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INJECTIVE DIMENSION OF GENERALIZED TRIANGULAR MATRIX RINGS

By

Kazunori SAKANO

Throughout this paper, let R and S denote rings with identity, M an (S, R)-bimodule, and Λ a generalized triangular matrix ring defined by ${}_SM_R$, i. e.,

$$\Lambda=\begin{bmatrix} R & 0 \\ M & S \end{bmatrix}$$

with the addition by element-wise and the multiplication by

$$\begin{bmatrix} r & 0 \\ m & s \end{bmatrix}\cdot\begin{bmatrix} r' & 0 \\ m' & s' \end{bmatrix}=\begin{bmatrix} rr' & 0 \\ mr'+sm' & ss' \end{bmatrix}.$$

The main purpose of the present paper is to estimate id-Λ_Λ, the injective dimension of Λ_Λ, in terms of those of R_R, M_R, and S_S. In fact, if we assume that fd-${}_SM$, the flat dimension of ${}_SM$, is finite, then there hold the inequalities

$$\max(\text{id-}R_R,\ \text{id-}M_R,\ \text{id-}S_S-\text{fd-}{}_SM)\leqq \text{id-}\Lambda_\Lambda\leqq$$
$$\max(\max(\text{id-}R_R,\ \text{id-}M_R)+\text{fd-}{}_SM,\ \text{id-}S_S-1)+1\,.$$

In this connection, we investigate the case when the left-hand or the right-hand side equality holds under the condition that ${}_SM$ is flat.

In [7], Zaks shows that the injective dimension of an $n\times n$ lower triangular matrix ring over a semiprimary ring R is just equal to id-R_R+1. An example is constructed to show that the condition on R benig semiprimary is redundant in his theorem.

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Let $e=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\in\Lambda$ and $e'=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\in\Lambda$. Then $R\cong e\Lambda e$, $M\cong e'\Lambda e$, and $S\cong e'\Lambda e'$.

LEMMA 1. *Let X be a right Λ-module with $X=Xe$.*

(1) *If X_R is projective, then X_Λ is projective.*

(2) $\mathrm{Ext}^i_\Lambda(X_\Lambda, \Lambda_\Lambda)\cong\mathrm{Ext}^i_\Lambda(X_R, \Lambda e_R)$.

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PROOF. (1) This is found in [2, Theorem 2.2].

(2) By [4, Exercise 22, p. 114], we have the natural isomorphism

$$\mathrm{Hom}_\Lambda(X_\Lambda, \Lambda_\Lambda) \cong \mathrm{Hom}_R(X_R, \Lambda e_R).$$

It follows that

$$\mathrm{Ext}^i_\Lambda(X_\Lambda, \Lambda_\Lambda) \cong \mathrm{Ext}^i_R(X_R, \Lambda e_R),$$

for a projective resolution of X_R may be viewed as one of X_Λ by (1).

LEMMA 2. *Let Y be a right Λ-module.*

(1) *If Y_Λ is projective, then Ye'_S is projective.*

(2) $\mathrm{Ext}^i_\Lambda(Y_\Lambda, e'\Lambda/e'\Lambda e_\Lambda) \cong \mathrm{Ext}^i_S(Ye'_S, S_S)$.

PROOF. (1) This is found in [4, Exercise 19, p. 114].

(2) By [4, Exercise 22, p. 114], we have the natural isomorphism

$$\mathrm{Hom}_\Lambda(Y_\Lambda, e'\Lambda/e'\Lambda e_\Lambda) \cong \mathrm{Hom}_S(Ye'_S, S_S).$$

Note that, if $P'_\Lambda \to P_\Lambda \to P''_\Lambda$ is an exact sequence of projective Λ-modules, then so is $P'e'_S \to Pe'_S \to P''e'_S$ of projective S-modules in view of (1). Thus

$$\mathrm{Ext}^i_\Lambda(Y_\Lambda, e'\Lambda/e'\Lambda e_\Lambda) \cong \mathrm{Ext}^i_S(Ye'_S, S_S).$$

