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DIMENSION OF SPECIAL μ-SPACES

By

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K. Nagami in [5] called a space X a μ-space if X is embedded in the countable product of paracompact σ-metric spaces. Especially he called X a cubic μ-space if X is the countable product itself of paracompact σ-metric spaces and proved that if X is a cubic μ-space, then the following statements are equivalent: (1) $\dim X\leqq n$, (2) $X=\bigcup_{i=1}^{n+1} X_i$, where $\dim X_i\leqq 0$ for each i, (3) $\operatorname{Ind} X\leqq n$ and (4) there exists a closed mapping f of a μ-space Z with $\dim Z\leqq 0$ onto X such that $\operatorname{ord} f\leqq n+1$, Recently he in [7] defined the class of free L-spaces between those of Lašnev and μ-spaces, and proved there that if $\mathcal{C}$ is the class of free L-spaces and $X\in\mathcal{C}$, then the above (1), (2), (3) and the following (4)′ are equivalent: (4)′ there exists a closed mapping f of $Z\in\mathcal{C}$ with $\dim Z\leqq 0$ onto X such that $\operatorname{ord} f\leqq n+1$. In this paper we define the class $\mathcal{C}$ of special μ-spaces which are spaces which can be embedded in the countable product of special σ-metric spaces and study the dimension of such spaces. In Theorem 2 it is proved that every free L-space is a special μ-space and in Theorem 3 that every special μ-space is the perfect image of a free L-space. Though it is proved by K. Nagami that a space X is a free L-space if and only if X is embedded in the conutable product of almost metric spaces, it is proved in Corollary 3 to Theorem 3 that if X is a special μ-space, then $X\subset\prod_{i=1}^{\infty} X_i$, where each X_i is an almost metric space plus one point. In Theorem 4 the above (1), (2), (3), (4)′ are shown to be equivalent even if $\mathcal{C}$ is the class of special μ-spaces.

All spaces are assumed to be Hausdorff, otherwise the contrary is stated. All mappings are assumed to be continuous. N always denotes the positive integers

DEFINITION 1 (K. Nagami [4]). A space X is called a *σ-metric space* if $X=\bigcup_{i=1}^{\infty} X_i$, where each X_i is a closed metrizable subspace of X. Such $\{X_i\}$ is called a *scale of* X. A scale $\{X_i\}$ is called *monotone* if $X_i\subset X_{i+1}$ for every $i\in N$.

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The following argument is due to K. Nagami [4]: If X is a paracompact σ-metric space with a monotone scale $\{X_i\}$, then there exists a contraction ρ of X onto a metric space $\hat{X}$ such that $\rho|X_i$ is a homeomorphism onto a closed subspace $\rho(X_i)$ of $\hat{X}$ for each $i\in N$. Such a pair $(\hat{X}, \rho)$ is called a *replica of X*.

DEFINITION 2 (K. Nagami [6, Definition 4.4, 1.1]). Let F be a closed set of a space X. An open cover of $X-F$ is called an *anti-cover* of F. An anti-cover $\mathcal{U}$ of F is said to be *uniformly approaching* (*to* F) if for every open set G of X, $\overline{S(X-G, \mathcal{U})}\cap F\cap G=\emptyset$. $\mathcal{U}$ is said to be *approaching* (*to* F) if for every open neighborhood G of F, $\overline{S(X-G, \mathcal{U})}\cap F=\emptyset$.

Every closed set of a metrizable space has a uniformly approaching anti-cover [6, Remark 4.5]. This fact is used frequently in the later discussion.

DEFINITION 3. A σ-metric space X is said to *have a special scale* $\{X_i: i\in N\}$ if $\{X_i\}$ is a scale of X such that each X_i has a uniformly approaching anti-cover. A space X is called a *special σ-metric space* if X is a paracompact σ-metric space with a special scale. A space is called a *special μ-space* if it is embedded in the countable product of special σ-metric spaces.

