|^2})$, $|p|^2=x^2+y^2$. $\rho^2(p)$ is not convex, on the other hand $Hess(|p|^2)=2/(1+|p|^2)\cdot(dx^2+dy^2)$, that is, $|p|^2$ is strictly convex, Note that $K(p)$ has the same order as $1/\{\rho(p)^2\}$ at infinity $(\rho(p)\to\infty)$. Hence $\int_0^\infty s\cdot\bar{K}(s)ds$ diverges.

We observed that ρ^2 is not always strictiy convex. However, we can find a new metric g^* from a conformal change of the given g such that ρ^{*2} is strictly convex.

PROPOSITION 4. *Let (M, g, o) be a Riemannian manifold with a pole. Assume that the radial curvature K is bounded above by a suitable smooth function of ρ. Then there is a continuous function $f: \boldsymbol{R}\to\boldsymbol{R}$ such that $f\circ\rho\in C^{\infty}(M)$ and* (1) *(M, g^*, o) is also a Riemannian manifold with a pole o, where $g^*=e^{2f\circ\rho}g$,* (2) *the radial curvature of g^* is nonpositive and hence* (3) *the square of the distance function ρ^{*2} is strictly convex with respect to g^*.*

Before proving this, we show following two statements by considering geometrical aspects of a metric g^* defined by $g^*=e^{2f\circ\rho}g$.

PROPOSITION 5. *Let γ is a normal g-geodesic issuing from o. Then a curve: $s\mapsto\gamma(t(s))$ is a normal g^*-geodesic, where $t(s)$ is the inverse function of $s(t)=\int_0^t e^{f(u)}du$.*

PROOF. We apply the formulae of the covariant derivative with respect to a conformal change $g^*=e^{2f\circ\rho}g$ ([5]) to the curve $c(s)=\gamma(t(s))$:

$$\begin{aligned}&\nabla^*_X Y=\nabla_X Y+d\sigma(X)Y+d\sigma(Y)X-g(X, Y)\, grad\, \sigma\,,\\&\frac{\nabla^*}{ds}Y=\frac{\nabla}{ds}Y+\frac{d}{ds}\sigma(c(s))Y+(Y\sigma)\dot{c}(s)-g(\dot{c}(s), Y)\, grad\, \sigma\end{aligned}\tag{4}$$

$(\sigma=f\circ\rho)$.

Since $g^*(\dot{c}(s), \dot{c}(s))=1$, we have $(\nabla^*/ds)\dot{c}\perp\dot{c}$ by covariantly differentiating both sides. Let Y_i, $1\leqq i\leqq n$, be orthonormal g-parallel vector fields such that $Y_n=\dot{\gamma}$. We shall show $(\nabla^*/ds)\dot{c}(s)\perp Y_i(t(s))$, $1\leqq i\leqq n-1$. By covariantly differentiating $g^*(\dot{c}(s), Y_i(t(s))=0$, we have

$$0=g^*\Big(\frac{\nabla^*}{ds}\dot{c}(s), Y_i(t(s))\Big)+g^*\Big(\dot{c}(s), \frac{\nabla^*}{ds}Y_i(t(s))\Big).$$

From (4) together with the relations $\dot{c}(s)=(dt/ds)\dot{\gamma}$ and $grad\, \rho(s(t))=\dot{\gamma}(t)$,

$$\frac{\nabla^*}{ds}Y_i(t(s))=\frac{d}{ds}(f\circ\rho)(c(s))Y_i(s).$$

Hence we derive that $g^*((\nabla^*/ds)\dot{c}(s), Y_i(s))=-g^*(\dot{c}(s), (\nabla^*/ds)Y_i(s))=0$. Thus, we have $(\nabla^*/ds)\dot{c}(s)\perp Y_i(s)$, $1\leqq i\leqq n$, that is, $(\nabla^*/ds)\dot{c}(s)=0$.