LEMMA 3 [4, Proposition 4.1]. *Every right ideal of Λ has the from of $\begin{bmatrix} X & 0 \\ & K \end{bmatrix}$, where K is a right ideal of S and $\begin{bmatrix} 0 \\ KM \end{bmatrix}_R \subseteq X_R \subseteq \begin{bmatrix} R \\ M \end{bmatrix}_R$.*

THEOREM 4. *Assume that* fd-$_SM$ *is finite. Then we have*

$$\max(\text{id-}R_R, \text{id-}M_R, \text{id-}S_S - \text{fd-}_SM) \leqq \text{id-}\Lambda_\Lambda \leqq$$

$$\max(\max(\text{id-}R_R, \text{id-}M_R) + \text{fd-}_SM, \text{id-}S_S - 1) + 1.$$

PROOF. Suppose $\max(\max(\text{id-}R_R, \text{id-}M_R) + \text{fd-}_SM, \text{id-}S_S - 1) + 1 = t$. Let $\begin{bmatrix} X & 0 \\ & K \end{bmatrix}$ be a right ideal of Λ. Since R can be considered as a left Λ-module via ρ: $\Lambda \to R\left(\begin{bmatrix} r & 0 \\ m & s \end{bmatrix} \mapsto r\right)$, the exact sequence

$$0 \longrightarrow {}_\Lambda\begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix} \longrightarrow {}_\Lambda\Lambda \longrightarrow {}_\Lambda R \longrightarrow 0$$

induces

$$\mathrm{Tor}^\Lambda_{i+1}(C, R) \cong \mathrm{Tor}^\Lambda_i\left(C, \begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix}\right) \qquad (i \geqq 1)$$

for every right Λ-module C. It follows that $\text{fd-}_\Lambda\begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix} + 1 = \text{fd-}_\Lambda R$. Moreover, since $_\Lambda S$ is flat, $\text{fd-}_SM = \text{fd-}_\Lambda\begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} = \text{fd-}_\Lambda\begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix}$ by [1, Proposition 4.1.1, p. 117].

Therefore $\text{fd-}_{\Lambda}R=\text{fd-}_{\Lambda}\begin{bmatrix}0 & 0\\ M & S\end{bmatrix}+1=\text{fd-}_{S}M+1$. The exact sequence of right Λ-modules

$$0\longrightarrow\begin{bmatrix}R & 0\\ M & 0\end{bmatrix}\longrightarrow\Lambda\longrightarrow\Lambda/\begin{bmatrix}R & 0\\ M & 0\end{bmatrix}\longrightarrow 0$$

yields the following exact sequence

$$\text{Ext}_{\Lambda}^{t}\left(\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},\begin{bmatrix}R & 0\\ M & 0\end{bmatrix}\right)\rightarrow\text{Ext}_{\Lambda}^{t}\left(\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},\Lambda\right)\rightarrow\text{Ext}_{\Lambda}^{t}\left(\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},\Lambda/\begin{bmatrix}R & 0\\ M & 0\end{bmatrix}\right).$$

Since

$$\text{Hom}_{\Lambda}\left(\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},\begin{bmatrix}R & 0\\ M & 0\end{bmatrix}\right)\cong\text{Hom}_{\Lambda}\left(\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},\text{Hom}_{R}\left(R,\begin{bmatrix}R & 0\\ M & 0\end{bmatrix}\right)\right)$$
$$\cong\text{Hom}_{R}\left(\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix}\otimes_{\Lambda}R,\begin{bmatrix}R & 0\\ M & 0\end{bmatrix}\right),$$

the resulting spectral sequence is

$$E_2^{p,q}=\text{Ext}_R^{q}\left(\text{Tor}_p^{\Lambda}\left(\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},R\right),\begin{bmatrix}R & 0\\ M & 0\end{bmatrix}\right)\underset{q}{\Rightarrow}\text{Ext}_{\Lambda}^{n}\left(\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},\begin{bmatrix}R & 0\\ M & 0\end{bmatrix}\right).$$