By a routine check it is easily seen that a space X has a special scale if and only if X has a special and monotone scale. Therefore in the below discussion, we do not distinguish between usual and monotone scales. As seen in [2, Example 1] every paracompact σ-metric space need not be special σ-metric, and every special σ-metric space is M_1, but not the converse.

DEFINITION 4 (K Nagami [7, Definition 1.2]). For a space X, consider a pair $\mathcal{P}=(\mathcal{F}, \{\mathcal{U}_F: F\in\mathcal{F}\})$ of a σ-discrete closed collection $\mathcal{F}$ of X and a collection of anti-covers $\mathcal{U}_F$ of $F\in\mathcal{F}$. $\mathcal{P}$ is called a *free L-structure* if for each point $p\in X$ and each open neighborhood U of p, there exist a finite collection $F_1, \cdots, F_k$ of $\mathcal{F}$ and a canonical neighborhood U_i of each F_i with $p\in \bigcap_{i=1}^{k} F_i\subset \bigcap_{i=1}^{k} U_i\subset U$. X is called a *free L-space* if X is a paracompact space with a free L-structure. (U is called a *canonical neighborhood of F* with respect to $\mathcal{U}_F$ if U is an open neighborhood of F such that for each $i\in N$ $\overline{S^i(X-U, \mathcal{U}_F)}\cap F=\emptyset$. Especially when the relation holds for $i=1$, U is called a *semi-canonical neighborhood of F*.)

DEFINITION 5 (K. Nagami [7, Definition 3.1]). Let X be a space. The set of all points of X which have the metric neighborhoods is said to be the *metric part of X*. The complement of the metric part is said to be the *nonmetric part*.

X is said to be an *almost metric space* if the following three conditions are satisfied:

a) X is perfectly normal and paracompact.

b) The collection of points of the nonmetric part X_0 is discrete.

c) X_0 has an anti-cover approaching to X_0.

THEOREM 1 (The embedding theorem for free L-spaces [7, Theorem 3.4]). *A space X is a free L-space if and only if X is embedded in the countable product of almost metric spaces.*

THEOREM 2. *Every free L-space is a special μ-space.*

PROOF. It is proved that every almost metric space is a special σ-metric space, and therefore by Theorem 1 every free L-space is embedded in the countable product of special σ-metric spaces. Indeed, let X be an almost metric space with its nonmetric part X_0. Let $\{U(p): p\in X_0\}$, $\{V(p): p\in X_0\}$ be two discrete open collections of X such that

$$p\in V(p)\subset \overline{V(p)}\subset U(p) \qquad \text{for every } p\in X_0 .$$

Let $\mathcal{U}$ be an approaching anti-cover of X_0 and set

$$\mathcal{U}_0=(X-\cup\{\overline{V(p)}: p\in X_0\})\cup(\cup\{\mathcal{U}\,|\,U(p): p\in X_0\}) .$$

Then $\mathcal{U}_0$ is a uniformly approaching anti-cover of X_0. Let

$$X_0=\bigcap_{i=1}^{\infty} G_n , \qquad \overline{G_{n+1}}\subset G_n \qquad \text{for every } n\in N,$$

where each G_n is open in X. Let $\mathcal{V}_n$ be a uniformly approaching anti-cover of $X_n=X-G_n$ in the metric subspace $X-X_0$. Set

$$\mathcal{U}_n=\mathcal{V}_n\cup\{G_{n+1}\} .$$

Then $\mathcal{U}_n$ is a uniformly approaching anti-cover of X_n. Therefore $\{X_n: n=0, 1, \cdots\}$ is a special scale of X.