NOTE. If $\lim_{t\to\infty}\int_0^t e^{f(u)}du=+\infty$, $c(s)$ is defined on the whole $\boldsymbol{R}$ and therefore $\exp: T_oM\to M$ is a diffeomorphism with respect to g^*, that is, (M, g^*) has a pole o. The distance function $\rho^*(\cdot)=d^*(\cdot, o)$ with respect to g^* is given by

$$\rho^*(p)=\int_0^{\rho(p)} e^{f(u)}du\,, \qquad p\in M\,.$$

Now we shall consider the radial curvature of g^*. Let R and R^* be the curvature tensors of g and g^* respectively. Then we have ([5])

$$\begin{aligned} g^*(R^*(X, Y)Y, X)=e^{2f\circ\rho}g(R(X, Y)Y, X)+2S_{f\circ\rho}(X, Y)g^*(X, Y) \\ -S_{f\circ\rho}(Y, Y)g^*(X, X)-S_{f\circ\rho}(X, X)g^*(Y, Y) \end{aligned} \tag{5}$$

where

$$S_\sigma=Hess\,(\sigma)-d\sigma\otimes d\sigma+\frac{1}{2}\|grad\,\sigma\|^2 g\,, \qquad \sigma\in C^\infty(M)\,.$$

The radial curvature $K^*(Y\wedge grad\,\rho)$ at p $(Y\perp grad\,\rho)$ with respect to g^* is written as

$$\begin{aligned} K^*(Y\wedge grad\,\rho)=e^{-2f\circ\rho}\{K(Y\wedge grad\,\rho)-f''(\rho(p)) \\ -f'(\rho(p))\,Hess\,(\rho)(Y, Y)/\|Y\|^2\}\,. \end{aligned} \tag{6}$$

The above formula is obtained as follows. Since $Y\perp grad\,\rho$,

$$\begin{aligned} K^*(Y\wedge grad\,\rho) &= \frac{1}{e^{4f\circ\rho}\|Y\|^2\cdot\|grad\,\rho\|^2}\,g^*(R^*(Y, grad\,\rho)\,grad\,\rho, Y) \\ &= e^{-2f\circ\rho}\{K(Y\wedge grad\,\rho)-S_{f\circ\rho}(grad\,\rho, grad\,\rho)/\|grad\,\rho\|^2 \\ &\quad -S_{f\circ\rho}(Y, Y)/\|Y\|^2\}\,. \end{aligned}$$

On the other hand, $S_{f\circ\rho}=f'\,Hess\,(\rho)+\{f''-f'^2\}\,d\rho\otimes d\rho+1/2(f'^2)g$, hence we have (6).

PROPOSITION 6. *There is a function $f\circ\rho\in C^\infty(M)$ such that the radial curvrture is nonpositive eveywhere with respect to $g^*=e^{2f\circ\rho}g$.*

PROOF. By the assumption of Proposition 4, we can choose smooth functions $\bar{K}(t)$ from $\boldsymbol{R}^+$ to $\boldsymbol{R}$ which satisfies

$$\bar{K}(t)\geqq\max\{0, \text{ radial curvature at } x,\ \rho(x)=t\}\,.$$

Set $\bar{H}(t)=-\int_0^t\bar{K}(t)dt$, then $\bar{H}$ is also smooth and satisfies that

$$\bar{H}(t)\leqq\min\,\{Hess\,(\rho)(Y, Y) \text{ at } x,\ \rho(x)=t,\ Y\in M_x,\ \|Y\|=1\}\,.$$

The nonnegative function $\bar{u}(t)=\exp\left(-\int_0^t\bar{H}dt\right)\cdot\int_0^t\bar{K}(t)\exp\left(\int\bar{H}dt\right)dt$ is a solution of $d\bar{u}/dt+\bar{H}\bar{u}-\bar{K}=0$. Then we have for $\bar{u}$,

$$\frac{d\bar{u}}{dt}(t)+\bar{u}(t)\, Hess(\rho)(Y, Y)/\|Y\|^2-K(Y\wedge grad\ \rho)$$
$$=\bar{u}(t)\{Hess(\rho)(Y, Y)/\|Y\|^2-\bar{H}(t)\}+\{\bar{K}(t)-K(Y\wedge grad\ \rho)\}\leqq 0,$$

for each $Y\in M_x$, $\rho(x)=t$. Therefore, if we set $f(t)=\int_0^t \bar{u}(t)dt$, then (M, g^*), $g^*=e^{2f\circ\rho}g$ has nonpositive radial curvature from (6).