Since $E_2^{p,q}=0$ for either $q>\max(\text{id-}R_R,\ \text{id-}M_R)$ or $p>\text{fd-}_{S}M$, we have $\text{Ext}_{\Lambda}^{n}\left(\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},\begin{bmatrix}R & 0\\ M & 0\end{bmatrix}\right)=0$ for $n>\max(\text{id-}R_R,\ \text{id-}M_R)+\text{fd-}_{S}M$. Since

$$\text{Ext}_{\Lambda}^{t}\left(\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},\Lambda/\begin{bmatrix}R & 0\\ M & 0\end{bmatrix}\right)\cong\text{Ext}_{\Lambda}^{t}\left(\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},\begin{bmatrix}0 & 0\\ M & S\end{bmatrix}/\begin{bmatrix}0 & 0\\ M & 0\end{bmatrix}\right)$$
$$\cong\text{Ext}_{S}^{t}(K,S)=0$$

by Lemma 2, we have $\text{Ext}_{\Lambda}^{t}\left(\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},\Lambda\right)=0$. It follows that $\text{id-}\Lambda_{\Lambda}\leqq t$ from the exactness of the sequence

$$\text{Ext}_{\Lambda}^{t}\left(\begin{bmatrix}X & 0\\ & K\end{bmatrix}/\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},\Lambda\right)\rightarrow\text{Ext}_{\Lambda}^{t}\left(\begin{bmatrix}X & 0\\ & K\end{bmatrix},\Lambda\right)\rightarrow\text{Ext}_{\Lambda}^{t}\left(\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},\Lambda\right),$$

and from the fact that

$$\text{Ext}_{\Lambda}^{t}\left(\begin{bmatrix}X & 0\\ & K\end{bmatrix}/\begin{bmatrix}0 & 0\\ KM & K\end{bmatrix},\Lambda\right)\cong\text{Ext}_{R}^{t}(X/KM,\ R\oplus M)=0$$

by Lemma 1.

Conversely, suppose $\text{id-}\Lambda_{\Lambda}=m$. Then Lemma 1 forces that $\text{id-}R_R\leqq m$ and id-$M_R \leqq m$. Now, let K be a right ideal of S. Since

$$\mathrm{Hom}_\Lambda\left(S/K \otimes_S \begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix}, \Lambda\right) \cong \mathrm{Hom}_S\left(S/K, \mathrm{Hom}_\Lambda\left(\begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix}, \Lambda\right)\right)$$
$$\cong \mathrm{Hom}_S(S/K, S)$$

and $\mathrm{Ext}^i_\Lambda\left(\begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix}, \Lambda\right)=0$ for $i>0$, the resulting spectral sequence is

$$E_2^{p,q}=\mathrm{Ext}^q_\Lambda\left(\mathrm{Tor}^S_p\left(S/K, \begin{bmatrix} 0 & 0 \\ M & S \end{bmatrix}\right), \Lambda\right) \underset{q}{\Rightarrow} \mathrm{Ext}^n_S(S/K, S).$$

Since $E_2^{p,q}=0$ for either $q>\text{id-}\Lambda_\Lambda$ or $p>\text{fd-}_SM$, we have $\mathrm{Ext}^n_S(S/K, S)=0$ for $n>\text{id-}\Lambda_\Lambda+\text{fd-}_SM$. Thus id-$S_S-$fd-$_SM \leqq$ id-Λ_Λ.

The following is essentially in [1, p. 346].