THEOREM 3. *If X is a special μ-space, then X is the perfect image of a free L-space.*

PROOF. Part 1: As a special case, we shall show that if X is a special σ-metric space, then X is the perfect image of a free L-space. Let $\{X_i: i\in N\}$ be a special scale of X. Let $(\hat{X}, \rho)$ be its replica with respect to $\{X_i\}$. $\hat{X}$ is the image of a metric space Y with $\dim Y\leqq 0$ under a perfect mapping g. Construct

Z as follows:

$$Z=\{(x, y)\in X\times Y: \rho(x)=g(y)\}.$$

Let f, σ be the restrictions of the projections onto X, Y, respectively. Then Z is a paracompact σ-metric space with a scale $\{Z_i=f^{-1}(X_i): i\in N\}$ such that $\dim Z\leqq 0$ and $\sigma|Z_i$ is a homeomorphism of Z_i onto a closed subspace $\sigma(Z_i)$ of Y. Let $\{\mathcal{U}_i: i\in N\}$ be a sequence of uniformly approaching anti-covers of each X_i in X. Since every closed set of a metric space has a uniformly approaching anti-cover, there exists a uniformly approaching anti-cover $\mathcal{V}_i$ of $\sigma(Z_i)$ in Y. Set

$$\mathcal{W}_i=(\mathcal{U}_i\times\mathcal{V}_i)|Z.$$

Then $\mathcal{W}_i$ is an anti-cover of Z_i with the following property:

(*) If P, Q is a pair of open sets of X, Y, respectively, then

$$\overline{S(Z-P\times Q, \mathcal{W}_i)}^Z\cap(P\times Q)\cap Z_i=\emptyset.$$

We shall show that Z is a free L-space. Assume that each $\mathcal{W}_i=\{W_\alpha: \alpha\in A_i\}$ is locally finite in $Z-Z_i$ and finitely multiplicative, that is, every finite intersection of members of $\mathcal{W}_i$ belongs to $\mathcal{W}_i$. Let $p_i: Z-Z_i\to K_i$ be the canonical mapping such that

$$p_i(z)=\sum\{\phi_\alpha(z)\alpha: \alpha\in A_i\},$$

where each K_i is the nerve of $\mathcal{W}_i$ and ϕ_α is a continuous mapping of $Z-Z_i$ onto $[0, 1]$ such that $W_\alpha=\mathrm{coz}(\phi_\alpha)$ and $\{\phi_\alpha: \alpha\in A_i\}$ is a partition of unity. We define the topology $\mathcal{T}_i$ of the disjoint sum $T_i=K_i\cup Z_i$ as follows: Let $\mathcal{T}_i(K_i)$ be the metric topology of K_i and for an open set V of Z let $\Delta(V)$ be the totality of subsets δ of A_i such that

$$L(V, \delta)=(V\cap Z_i)\cup(\cup\{W_\alpha: \alpha\in\delta\})$$

is an open set of Z. For each $\delta\in\Delta(V)$, set

$$M(V, \delta)=(V\cap Z_i)\cup(\cup\{St(\alpha): \alpha\in\delta\}),$$

where $St(\alpha)$ means the star of the vertex α in K_i. Thus $\mathcal{T}_i$ is defined to be the topology having as its base

$$\mathcal{T}(K_i)\cup\{M(V, \delta): \delta\in\Delta(V),\ V \text{ open in } Z\}.$$

Indeed, if $\delta_1\in\Delta(V_1)$, $\delta_2\in\Delta(V_2)$, where V_1, V_2 are open in Z, then

$$M(V_1, \delta_1)\cap M(V_2, \delta_2)=((V_1\cap V_2)\cup Z_i)\cup(\cup\{St(\alpha): \alpha\in\delta\})$$

for some $\delta\subset A_i$ such that

$$((V_1\cap V_2)\cap Z_i)\cup(\cup\{W_\alpha: \alpha\in\delta\})=L(V_1, \delta_1)\cap L(V_2, \delta_2)$$

is open in Z. Hence $\delta\in\Delta(V_1\cap V_2)$. Define a transformation $f_i: Z\to(T_i, \mathcal{T}_i)$ as follows:

$$f_i|Z_i=id_{Z_i}, \qquad f_i|(Z-Z_i)=p_i .$$

Then from the construction of $\mathcal{T}_i$, f_i is easily seen to be continuous and onto. Define a transformation $f: Z\to\prod_{i=1}^{\infty}(T_i, \mathcal{T}_i)$ as follows:

$$f(x)=(f_i(x))_{i\in N}, \qquad x\in Z .$$

Then f is a continuous mapping. Suppose $x\neq y$, $x, y\in Z$. Then $x, y\in Z_i$ for some $i\in N$. Then $f_i(x)\neq f_i(y)$, implying $f(x)\neq f(y)$. Hence f is one-to-one. We shall show that f is an open mapping of Z onto $f(Z)$. Let V be an open set of Z. Suppose $f(x)\in f(V)$. Then for some $m\in N$ $x\in Z_m\cap V\subset V$. There exists a pair P, Q of open sets of X, Y, respectively, such that $x\in(P\times Q)\cap Z\subset V$. Apply (*) for P, Q to get $\delta\subset A_m$ such that $L((P\times Q)\cap Z, \delta)$ is an open neighborhood of x in Z, and $L((P\times Q)\cap Z, \delta)\subset(P\times Q)\cap Z$. Then $M((P\times Q)\cap Z, \delta)$ is an open neighborhood of $f_m(x)$. Set

$$O=(\prod_{i=1}^{\infty}O_i)\cap f(Z),$$

$$O_i=T_i \quad \text{if } i\neq m,\ O_m=M((P\times Q)\cap Z, \delta).$$

Then O is an open neighborhood of $f(x)$ in $f(Z)$ such that $O\subset f(V)$. Thus $f(V)$ is an open set of $f(Z)$, proving that f is an open mapping. It remains to prove that each $(T_i, \mathcal{T}_i)$ is a free L-space, but this is proved in Part 3.

Part 2: For the later use, we shall show that $Z_i\subset(T_i, \mathcal{T}_i)$ is the countable intersection of closures in T_i of open sets containing Z_i. From the construction of Z, $Z_i=\bigcap_{n=1}^{\infty}V_n$, where each V_n is open in Z. By the repeated use of (*) there exists a $\delta_n\in\Delta(V_n)$ such that

$$Z_i\subset L(V_n, \delta_n)\subset V_n, \qquad S(L(V_n, \delta_n), \mathcal{W}_i)\subset V_n .$$

Then it follows that

$$Z_i=\bigcap_{n=1}^{\infty}M(V_n, \delta_n)=\bigcap_{n=1}^{\infty}\overline{M(V_n, \delta_n)}^{T_i} .$$

Part 3: We shall show that each $(T_i, \mathcal{T}_i)$ can be embedded in the countable product of almost metric spaces. Since Z_i is a zero-dimensional metric space, there exists a sequence $\{\mathcal{H}_n: n\in N\}$ of covers of Z_i such that

(i) each $\mathcal{H}_n=\{H_\lambda: \lambda\in\Lambda_n\}$ is a discrete collection of closed and open sets of Z_i,

(ii) $\mathcal{H}=\bigcup_{n=1}^{\infty}\mathcal{H}_n$ is an open base for Z_i.

Let R_n be the disjoint sum of K_i and Λ_n and define a transformation $q_n: T_i\to R_n$ as follows:

$$q_n(p)=p \qquad \text{if } p\in K_i\,,$$

$$q_n(p)=\lambda \qquad \text{if } p\in H_\lambda,\ \lambda\in\Lambda_n\,.$$

Let the topology of R_n be the quotient one with respect to $q_n: T_i\to R_n$. Define a transformation $q: T_i\to\prod_{n=1}^{\infty}R_n$ by

$$q(x)=(q_n(x))_{n\in N}\,, \qquad x\in T_i\,.$$

Then q is a continuous mapping. Suppose $x\neq y$, x, $y\in T_i$. If x, $y\in K_i$, then $x=q_n(x)\neq q_n(y)=y$ for every $n\in N$. If x, $y\in Z_i$, then there exists an $n\in N$ such that

$$x\in H_\lambda,\ y\in H_{\lambda'}\,, \qquad \lambda\neq\lambda',\ \lambda,\ \lambda'\in\Lambda_n\,.$$