From these propositions, we have a required function $f\circ\rho$ in Proposition 4, since $\lim_{t\to\infty}\int_0^t e^{f(a)}da=\infty$ by $f'=\bar{u}\geqq 0$. Thus Proposition 4 is proved.

At the last part of ***2***, we find a necessary condition for the existence of a strictly convex radial function, by a group-theoretical version. The following proposition is a generalization of E. Cartan's fixed point theorem [2].

PROPOSITION 7 (*Fixed Point Theorem*). *Let (M, o) be a Riemannian manifold with a pole o. Let K be a compact Lie group which acts on M as isometries. If there is a strictly convex increasing radial function $f\circ\rho$, then K has a common fixed point.*

REMARK. If M is of negative curvature, then ρ^2 is strictly convex by comparing M with a Euclidean space. Thus we have the well known E. Cartan's fixed point theorem [2]: A compact Lie group which acts as isometries on a simply connected complete Riemannian manifold of negative curvature has a common fixed point.

PROOF. Let dk denote the Haar measure on K, normalized by $\int_K dk=1$. Consider the real function F on M given by $F(x)=\int_K f\circ\rho(k\cdot x)dk$. Then F is a nonnegative continuous function. Since $f\circ\rho$ is exhaustion and the orbit of o is compact, there is a ball $B_r(o)$ such that $F(x)>F(o)$ for all $x\in B_r(o)$. The closure of $B_r(o)$ contains a minimum point x_0 for F. The point x_0 is also a minimum for F on M. Since $F(k\cdot x_0)=F(x_0)$ for $k\in K$, in order to prove the existence of the fixed point, it is sufficient to show that $F(x)>F(x_0)$ if $x\neq x_0$. But this is derived by the strictly convexity of F, since $F(\gamma(t))''=\int_K \{f\circ\rho(k\cdot\gamma(t))\}''dk$ for every geodesic γ.

3. Let M be a complete open Kähler manifold. As in function theory, the *order* $\gamma(f)$ of a holomorphic function f is defined by

$$\gamma(f)=\limsup_{r\to+\infty}\ \log M(f, r)/\log r,$$

where $M(f, r)=\sup\{|f(x)|\,;\, x\in M, d(o, x)=r, o$ is a fixed point$\}$ [7]. The definition of $\gamma(f)$ does not depend on the choice of o. If $\gamma(f)$ is positive finite, then for each $\varepsilon>0$, there are $C>0$ and $\nu>0$ such that $\gamma(f)\leqq\nu<\gamma(f)+\varepsilon$ and $|f(x)|\leqq C(1+\rho(x))^{\nu}$ for all $x\in M$ ($\rho(x)=d(x, o)$).

We discuss some aspects of $\gamma(f)$.

Let (M, o) be a Kähler manifold with a pole o and (N, p) a model space, $\dim M=\dim N=n$, which satisfy the radial curvature $K_M(t)\leqq$the radial curvature $K_{N(t)}$ for all $t>0$. By a *model* we mean a Riemannian manifold (N, p) with a pole p such that every linear isometry $\phi: N_p\to N_p$ is realized as the differential of an isometry $\Phi: N\to N$ ([1]). Let $V_M(r)$ and $V_N(r)$ be the volumes of the open balls $B_M(r)$ and $B_N(r)$ of radius r around o and p in M and N respectively. Note that by the sub-mean value property, $V_M(r)\geqq V_N(r)$.