LEMMA 5. *Let* A_S, ${}_SB_\Lambda$, *and* C_Λ *be modules such that* $\mathrm{Ext}^i_\Lambda(B, C)=0$ $(i>0)$ *and* $\mathrm{Tor}^S_i(A, B)=0$ $(i>0)$. *Then there holds*

$$\mathrm{Ext}^n_S(A, \mathrm{Hom}_\Lambda(B, C)) \cong \mathrm{Ext}^n_\Lambda(A \otimes_S B, C).$$

LEMMA 6. *Assume that* ${}_SM$ *is flat. Let*

$$f^\#_i=\mathrm{Ext}^i_\Lambda(f, 1_\Lambda): \mathrm{Ext}^i_\Lambda\left(\Lambda/\begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right) \to \mathrm{Ext}^i_\Lambda\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix}\Big/\begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right)$$

be the induced map by

$$f: \begin{bmatrix} R & 0 \\ M & K \end{bmatrix}\Big/\begin{bmatrix} R & 0 \\ KM & K \end{bmatrix} \hookrightarrow \Lambda/\begin{bmatrix} R & 0 \\ KM & K \end{bmatrix},$$

where K *is a right ideal of* S. *Then* $\mathrm{Im}\, f^\#_i$ *is contained in*

$$\mathrm{Ext}^i_\Lambda\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix}\Big/\begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, e'\Lambda\right),$$

a direct summand of

$$\mathrm{Ext}^i_\Lambda\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix}\Big/\begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right).$$

PROOF. Let

$$\longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow S/K \longrightarrow 0$$

be a free resolution of S/K, and

$$\longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow \begin{bmatrix} R & 0 \\ M & K \end{bmatrix}\Big/\begin{bmatrix} R & 0 \\ KM & K \end{bmatrix} \longrightarrow 0$$

a projective resolution of $\begin{bmatrix} R & 0 \\ M & K \end{bmatrix} \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}$. Then

$$\longrightarrow P_n \otimes_S e'\Lambda \longrightarrow P_{n-1} \otimes_S e'\Lambda \longrightarrow \cdots \longrightarrow P_0 \otimes_S e'\Lambda \longrightarrow S/K \otimes_S e'\Lambda \longrightarrow 0$$

is a projective resolution of $S/K \otimes_S e'\Lambda$, since ${}_SM$ is flat. Consider the following exact commutative diagram

$$\begin{array}{ccccccccccc}
\longrightarrow & P_n \otimes_S e'\Lambda & \longrightarrow & P_{n-1} \otimes_S e'\Lambda & \longrightarrow \cdots \longrightarrow & P_0 \otimes_S e'\Lambda & \longrightarrow & P_n \otimes_S e'\Lambda & \longrightarrow & 0 \\
 & \uparrow f_n & & \uparrow f_{n-1} & & \uparrow f_0 & & \wr\uparrow g & & \\
 & & & & & & & \Lambda \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix} & & \\
 & & & & & & & \uparrow f & & \\
\longrightarrow & Q_n & \longrightarrow & Q_{n-1} & \longrightarrow \cdots \longrightarrow & Q_0 & \longrightarrow & \begin{bmatrix} R & 0 \\ M & K \end{bmatrix} \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix} & \longrightarrow & 0,
\end{array}$$

where (f_i) is a map over $g \circ f$. Now, every element of $\mathrm{Hom}_\Lambda(e'\Lambda, \Lambda)$ is given by the left multiplication of $\Lambda e'$, so

$$\mathrm{Hom}_\Lambda(e'\Lambda, \Lambda) = \mathrm{Hom}_\Lambda(e'\Lambda, \Lambda e'\Lambda) = \mathrm{Hom}_\Lambda(e'\Lambda, e'\Lambda).$$

It follows that

$$\begin{aligned} \mathrm{Hom}_\Lambda(P_n \otimes_S e'\Lambda, \Lambda) &= \mathrm{Hom}_\Lambda(S^{(I_n)} \otimes_S e'\Lambda, \Lambda) \\ &\cong \mathrm{Hom}_\Lambda(e'\Lambda^{(I_n)}, \Lambda) \\ &\cong \mathrm{Hom}_\Lambda(e'\Lambda^{(I_n)}, e'\Lambda) \\ &\cong \mathrm{Hom}_\Lambda(P_n \otimes_S e'\Lambda, e'\Lambda), \end{aligned}$$

hence that

$$\mathrm{Im}\,\mathrm{Hom}_\Lambda(f_n, 1_\Lambda) \subset \mathrm{Hom}_\Lambda(Q_n, e'\Lambda).$$

