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ON PROJECTIVE NORMALITY AND DEFINING EQUATIONS OF A PROJECTIVE CURVE OF GENUS THREE EMBEDDED BY A COMPLETE LINEAR SYSTEM

By

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Introduction. Let $\phi_L: C \hookrightarrow \boldsymbol{P}^{h^0(L)-1}$ be the projective embedding of a complete non-singular curve C of genus g by means of $\Gamma(L)$, where L is a very ample invertible sheaf on C. We will study the homogeneous coordinate ring and the ideal of definition $I(L)$ of $\phi_L(C)$ in the case $g=3$. Our results are summarized in the following table. (If the genus of C is less than three, answers to the same kind of problems are easy.) In the table we will say that the homogeneous ideal $I(L)$ is generated strictly by its elements of degrees $\nu_1, \cdots, \nu_m$ if $I(L)$ is generated by its elements of degrees $\nu_1, \cdots, \nu_m$ and $I(L)$ is not generated by its elements of degrees $\nu_1, \cdots, \hat{\nu}_j, \cdots, \nu_m$ for any ν_j $(1 \leqq j \leqq m)$, where $\hat{\nu}_j$ means that ν_j is omitted.

$d \leqq 3$	There is no very ample invertible sheaf of degree $d \leqq 3$ on C.
$d=4$	If C is hyperelliptic, then C has no very ample invertible sheaf of degree 4. If C is non-hyperelliptic, then there is only one very ample invertible sheaf of degree 4 on C, which is the canonical sheaf ω_C. $\phi_{\omega_C}(C)$ is projectively normal. The homogeneous ideal $I(\omega_C)$ is generated strictly by its element of degree 4.
$d=5$	There is no very ample invertible sheaf of degree 5 on C.
$d=6$	The set of very ample invertible sheaves of degree 6 on C coincides with $\mathrm{Pic}^6(C) - \{\omega_C(P+Q) \mid P, Q \in C\}$. If C is hyperelliptic, then for a very ample invertible sheaf L of degree 6 on C, $\phi_L(C)$ is not projectively normal and the homogeneous ideal $I(L)$ generated strictly by its elements of degrees 2 and 4. If C is non-hyperelliptic, then for a very ample invertible sheaf L of degree 6 on C, $\phi_L(C)$ is projectively normal and the homogeneous ideal $I(L)$ is generated strictly by its elements of degree 3.

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$d=7$	Any invertible sheaf of degree 7 on C is very ample. For an invertible sheaf L of degree 7 on C, $\phi_L(C)$ is projectively normal and the homogeneous ideal $I(L)$ is generated strictly by its elements of degrees 2 and 3.
$d\geqq 8$	Any invertible sheaf of degree $d\geqq 8$ on C is very ample. For an invertible sheaf L of degree $d\geqq 8$, $\phi_L(C)$ is projectively normal and the homogeneous ideal $I(L)$ is generated strictly by its elements of degree 2.

Notation and Terminology. We fix an algebraically closed field K. We use the word "curve" to mean a complete non-singular curve over K. For a finite dimensional vector space V, $S^m V$ means the m-th symmetric power of V. Let L be an invertible sheaf on a curve C. We denote by L^m the m-th tensor product $L^{\otimes m}$. For the vector space of global sections $\Gamma(L)$, we define $I_m(L)$ (or simply I_m) and $I(L)$, by

$$I_m(L)=\mathrm{Ker}\,[S^m\Gamma(L) \longrightarrow \Gamma(L^m)]$$

and

$$I(L)=\bigoplus_{m\geq 0} I_m(L)\,.$$

We denote by ω_C the canonical invertible sheaf on C, and by $\mathrm{Pic}^d(C)$ the set of invertible sheaves of degree d on C. For a coherent sheaf $\mathcal{F}$ on C, $h^i(\mathcal{F})$ is the dimension of the vector space $H^i(C, \mathcal{F})$ over K.

§1. Known facts.

This section consists of two parts. In the first part we will state some general facts concerning our problems. In the second part we will determine the set of very ample invertible sheaves on a curve of genus three.