Thus $q_n(x)=\lambda\neq q_n(y)=\lambda'$. If $x\in K_i$, $y\in Z_i$, then for each $n\in N$ and for some $\lambda\in\Lambda_n$ $q_n(x)=x\neq\lambda=q_n(y)$. Hence q is one-to-one. To see that q is an open mapping of T_i onto $q(T_i)$, let V be an open set of T_i and $q(x)\in q(V)$. Without loss of generality we can assume $x\in V\cap Z_i$. Since $\mathcal{H}$ is an open base for Z_i, there exists an $n\in N$ such that $x\in H_\lambda\subset V\cap Z_i$ for some $\lambda\in\Lambda_n$. Let H'_λ be an open set of Z with $H'_\lambda\cap Z_i=H_\lambda$ and $H'_\lambda\subset f_i^{-1}(V)$. By the property (*) there exists a $\delta\in\Delta(H'_\lambda)$ such that $H_\lambda\subset L(H'_\lambda,\delta)\subset H'_\lambda$. Then $q_n(M(H'_\lambda,\delta))$ is an open neighborhood of λ in R_n such that $q_n(M(H'_\lambda,\delta))\subset q_n(V)$. Set

$$O=(\prod_{j=1}^{\infty}O_j)\cap q(T_i)\,,$$

$$O_j=R_j \qquad \text{if } j\neq n,\ O_n=q_n(M(H'_\lambda,\delta))\,.$$

Then O is an open neighborhood of $q(x)$ such that $O\subset q(V)$. Hence q is an embedding.

Part 4: Each R_n is an M_1-space. Obviously R_n is Hausdorff. To see that R_n is regular, suppose $p\in V$ for an open set V of R_n and a point $p\in R_n$. As seen in part 2, Λ_n is written as

$$\Lambda_n=\bigcap_{m=1}^{\infty}W_m=\bigcap_{m=1}^{\infty}\overline{W}_m^{R_n}\,,$$

where each W_m is an open set of R_n. If $p\in K_i$, then $p\notin\overline{W}_m^{R_n}$ for some $m\in N$ and by the regularity of K_i, there exists an open set N such that

$$p\in N\subset\bar{N}^{R_n}\subset(R_n-W_m)\cap V\,.$$

Consider the case $p=\lambda$. Since $H_\lambda \subset f_i^{-1}(q_n^{-1}(V))$, there exists a closed and open set U of Z such that

$$U \cap Z_i = H_\lambda, \quad U \subset f_i^{-1}(q_n^{-1}(V)).$$

By (*) there exists a $\delta \in \Delta(U)$ such that

$$L(U, \delta) \subset U, \qquad S(L(U, \delta), \mathcal{W}_i) \subset U.$$

Then $N = q_n(M(U, \delta))$ is an open neighborhood of p such that $\bar{N}^{R_n} \subset V$. Hence R_n is regular.

Since Z is paracompact, $\dim Z \leqq 0$ and each $\mathcal{H}_n$ is a discrete collection of closed and open sets of Z_i, there exists a discrete collection $\{H'_\lambda : \lambda \in \Lambda_n\}$ of closed and open sets of Z such that $H'_\lambda \cap Z_i = H_\lambda$ for each $\lambda \in \Lambda_n$. For each $\lambda \in \Lambda_n$, take a $\delta_\lambda \in \Delta(H'_\lambda)$ such that

$$L(H'_\lambda, \delta_\lambda) \subset H'_\lambda, \qquad S(L(H'_\lambda, \delta), \mathcal{W}_i) \subset H'_\lambda.$$