Thus

$$\mathrm{Im}\,\mathrm{Ext}^i_\Lambda(f, 1_\Lambda) \subset \mathrm{Ext}^i_\Lambda\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix} \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, e'\Lambda\right).$$

PROPOSITION 7. *Assume that* ${}_SM$ *is flat and put* $\max(\text{id-}R_R, \text{id-}M_R) = i$.

(1) *If* $\text{id-}S_S > i$, *then* $\text{id-}\Lambda_\Lambda = \text{id-}S_S$.

(2) *If* $\text{id-}S_S < i \neq 0$, *then* $\text{id-}\Lambda_\Lambda = i$ *if and only if* $\mathrm{Ext}^i_R(M/KM, R \oplus M) = 0$ *for every right ideal* K *of* S.

(3) *If* $\text{id-}S_S = i \neq 0$ *and if* $\mathrm{Ext}^i_R(M/KM, R \oplus M) = 0$ *for every right ideal* K *of* S, *then* $\text{id-}\Lambda_\Lambda = i$.

(4) *If* $\text{id-}S_S = i \neq 0$ *and if* $\mathrm{Ext}^i_R(M/RM, R) \neq 0$ *for some right ideal* K *of* S, *then* $\text{id-}\Lambda_\Lambda = i + 1$.

PROOF. (1) This directly follows from Theorem 4.

(2) Let $\begin{bmatrix} X & 0 \\ & K \end{bmatrix}$ be a right ideal of Λ. Since

$$\mathrm{Ext}_\Lambda^{i+1}\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix}\Big/\begin{bmatrix} X & 0 \\ & K \end{bmatrix}, \Lambda\right) \cong \mathrm{Ext}_R^{i+1}((R\oplus M)/X, R\oplus M)=0$$

and

$$\begin{aligned}\mathrm{Ext}_\Lambda^{i}\left(\Lambda\Big/\begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right) &\cong \mathrm{Ext}_\Lambda^{i}(S/K\otimes_S e'\Lambda, \Lambda) \\ &\overset{\Phi}{\cong} \mathrm{Ext}_S^{i}(S/K, \mathrm{Hom}_\Lambda(e'\Lambda, \Lambda)) \\ &\cong \mathrm{Ext}_S^{i}(S/K, S)=0,\end{aligned}$$

where Φ is an isomorphism by Lemma 5, we obtain the following exact sequences

$$\mathrm{Ext}_\Lambda^{i+1}\left(\Lambda\Big/\begin{bmatrix} R & 0 \\ M & K \end{bmatrix}, \Lambda\right) \longrightarrow \mathrm{Ext}_\Lambda^{i+1}\left(\Lambda\Big/\begin{bmatrix} X & 0 \\ & K \end{bmatrix}, \Lambda\right) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}_\Lambda^{i+1}\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix}\Big/\begin{bmatrix} X & 0 \\ & K \end{bmatrix}, \Lambda\right)=0$$

and

$$0=\mathrm{Ext}_\Lambda^{i}\left(\Lambda\Big/\begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right) \longrightarrow \mathrm{Ext}_\Lambda^{i}\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix}\Big/\begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}_\Lambda^{i+1}\left(\Lambda\Big/\begin{bmatrix} R & 0 \\ M & K \end{bmatrix}, \Lambda\right) \longrightarrow \mathrm{Ext}_\Lambda^{i+1}\left(\Lambda\Big/\begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right)=0,$$

from which it follows that, for every right ideal K of S,

$$\begin{aligned}\text{id-}\Lambda_\Lambda=i &\Leftrightarrow \mathrm{Ext}_\Lambda^{i+1}\left(\Lambda\Big/\begin{bmatrix} R & 0 \\ M & K \end{bmatrix}, \Lambda\right)=0 \\ &\Leftrightarrow \mathrm{Ext}_R^{i}(M/KM, R\oplus M) \cong \mathrm{Ext}_\Lambda^{i}\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix}\Big/\begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right)=0.\end{aligned}$$