Let L be an invertible sheaf on a projective variety X. According to Mumford [4], we say that L is *normally generated* if L is ample and the natural map $\Gamma(L)^{\otimes m}\to\Gamma(L^m)$ is surjective for any positive integer m. Obviously, $\Gamma(L)^{\otimes m}\to\Gamma(L^m)$ is surjective for all $m\geqq 1$ if and only if $\Gamma(L^m)\otimes\Gamma(L)\to\Gamma(L^{m+1})$ is surjective for all $m\geqq 1$. If X is a normal variety and L is normally generated, then L is very ample and $\phi_L(C)$ is projectively normal, and the converse is true too.

The following theorem was proved by Mumford [4, Corollary to Theorem 6].

THEOREM 1.1. *Let L be an invertible sheaf of degree d on a curve of genus g. If $d\geqq 2g+1$, then L is normally generated.*

A proof of the following "Noether's Theorem" is found in [6].

THEOREM 1.2. *Let C be a curve. Then the following conditions are eqivalent:*
(1) *C is non-hyperelliptic, and*
(2) *the canonical sheaf ω_C is normally generated.*

Concerning the ideal of definition $I(L)$ of $\phi_L(C)$, Saint-Donat [5] proved,

THEOREM 1.3. *Let L be an invertible sheaf of degree d on a curve of genus g.*
(a) *If $d \geqq 2g+1$, then $I(L)$ is generated by I_2 and I_3.*
(b) *If $d \geqq 2g+2$, then $I(L)$ is generated by I_2.*

In the previous paper [3], we learned a slight generalization of Theorem 1.3(a):

THEOREM 1.4. *If L is a normally generated invertible sheaf on a curve C with $H^1(C, L)=(0)$, then $I(L)$ is generated by I_2 and I_3.*

An invertible sheaf L on C is very ample if and only if $\Gamma(L)$ separates two distinct points and infinitely near points, so we have:

PROPOSITION 1.5. *An invertible sheaf on C is very ample if and only if*

$$h^0(C, L(-P-Q))=h^0(C, L)-2$$

for any $P, Q \in C$ (including the case $P=Q$).

A precise proof of Proposition 1.5 can be found in [2, IV Proposition 3.1].

COROLLARY 1.5.1. *If L is an invertible sheaf on a curve of genus g, whose degree is not less than $2g+1$, then L is very ample.*

COROLLARY 1.5.2. *An invertible sheaf L of degree $2g$ on a curve C of genus g is not very ample if and only if L is isomorphic to $\omega_C(P+Q)$ for some points $P, Q \in C$ (may be $P=Q$).*

The following two propositions are useful to determine the set of very ample invertible sheaves on a curve of genus three. The first one is "Halphen's Theorem" [2, IV Proposition 6.1], and the second one is famous as "Clifford's Theorem".

PROPOSITION 1.6. *Let C be a curve of genus $g \geqq 2$, and let d be an integer. Then C has a very ample invertible sheaf L of degree d with $h^1(L)=0$ if and only if $d \geqq g+3$.*

PROPOSITION 1.7. *Let L be an invertible sheaf on C with $h^0(L)>0$ and $h^1(L)>0$. Then*

$$2(h^0(L)-1)\leqq \deg L .$$

Furthermore, equality occurs if and only if either $L\cong\mathcal{O}_C$ or $L\cong\omega_C$ or C is hyperelliptic and $L\cong(f^*\mathcal{O}_{P^1}(1))^{\otimes r}$ $(0\leqq r\leqq g-1)$, where $f: C\to \boldsymbol{P}^1$ is a double covering.

COROLLARY 1.7.1. *Let C be a curve of genus $g\geqq 1$, and let L be an invertible sheaf on C with $h^0(L)>0$ and $h^1(L)>0$. Then*

$$h^0(L)\leqq g .$$

Furthermore, equality occurs if and only if $L\cong\omega_C$.

REMARK 1.8. Let L be an invertible sheaf on a curve of genus $g\leqq 2$. Then L is very ample if and only if $\deg L\geqq 2g+1$.

PROOF. In the case of $g=0$ or 1, our remark can be proved easily. If $g=2$ and L is very ample, then we have $h^1(L)=0$ by Corollary 1.7.1. Therefore our remark follows from Corollary 1.5.1 and Proposition 1.6.

PROPOSITION 1.9. *Let C be a curve of genus three. Then we have,*

d	The set of very ample invertible sheaves of degree d on C.
$d\leqq 3$	None.
$d=4$	None, if C is hyperelliptic. $\{\omega_C\}$, if C is non-hyperelliptic.
$d=5$	None.
$d=6$	$\mathrm{Pic}^6(C)-\{\omega_C(P+Q)\mid P, Q\in C\}$.
$d\geqq 7$	$\mathrm{Pic}^d(C)$.