Then $\{M(H'_\lambda, \delta_\lambda) : \lambda \in \Lambda_n\}$ is discrete in T_i, for if

$$St(\alpha) \cap M(H'_{\lambda_1}, \delta_{\lambda_1}) \neq \emptyset, \qquad St(\alpha) \cap M(H'_{\lambda_2}, \delta_{\lambda_2}) \neq \emptyset,$$

then

$$St(\alpha) \cap St(\beta_1) \neq \emptyset, \qquad St(\alpha) \cap St(\beta_2) \neq \emptyset$$

for some $\beta_1 \in \delta_{\lambda_1}$, $\beta_2 \in \delta_{\lambda_2}$. These mean

$$W_\alpha \cap W_{\beta_1} \neq \emptyset, \qquad W_\alpha \cap W_{\beta_2} \neq \emptyset.$$

Hence $L(H'_{\lambda_1}, \delta_{\lambda_1}) \cap L(H'_{\lambda_2}, \delta_{\lambda_2}) \neq \emptyset$, a contradiction. Since K_i is paracompact, there exists a locally finite (in K_i) open cover $\mathcal{V} = \{V_\xi : \xi \in \Xi\}$ of K_i refining $\{St(\alpha) : \alpha \in A_i\}$. For each $\lambda \in \Lambda_n$, let $\Delta_0(\lambda)$ be the collection of all subsets δ of Ξ such that

$$H(\lambda, \delta) = H_\lambda \cup (\cup \{V_\xi : \xi \in \delta\})$$

is an open set of T_i and $H(\lambda, \delta) \subset M(H'_\lambda, \delta_\lambda)$. Then $\{H(\lambda, \delta) : \delta \in \Delta_0(\lambda)\}$ is closure-preserving in T_i and therefore by the discreteness of $\{M(H'_\lambda, \delta_\lambda) : \lambda \in \Lambda_n\}$ it follows that $\{H(\lambda, \delta) : \delta \in \Delta_0(\lambda), \lambda \in \Lambda_n\}$ is closure-preserving in T_i. Since as seen in part 2, Λ_n is the countable intersection of closures in R_n of open neighborhoods of Λ_n and K_i is metrizable, there exists a σ-locally finite (in R_n) open collection $\mathcal{B}_0$ of R_n which is an open base for the subspace K_i. Put

$$\mathcal{B}_n = \{B(\lambda, \delta) = \{\lambda\} \cup (\cup \{V_\xi : \xi \in \delta\}) : \delta \in \Delta_0(\lambda), \lambda \in \Lambda_n\},$$

$$\mathcal{B} = \mathcal{B}_0 \cup (\bigcup_{n=1}^{\infty} \mathcal{B}_n).$$

Then $\mathcal{B}$ is a σ-closure-preserving open collection of R_n. To see that $\mathcal{B}$ is an open base for R_n, let $p \in V$ for an open set V and a point $p \in R_n$. If $p \in K_i$, then obviously $p \in B \subset V$ for some $B \in \mathcal{B}_0$. Suppose $p = \lambda \in \Lambda_n$. There exists a $\delta_0 \in \Delta(H'_\lambda)$ such that

$$S(M(H'_\lambda, \delta_0), \{St(\alpha) : \alpha \in A_i\}) \subset q_n^{-1}(V) \cap M(H'_\lambda, \delta_\lambda).$$

$\mathcal{V} < \{St(\alpha) : \alpha \in A_i\}$ implies that there exists a $\delta \in \Delta_0(\lambda)$ such that

$$S(M(H'_\lambda, \delta_0), \mathcal{V}) = H(\lambda, \delta) \subset q_n^{-1}(V) \cap M(H'_\lambda, \delta_\lambda).$$

Hence $\lambda \in B(\lambda, \delta) \subset V$. This completes the proof of part 4.