(3) Let $\begin{bmatrix} X & 0 \\ & K \end{bmatrix}$ be a right ideal of Λ. Considering the following exact sequences in the similar manneras in (2)

$$\mathrm{Ext}_\Lambda^{i+1}\left(\Lambda\Big/\begin{bmatrix} R & 0 \\ M & K \end{bmatrix}, \Lambda\right) \longrightarrow \mathrm{Ext}_\Lambda^{i+1}\left(\Lambda\Big/\begin{bmatrix} X & 0 \\ & K \end{bmatrix}, \Lambda\right) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}_\Lambda^{i+1}\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix}\Big/\begin{bmatrix} X & 0 \\ & K \end{bmatrix}, \Lambda\right)=0$$

and

$$\mathrm{Ext}_\Lambda^i\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix} \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right) \longrightarrow \mathrm{Ext}_\Lambda^{i+1}\left(\Lambda \Big/ \begin{bmatrix} R & 0 \\ M & K \end{bmatrix}, \Lambda\right) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}_\Lambda^{i+1}\left(\Lambda \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right) = 0,$$

we conclude that id-$\Lambda_\Lambda = i$ if $\mathrm{Ext}_R^i(M/KM, R\oplus M) \cong \mathrm{Ext}_\Lambda^i\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix} \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right)$ $=0$ for every right ideal K of S.

(4) Let K be a right ideal of S such that $\mathrm{Ext}_R^i(M/KM, R) \neq 0$. Let

$$f: \begin{bmatrix} R & 0 \\ M & K \end{bmatrix} \Big/ \begin{bmatrix} R & 0 \\ M & K \end{bmatrix} \hookrightarrow \Lambda \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}.$$

Then f induces a non-epimorphism

$$f_i^\#: \mathrm{Ext}_\Lambda^i\left(\Lambda \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right) \longrightarrow \mathrm{Ext}_\Lambda^i\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix} \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right) =$$

$$\mathrm{Ext}_\Lambda^i\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix} \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, e'\Lambda\right) \oplus$$

$$\mathrm{Ext}_\Lambda^i\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix} \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, e\Lambda\right)$$

by the preceding Lemma 6, It follows that $\mathrm{Ext}_\Lambda^{i+1}\left(\Lambda \Big/ \begin{bmatrix} R & 0 \\ M & K \end{bmatrix}, \Lambda\right) \neq 0$ from the exactness of the following sequence

$$\mathrm{Ext}_\Lambda^i\left(\Lambda \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right) \xrightarrow{f_i^\#} \mathrm{Ext}_\Lambda^i\left(\begin{bmatrix} R & 0 \\ M & K \end{bmatrix} \Big/ \begin{bmatrix} R & 0 \\ KM & K \end{bmatrix}, \Lambda\right) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}_\Lambda^{i+1}\left(\Lambda \Big/ \begin{bmatrix} R & 0 \\ M & K \end{bmatrix}, \Lambda\right),$$

hence that id-$\Lambda_\Lambda = i+1$ together with Theorem 4.

It is remaining the case when R_R, M_R, and S_S are all injective. Since ${}_SM_R$ can be considered as an $(R\oplus S, R\oplus S)$-bimodule in the natural way, i. e., $(r, s)m = sm$ and $m(r, s) = mr$, Λ can be regarded as the trivial extension of the ring $R\oplus S$ by the $(R\oplus S, R\oplus S)$-bimodule M. Thus [6, Theorem 1. 4. 1] can be applied to the above, namely, PROPOSITION 8. *Let* $\mu: S \rightarrow \mathrm{End}(M_R)$ *be the canonical map. Then* Λ_Λ *is injective iff*

(1) R_R, M_R, *and* $\boldsymbol{l}_S(M)_S = \{s \in S\,;\ sm = 0$ *for every* $m \in M\}$ *are all injective.*

(2) μ *is an epimorphism.*

(3) $\mathrm{Hom}_R(M_R, R_R) = 0$.