PROOF. In the case of $d\geqq 6$, our results follows from Corollaries 1.5.1 and 1.5.2. By Halphen's Theorem there is no very ample invertible sheaf L of degree $d\leqq 5$ with $h^1(L)=0$. By virtue of Corollary 1.7.1, a possibility of a very ample invertible sheaf L of degree $d\leqq 5$ with $h^1(L)>0$ is only the canonical invertible sheaf ω_C. On the other hand, ω_C is very ample if and only if C is non-hyperelliptic. This completes the proof.

§ 2. Projective normality.

In this section we will determine the set of normally generated invertible sheaves on a curve C of genus three. The answer to the same kind of problem for a curve of genus $g \leqq 2$ is easy. Indeed, by Remark 1.8 and by Theorem 1.1 an invertible sheaf L is normally generated if and only if L is very ample.

In the case of genus three, by Theorem 1.1 an invertible sheaf L is normally generated if $\deg L \geqq 7$, and by Theorem 1.2 the canonical invertible sheaf ω_C is normally generated if C is non-hyperelliptic. Therefore, to show our table it suffices to prove the following theorem.

THEOREM 2.1. *Let C be a curve of genus three, and let L be a very ample invertible sheaf of degree* 6 *on C. Then L is normally generated if and only if C is non-hyperelliptic.*

PROOF. (Step 1) First we will show that L is normally generated if and only if $\phi_L(C)$ is not contained any quadric surface in $\boldsymbol{P}^3$. Indeed, L is normally generated if and only if $\Gamma(L^m)\otimes\Gamma(L)\rightarrow\Gamma(L^{m+1})$ is surjective for all $m \geqq 1$. By the lemma of Castelnuovo [4], these maps are surjective when $m \geqq 2$. Hence,

$$
\begin{aligned}
&L \text{ is normally generated},\\
&\Leftrightarrow \Gamma(L)\otimes\Gamma(L) \longrightarrow \Gamma(L^2) \text{ is surjective},\\
&\Leftrightarrow S^2\Gamma(L) \longrightarrow \Gamma(L^2) \text{ is surjective}.
\end{aligned}
$$

Since $\dim S^2\Gamma(L)=\dim\Gamma(L^2)$, these conditions are equivalent to the condition that $S^2\Gamma(L)\rightarrow\Gamma(L^2)$ is injective. The last condition means that $\phi_L(C)$ is not contained any quadric surface in $\boldsymbol{P}^3$.

It is well known that a quadric surface in $\boldsymbol{P}^3$ is a union of planes (may be non-reduced) or an irreducible quadric cone, which is a projective cone of a 2-uple embedding of $\boldsymbol{P}^1$, or a non-singular quadric surface, which is a Segre embedding of $\boldsymbol{P}^1\times\boldsymbol{P}^1$ into $\boldsymbol{P}^3$. Obviousely, a union of planes dose not contain $\phi_L(C)$. In the next step, we will show that an irreducible quadric cone does not contain $\phi_L(C)$, either.

(Step 2) Let F be an irreducible quadric cone with vertex O in $\boldsymbol{P}^3$. Let

$$\boldsymbol{P}^3\times\boldsymbol{P}^2\supset\tilde{F}\xrightarrow{\pi}F\subset\boldsymbol{P}^3$$

be the monoidal transformation of F with center O. Then $p_2|\tilde{F}$ factors through a 2-uple embedding of $\boldsymbol{P}^1$:

$$\begin{tikzcd} \tilde{F} \arrow[rr, "p_2|\tilde{F}"] \arrow[dr, "q"'] & & P^2 \\ & P^1 \arrow[ur, hook] & \end{tikzcd}$$

and then $\tilde{F} \xrightarrow{q} P^1$ coincides with the geometrically ruled surface $\mathrm{Proj}\,(S(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-2))) \to P^1$ [2, V(5) Example 2.11.4].