Now we show the following

PROPOSITION 8. *Assume that* $V_M(r)\sim r^{\alpha}$, $V_N(r)\sim r^{\beta}$, $\beta\geqq 1$ $(r\to\infty)$. *If a holomorphic function* f *has* $\gamma(f)<1+(\beta-\alpha)/2$, *then* $df=0$ *at* o.

REMARK. If (M, o) is of nonpositive curvature and $\alpha<2n+2$ in the above proposition, then a bounded holomorphic function is constant, since every point gives a pole. Note that $V_N(r)\sim r^{2n}$ for $(N, p)=(\boldsymbol{C}^n, o)$ with a flat metric.

Before the proof of the proposition, we have some lemmas.

LEMMA (*Sub-mean-value Property*). *Let* ϕ *be a continuous nonnegative subharmonic function on* M, *then*

$$\int_{B_M(r)}\phi\geqq V_N(r)\phi(o)\qquad \text{for all } r>0\,.$$

For the proof, see Theorem B, [1].

LEMMA (*Integral Inequality of the Laplacian*). *Assume that* $(d/dr)V_N(r)$ *is an increasing function. Let* f *be a nonnegative subharmonic function. Then for all* λ, $0<\lambda<1$, *there is a constant* $\gamma=\gamma_\lambda>0$ *such that*

$$\int_{B_M(\lambda r)}\Delta f\leqq\frac{\gamma}{r^2}\int_{B_M(r)}f\,. \tag{7}$$

PROOF. Since $f\geqq 0$, we have, from (3, 6) in [1]

$$\int_{B_M(r)}\left[\Delta f\left(\int_{t=\rho}^{t=r}\frac{dt}{v_N(t)}\right)\right]dv\leqq\frac{1}{v_N(r)}\int_{S_M(r)}f\,d\omega(r)\,,$$

which implies

$$\int_0^r\int_t^r\Big(\int_{S_M(t)}\Delta f\,d\omega(t)\Big)\frac{ds}{v_N(s)}dt\leqq\frac{1}{v_N(r)}\int_{S_M(r)}f\,d\omega(r)\,,$$

where $v_N(r)=v(r)$ denotes the volume of the r-sphere $S_N(r)$ around p in N.

By using Fubini's theorem with respect to s and t on the left hand side, we have

$$\frac{1}{v(r)}\int_{S(r)}f\,d\omega(r)\geqq\int_0^r\Big[\int_0^s\Big(\int_{S(t)}\Delta f\,d\omega(t)\Big)dt\Big]\frac{ds}{v(s)}$$

$$=\int_0^r\Big(\frac{1}{v(s)}\int_{B(s)}\Delta f\Big)ds\,.$$

Multiply by $v(r)$ and integrate relative to r. Then

$$\int_{B(u)}f\leqq\int_0^u v(r)\Big[\int_0^r\Big(\frac{1}{v(s)}\int_{B(s)}\Delta f\Big)ds\Big]dr\,.$$

Since $\Delta f\geqq 0$,

$$\int_0^r\frac{1}{v(s)}\Big(\int_{B(s)}\Delta f\Big)ds\geqq\int_{\sqrt{\lambda}r}^r\frac{1}{v(s)}\Big(\int_{B(\sqrt{\lambda}r)}\Delta f\Big)ds=\Big(\int_{B(\sqrt{\lambda}r)}\Delta f\Big)\cdot\int_{\sqrt{\lambda}r}^r\frac{ds}{v(s)}$$

and that

$$\int_0^u v(r)\Big(\int_0^r\frac{1}{v(s)}\int_{B(s)}\Delta f\,ds\Big)dr\geqq\int_0^u v(r)\Big(\int_{B(\sqrt{\lambda}r)}\Delta f\Big)\Big(\int_{\sqrt{\lambda}r}^r\frac{ds}{v(s)}\Big)dr$$

$$\geqq\int_{\sqrt{\lambda}u}^u\Big(v(r)\cdot\int_{\sqrt{\lambda}r}^r\frac{ds}{v(s)}\Big)dr\cdot\Big(\int_{B(\lambda u)}\Delta f\Big)\geqq(1-\sqrt{\lambda})(1-\lambda)\frac{u^2}{2}\int_{B(\lambda u)}\Delta f\,,$$

where the last inequality follows from $v_N(r)$ being increasing. Hence we obtain the inequality (7).