REMARK 9. Let $A \ltimes N$ denote the trivial extension of the ring A by the (A, A)-bimodule N. It appeared in [3] concerning the injective dimension of $A \ltimes N_{A \ltimes N}$ that, if $\mathrm{Ext}_A^i(N_A, N_A) \cong \begin{cases} A\,(i=0) \\ 0\,(i>0) \end{cases}$, then id-$N_A$=id-$A \ltimes N_{A \ltimes N}$. This yields, however, only a trivial result for our situations, because $\mathrm{End}(M_{R \oplus S}) \cong R \oplus S$ iff $R = M = S = 0$.

REMARK 10. In view of Theorem 4, we may consider the following five cases concerning the relationships between id-R_R, id-M_R, and id-S_S under the condition that ${}_SM$ is flat.

Case 1. id-R_R=id-M_R=id-S_S=id-Λ_Λ.

Case 2. id-R_R=id-M_R=id-S_S=id-$\Lambda_\Lambda - 1$.

Case 3. Each of (id-R_R, id-M_R, id-S_S) does not equal to the other and max(id-R_R, id-M_R, id-S_S)=id-Λ_Λ.

Case 4. Each of (id-R_R, id-M_R, id-S_S) does not equal to the other and max(id-R_R, id-M_R, id-S_S)=id-$\Lambda_\Lambda - 1$.

Case 5. The other cases.

The following Examples are given to show the existence of each of the above cases.

Example of Case 1. Let R be an infinite direct product of fields, I a maximal ideal containing their direct sum, and $M = R/I$. Let

$$\Lambda = \begin{bmatrix} R & 0 \\ M & \mathrm{End}(M_R) \end{bmatrix}.$$

Since R is a V-ring, M_R is injective. Moreover, $\mathrm{Hom}_R(M_R, R_R) = 0$. Thus Λ_Λ is injective by Proposition 8.

Example of Case 2. Let Λ_2 be a 2×2 lower triangular matrix ring over a ring $R \neq 0$ with id-$R_R = i < +\infty$. Since $\mathrm{Ext}_R^i(R/I, R) \neq 0$ $(i>0)$ for some right ideal I of R and $\mathrm{Hom}_R(R_R, R_R) \neq 0$, id-$(\Lambda_2)_{\Lambda_2}$=id-$R_R + 1$ by Theorem 4, Propositions 7, and 8.

Example of Case 3. Let

$$\Lambda=\left[\begin{array}{cc:c} \boldsymbol{Z} & 0 & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} & 0 \\ \hdashline \boldsymbol{Q} & \boldsymbol{Q} & \boldsymbol{Z} \end{array}\right], \quad R=\begin{bmatrix} \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & \boldsymbol{Q} \end{bmatrix}.$$

Then id-$R_R=2$, id-$(\boldsymbol{Q}\ \boldsymbol{Q})_R=0$, and id-$\boldsymbol{Z}_{\boldsymbol{Z}}=1$. Since $\mathrm{Ext}_R^2((\boldsymbol{Q}\ \boldsymbol{Q})/K(\boldsymbol{Q}\ \boldsymbol{Q}), R\oplus(\boldsymbol{Q}\ \boldsymbol{Q}))=0$ for every right ideal K of $\boldsymbol{Z}$, we have id-$\Lambda_\Lambda=2$ by Proposition 7 (2).