If C_0 is the inverse image $\pi^{-1}(O)$ of O and f is a fibre of q of a point, then $\mathrm{Pic}\,(F)$ is isomorphic to $\boldsymbol{Z}C_0 \oplus \boldsymbol{Z}f$. A canonical divisor K_F of F is linearly equivalent to $-2C_0-4f$, and the intersection pairing on F is given by

$$C_0^2=-2, \qquad C_0\cdot f=1 \qquad \text{and} \quad f^2=0 \quad [1, \text{ p. } 33].$$

Let D be a curve of genus g on F, and let $\tilde{D}$ be the strict transform of D on $\tilde{F}$. Assume that $\tilde{D}$ is linearly equivalent to aC_0+bf. Then by the adjunction formula, we have

$$2g-2=-2(a^2-ab+b).$$

If the vertex O lies on D, then $1=\tilde{D}\cdot C_0=-2a+b$. Therefore we have $g=a(a-1)$, so g is even. If the vertex O does not lie on D, then $0=\tilde{D}\cdot C_0=-2a+b$. Therefore we have $g=(a-1)^2$, so g is a square number. We conclude that any curve of genus 3 does not lie on F.

(Step 3) In this step, we will show that if a curve C of genus 3 and degree 6 in P^3 lies on a non-singular quadric surface F, then C is hyperelliptic.

First, note that

$$\mathrm{Pic}\,(F)=\mathrm{Pic}\,(P^1\times P^1)=p_1^*\mathrm{Pic}\,(P^1)\oplus p_2^*\mathrm{Pic}\,(P^1)=\boldsymbol{Z}\oplus\boldsymbol{Z},$$

where $p_1^*\mathcal{O}_{P^1}(1)$ corresponds to (1, 0) and $p_2^*\mathcal{O}_{P^1}(1)$ corresponds to (0, 1). Obviously, a canonical divisor K_F corresponds to $(-2, -2)$, and a hyperplane section on F corresponds to (1, 1). The intersection pairing on F is given by $D\cdot D'=ab'+ba'$ for two divisors D and D' corresponding to (a, b) and (a', b') respectively.

Assume that C corresponds to (a, b). Then we have

$$6=\deg_{P^3}C=(C\cdot H)_{P^3}=(C\cdot H|_F)_F=a+b,$$

where H is a hyperplane of P^3, and

$$2\cdot 3-2=C\cdot(C+K_F)=2ab-2a-2b.$$

Hence, we have "$a=4$, $b=2$" or "$a=2$, $b=4$". Since $F=P^1\times P^1$, we may assume that C corresponds to (4, 2) Consider the diagram:

$$C \subset F = \boldsymbol{P}^1 \times \boldsymbol{P}^1 \xrightarrow{p_1} \boldsymbol{P}^1 .$$

If $f: C \to \boldsymbol{P}^1$ is defined by the restriction of p_1 to C, then f is surjective and then

$$\deg f = \deg f^*\mathcal{O}_{\boldsymbol{P}^1}(1) = \deg p_1^*\mathcal{O}_{\boldsymbol{P}^1}(1) \mid C = 2 .$$

Therefore C is hyperelliptic.

(Step 4) The final step, for a given hyperelliptic curve C of genus 3 and a given very ample invertible sheaf L of degree 6 on C, we construct a non-singular quadric surface in $\boldsymbol{P}^3$ containing $\phi_L(C)$.

Since C is hyperelliptic, there is a morphism $f: C \to \boldsymbol{P}^1$ of degree 2. We put $M_0 = f^*\mathcal{O}_{\boldsymbol{P}^1}(1)$, and $M = L \otimes M_0^{-1}$. Then the canonical map

$$(\sharp) \qquad \Gamma(M) \otimes \Gamma(M_0) \longrightarrow \Gamma(L)$$

is an isomorphism. To prove this, note that $\Gamma(M_0)$ is a base point free pencil. By the "base point free pencil trick" [6], we have an isomorphism $\operatorname{Ker}[\Gamma(M) \otimes \Gamma(M_0) \to \Gamma(L)] \cong \Gamma(M \otimes M_0^{-1})$. Assume that $\Gamma(M \otimes M_0^{-1}) \neq (0)$. Then there are two points P and Q on C such that $M \otimes M_0^{-1} \cong \mathcal{O}_C(P+Q)$. Hence $L \cong M_0^2(P+Q) \cong \omega_C(P+Q)$. This contradicts the very ampleness of L. Therefore the map $(\sharp)$ is injective. On the other hand, $\dim \Gamma(M) \otimes \Gamma(M_0) = \dim \Gamma(L)$, so the map $(\sharp)$ is an isomorphism. By the isomorphism $(\sharp)$ we obtain the following commutative diagram:

$$\begin{tikzcd}
C \arrow[r, hook, "\phi_M \otimes \phi_{M_0}"] \arrow[dr, hook, "\phi_L"'] & \boldsymbol{P}^1 \times \boldsymbol{P}^1 \arrow[d, hook, "\text{Segre embedding}"] \\
 & \boldsymbol{P}^3
\end{tikzcd}$$

This completes our proof.