LEMMA (*Cauchy's inequality for derivatives of holomorphic functions*). *For each holomorphic function f on M,*

$$\|df\|^2(0)\leqq\frac{\gamma}{V_N(r/2)r^2}\int_{B_M(r)}|f|^2\,.$$

PROOF. Since $\Delta|f|^2=\|df\|^2$ and $\Delta\|df\|^2=\|\nabla df\|^2$, from above lemmas,

$$\|df\|^2(0)\leqq\frac{1}{V_N(r/2)}\int_{B_M(r/2)}\|df\|$$

$$=\frac{\gamma}{V_N(r/2)r^2}\int_{B(r)}|f|^2\,,\qquad\text{where }\gamma=\gamma_{1/2}\,.$$

PROOF OF PROPOSITION 8. Since $\gamma(f)<1+1/2(\beta-\alpha)$, there is $\nu>0$ such that

$\gamma(f)<\nu<1+(\beta-\alpha)/2$, hence we have $|f(x)|<C(1+\rho(x))^{\nu}$ for some $C>0$. Then, from the above,

$$\|df\|^2(0)\leqq\frac{\gamma}{V_N(r/2)r^2}\int_{B(r)}|f|^2\leqq\frac{\gamma C^2}{V_N(r/2)r^2}(1+r)^{2\nu}V_M(r)\sim r^{(2\nu-2+\alpha-\beta)}.$$

Letting $r\to\infty$, we have $df=0$ at o.

As an application of the proposition, we have the following

COROLLARY 9. *Let $F=(f^1,\cdots,f^N)$; $M\to\boldsymbol{C}^N$ be a holomorphic mapping. If $\sum_{j=1}^{n}\gamma(f^{i_j})<n-n(\alpha-\beta)/2$ for each $1\leqq i_1<\cdots<i_n\leqq N$, then F is not of maximal rank at o.*

Moreover, if M is a Stein manifold and F; $M\to\boldsymbol{C}^N$ is a proper holomorphic imbedding, then $\sum_{j=1}^{n}\gamma(f^{i_j})\geqq n-n(\alpha-\beta)/2$ for some $1\leqq i_1<\cdots<i_n\leqq N$.

PROOF. Consider the holomorphic n-forms $df^{i_1}\wedge\cdots\wedge df^{i_n}$, $1\leqq i_1<\cdots<i_n\leqq N$. From the proposition, we have an estimate of the norm of $df^{i_1}\wedge\cdots\wedge df^{i_n}$;

$$\begin{aligned}\|df^{i_1}\wedge\cdots\wedge df^{i_n}\|^2(o)&\leqq\prod_{j=1}^{n}\|df^{i_j}\|^2(o)\leqq\prod_{j=1}^{n}\frac{\gamma}{V_N(r/2)r^2}\int_{B(r)}|f^{i_j}|^2\\&=\frac{\gamma^n}{V_N(r/2)^n\cdot r^{2n}}\prod_j\int_{B(r)}|f^{i_j}|^2\\&\leqq\gamma^n\prod C_j^2\cdot\frac{(1+r)^{2\Sigma\nu_j}\cdot V_M(r)^n}{V_N(r/2)^n r^{2n}},\end{aligned}$$

where $\nu_j>0$, $j=1,\cdots,n$, satisfy $\gamma(f^{i_j})<\nu_j<\gamma(f^{i_j})+\varepsilon_j$ and $\sum_j\nu_j<n-n(\alpha-\beta)/2$. By letting $r\to\infty$, we have $df^{i_1}\wedge\cdots\wedge df^{i_n}=0$ at o.

The last statement is easily derived from the above argument, since the F is of maximal rank everywhere.

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