Example of Case 4. Let

$$\Lambda=\left[\begin{array}{cc:c} \boldsymbol{Z} & 0 & 0 \\ \boldsymbol{Z} & \boldsymbol{Z} & 0 \\ \hdashline 0 & \boldsymbol{Q} & \boldsymbol{Z} \end{array}\right], \quad R=\begin{bmatrix} \boldsymbol{Z} & 0 \\ \boldsymbol{Z} & \boldsymbol{Z} \end{bmatrix}.$$

Then id-$R_R=2$ and id-$\boldsymbol{Z}_{\boldsymbol{Z}}=1$. Since $(0\ \boldsymbol{Q})_{\boldsymbol{Z}}$ (resp. ${}_{\boldsymbol{Z}}\boldsymbol{Z}$) can be considered as a right (resp. left) R-module via $\sigma: R\to\boldsymbol{Z}\left(\begin{bmatrix} z & 0 \\ z' & z'' \end{bmatrix}\mapsto z''\right)$, we have

$$(0\ \boldsymbol{Q})_R\cong\mathrm{Hom}_{\boldsymbol{Z}}({}_R\boldsymbol{Z}_{\boldsymbol{Z}}, (0\ \boldsymbol{Q})_{\boldsymbol{Z}})_R.$$

Since ${}_R\boldsymbol{Z}\cong{}_RRe'$ is flat and $(0\ \boldsymbol{Q})_{\boldsymbol{Z}}$ is injective, $(0\ \boldsymbol{Q})_R$ is injective. It follows that

$$\begin{aligned}\mathrm{Ext}_R^2((0\ \boldsymbol{Q}), R)&\cong\mathrm{Ext}_R^2((\boldsymbol{Q}\ \boldsymbol{Q})/(\boldsymbol{Q}\ 0), R)\\ &\cong\mathrm{Ext}_R^2((\boldsymbol{Q}\otimes_{\boldsymbol{Z}}(\boldsymbol{Z}\ \boldsymbol{Z}))/(\boldsymbol{Q}\otimes_{\boldsymbol{Z}}(\boldsymbol{Z}\ 0)), R)\neq 0\end{aligned}$$

from the proof of [7, Lemma B] together with $\mathrm{Ext}_{\boldsymbol{Z}}^1(\boldsymbol{Q}, \boldsymbol{Z})\neq 0$. Hence id-$\Lambda_\Lambda=3$ by Theorem 4 and Proposition 7 (2).

Example of Case 5. Let Λ_n $(n>2)$ be an $n\times n$ lower triangular matrix ring over a ring $R\neq 0$ with id-$R_R=i<+\infty$. Since Λ_n can be considered as

$$\left[\begin{array}{c:c} R & 0\cdots\cdots 0 \\ \hdashline R & \\ \vdots & \Lambda_{n-1} \\ R & \end{array}\right],$$

id-$(\Lambda_n)_{\Lambda_n}$=id-$(\Lambda_{n-1})_{\Lambda_{n-1}}$ by induction on n together with Proposition 7 (1). Hence id-$(\Lambda_n)_{\Lambda_n}$=id-R_R+1.

REMARK 11. (1) Example of Case 1 is due to T. Kato.

(2) T. Sumioka has also independently observed that the injective dimension of an $n\times n$ lower triangular matrix ring over a ring R has the injective dimension $\leqq$id-R_R+1.

Referencse

[1] Cartan, H. and Eilenberg, S., Homological Algebra. Princeton Univ. Press, Princeton, 1956.
[2] Chase, S., A generalization of the ring of triangular matrices. Nagoya Math. J., 18 (1961), 13-25.
[3] Fossum, R., Griffith, P. and Reiten, I., Trivial extensions of abelian categories. Lecture Notes in Math., vol. 456, Springer, Berlin-Heidelberg-New York, 1975.
[4] Goodearl, K. R., Ring Theory. Dekker, New York, 1976.
[5] Palmér, I. and Roos, J.-E., Explicit formulae for the global homological dimensions of trivial extensions of rings. J. Algebra, 27 (1973), 380-413.
[6] Reiten, I., Trivial extensions and Gorenstein rings. Thesis, University of Illinois, Urbana, 1971.
[7] Zaks, A., Injective dimension of semi-primary rings. J. Algebra, 13 (1969), 73-86.

Institute of Mathematics
University of Tsukuba
Ibaraki, 305 Japan