§ 3. Defining equations.

In this section we will study the homogeneous ideal $I(L)$ for a curve of genus $g \leqq 3$ with a very ample invertible sheaf L.

REMARK 3.1. Let C be a curve of genus $g \leqq 2$, and let L be a very ample invertible sheaf of degree d on C.

(a) If $d \geqq 2g+2$, then $I(L)$ is generated strictly by I_2.

(b) If $g=2$ and $d=5$ $(=2g+1)$, then $I(L)$ is generated strictly by I_2 and I_3.

(c) If $g=1$ and $d=3$ $(=2g+1)$, then $I(L)$ is generated strictly by I_3.

(d) If $g=0$ and $d=1$ $(=2g+1)$, then $I(L)=(0)$.

A proof of this remark is easy, so we omit it.

THEOREM 3.2. *Let L be a very ample invertible sheal of degree d on a curve C of genus three.*

(a) *If $d\geqq 8$, then $I(L)$ is generated strictly by I_2.*

(b) *If $d=7$, then $I(L)$ is generated strictly by I_2 and I_3.*

(c) *If C is non-hyperelliptic and $d=6$, then $I(L)$ is generated strictly by I_3.*

(d) *If C is non-hyperelliptic and $L=\omega_C$, then $I(\omega_C)$ is generated strictly by I_4.*

PROOF. (a) It is a special case of Theorem 1.3 (b).

(b) By Theorem 1.3 (a), $I(L)$ is generated by I_2 and I_3. Assume that $I(L)$ is generated by I_2. Since $\dim I_2(L)=3$, $\phi_L(C)$ is a complete intersections of three quadric hypersurfaces in $\boldsymbol{P}^4$. Therefore we have $\deg_{P^4}\phi_L(C)=8$. This contradicts the fact $\deg L=7$.

(c) By Theorem 1.4, $I(L)$ is generated by I_2 and I_3. On the other hand, by the proof of Theorem 2.1 we have $I_2(L)=(0)$.

(d) It is well known that $\phi_{\omega_C}(C)$ is plane quartic.

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By this theorem, to show our table it suffices to prove the following theorem.

THEOREM 3.3. *Let L be a very ample invertible sheaf of degree* 6 *on a hyperelliptic curve C of genus* 3. *Then $I(L)$ is generated strictly by I_2 and I_4.*

A proof of the theorem will be given at the last part of this section.

Let $M_1, \cdots, M_r$ be invertible sheaves on a projective variety. $\mathfrak{R}(M_1, \cdots, M_r)$ denotes the kernel of the canonical map:

$$\Gamma(M_1)\otimes\cdots\otimes\Gamma(M_r)\longrightarrow\Gamma(M_1\otimes\cdots\otimes M_r)\,.$$

LEMMA 3.4. *Let L be an ample invertible sheaf on a projective variety, and let m be a positive integer greater than* 1. *Assume that*

$$\Gamma(L)^{\otimes(m-1)}\longrightarrow\Gamma(L^{m-1})$$

and

$$\Gamma(L)^{\otimes m}\longrightarrow\Gamma(L^m)$$

are surjective. Then

$$\Gamma(L)\otimes\mathfrak{R}(L^{m-1}, L)\longrightarrow\mathfrak{R}(L^m, L)$$

is surjective if and only if

$$I_m(L)\otimes\Gamma(L) \longrightarrow I_{m+1}(L)$$

is surjective.

DEFINITION 3.5. (1) Let X be a normal closed subuariety of $\boldsymbol{P}^N$, and let $R(X)=\bigoplus_{i=0}^{\infty} R(X)_i$ be the homogeneous coordinate ring of X. $\widetilde{R(X)}$ denotes the normalization of $R(X)$. It is well known that $\widetilde{R(X)}$ is a graded ring too. We can define the non-negative integer $n(X\subset\boldsymbol{P}^N)$ by

$$n(X\subset\boldsymbol{P}^N)=\mathrm{Min}\ \{n\in\boldsymbol{N}\,|\,\widetilde{R(X)}_i=R(X)_i \quad \text{for all } i\geqq n\}\ .$$

(2) Let L be a very ample invertible sheaf on the normal projective variety X. We define the non-negative integer $n(L)$ by

$$n(L)=n(X \overset{\phi_L}{\hookrightarrow} \boldsymbol{P}^{h^0(L)-1})\,.$$

It is easy to show that

$$n(L)=\mathrm{Min}\ \{n\in\boldsymbol{N}\,|\,\Gamma(L)^{\otimes i}\to\Gamma(L^i) \quad \text{is surjective for all } i\geqq n\}\ .$$

COROLLARY 3.6. *Let L be a very ample invertible sheaf on an n-dimensional projective variety X. Assume that $H^i(X, L^j)=(0)$ for any integers $i, j>0$. If $\alpha=\mathrm{Max}\,(n+3,\ n(L)+1)$, then $I(L)$ is generated by $I_2, \cdots, I_\alpha$.*

The proofs of Lemma 3.4 and Corollary 3.6 are similar to those of [3, Proposition 1.2 and Corollary 1.3].

Next, we will calculate $n(L)$ for a very ample invertible sheaf L of degree 6 on a hyperelliptic curve of genus 3.

PROPOSITION 3.7. *Let L be a very ample invertible sheaf of degree 6 on a hyperelliptic curve C of genus 3. Then $\Gamma(L)^{\otimes m}\xrightarrow{\beta_m}\Gamma(L^m)$ is surjective for all $m\geqq 3$, i.e., $n(L)=3$.*

PROOF.(*) We prove the surjectivity of $\beta_m:\Gamma(L)^{\otimes m}\to\Gamma(L^m)$ $(m\geqq 3)$ by induction on m. For a given $m\geqq 3$, we consider the following commutative diagram:

(*) The author expresses his heartfelt thanks to the referee for a valuable suggestion, which simplified the proof.

$$\begin{tikzcd}
\Gamma(L)^{\otimes(m+1)} \arrow[rr, "\beta_m\otimes 1"] \arrow[d, "\beta_{m+1}"'] & & \Gamma(L^m)\otimes\Gamma(L) \arrow[dll, "\gamma_m"] \\
\Gamma(L^{m+1}) & &
\end{tikzcd}$$

By the induction hypothesis β_m is surjective, and also $\beta_m\otimes 1$ is surjective. By the lemma of Castelnuvo γ_m is surjective, and also β_{m+1} is surjective. Therefore, to prove our assertion, it suffices to prove the surjectivity of β_3. By Step 4 in the proof of Theorem 2.1, there is an irreducible quadric surface Q in $\boldsymbol{P}^3$ containing $\phi_L(C)$. The curve $\phi_L(C)$ can not be contained in a quadric surface other than Q, because $\phi_L(C)$ is not contained in any $\boldsymbol{P}^2$ and $\deg\phi_L(C)=6$. Hence we have $I_2(L)=K\cdot q$, where q is a quadratic form defining the quadric surface Q. If $\phi_L(C)$ is contained in an irreducible cubic surface H, then $\phi_L(C)$ coincides with the complete intersection $Q\cap H$, because $\phi_L(C)$ and $Q\cap H$ have degree 6. But the genus of a curve which is a complete intersection of surfaces of degrees 2 and 3 is equal to 4. This is a contradiction. Therefore, we have $I_3(L)=K\cdot q\circledast s_1\oplus\cdots\oplus K\cdot q\circledast s_4$, where $\{s_1,\cdots s_4\}$ is a basis of $\Gamma(L)$ and the symbel $\circledast$ means a symmetric product. Consider the exact sequence

$$0\longrightarrow I_3(L)\longrightarrow S^3\Gamma(L)\xrightarrow{\beta_3}\Gamma(L^3).$$

The left hand vector space has dimension 4 by the above result, the middle vector space has dimension 20, and right hand vector space has dimension 16 by the theorem of Riemann-Roch. So we conclude that β_3 is surjective.

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PROOF OF THEOREM 3.3. By Corollary 3.6 and Proposition 3.7, $I(L)$ is generated by I_2, I_3 and I_4. By the proof of Proposition 3.7, $I_2=K\cdot q$ and $I_3=K\cdot q\circledast s_1\oplus\cdots\oplus K\cdot q\circledast s_4$. Therefore, $I(L)$ is generated by I_2 and I_4. Obviously, I_2 does not generate $I(L)$. This completes the proof.

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