Part 5: R_n is shown to be an almost metric space. Obviously $\{\{\lambda\} : \lambda \in \Lambda_n\}$ is discrete in R_n. To see that Λ_n has an approaching anti-cover, it suffices to prove that Z_i has an approaching anti-cover $\mathcal{U} = \{St(\alpha) : \alpha \in A_i\}$. Let U be an open set of T_i such that $Z_i \subset U$. By (*) there exists a $\delta \subset A_i$ such that

$$W = Z_i \cup (\cup \{St(\alpha) : \alpha \in \delta\})$$

is an open neighborhood of Z_i such that $W \subset U$ and $S(W_\alpha, \mathcal{W}_i) \subset f_i^{-1}(U)$ for every $\alpha \in \delta$. It is easily seen that $W \cap S(T_i - U, \mathcal{U}) = \emptyset$. Hence R_n is an almost metric space.

Part 6: Let $X \subset \prod_{i=1}^{\infty} X_i$, where each X_i is a special σ-metric space. By the above argument, there exists a perfect mapping f_i of a free L-space Z_i onto X_i. Construct a mapping $f : \prod_{i=1}^{\infty} Z_i \to \prod_{i=1}^{\infty} X_i$ by

$$f(x) = (f_i(x_i))_{i \in N}, \qquad x = (x_i) \in \prod_{i=1}^{\infty} Z_i.$$

Let $Z_0 = f^{-1}(X)$ and $g = f | Z_0$. Then $g : Z_0 \to X$ is a perfect mapping of a free L-space onto X. This completes the proof.

Since every free L-space is M_1, every special μ-space is an image of an M_1-space by a perfect and irreducible mapping, and therefore by [1, Theorem 3.4] every special μ-space is M_1.

COROLLARY 1. *The following are equivalent:*

(1) *X is a free L-space with* $\dim X = 0$.

(2) $X \subset \prod_{i=1}^{\infty} X_i$, *where each X_i is a special σ-metric space with* $\dim X_i = 0$.

PROOF. (1)$\to$(2) follows immediately from [7, Theorem 3.8]. (2)$\to$(1): By the preceding proof each X_i is a free L-space. $\dim X = 0$ follows from [7, Lemma 3.7].

COROLLARY 2. *If X is a special σ-metric space with* $\dim X \leqq n$, *then there exists a closed mapping f of a free L-space Z with* $\dim Z \leqq 0$ *onto X snch that* $\mathrm{ord}\, f \leqq n+1$.

PROOF. In part 1 of the preceding proof, f is chosen to be a mapping with $\mathrm{ord}\, f \leqq n+1$.

In the next corollary, a space R_n^* is called an *almost metric space plus one point* if $R_n^* = R_n \cup \{p_n\}$ with $p_n \notin R_n$, where R_n is an almost metric space. Note that R_n^* need not be Hausdorff.

COROLLARY 3. *If X is a special μ-space, then X is embedded in the countable product of almost metric spaces plus one point.*

PROOF. It suffices to prove that if Z is a special σ-metric space then Z is embedded in the countable product of such spaces. Let $\{Z_i\}$ be the special scale and let $\mathcal{W}_i = \{W_\alpha : \alpha \in A_i\}$ be a uniformly approaching anti-cover of Z_i. Define a space $(T_i, \mathcal{T}_i)$ in the similar way to part 1 of the preceding proof. Then we have $Z \subset \prod_{i=1}^{\infty} (T_i, \mathcal{T}_i)$. Therefore it suffices to prove that each $(T_i, \mathcal{T}_i)$ can be embedded in the countable product of almost metric spaces plus one point. Let $\bigcup_{n=1}^{\infty} \mathcal{H}_n$ be an open base for the subspace Z_i, where each $\mathcal{H}_n = \{H_\lambda : \lambda \in \Lambda_n\}$ is discrete in Z_i. Let R_n^* be the disjoint sum of K_i, Λ_n and $\{r_n\}$, where $r_n \notin K_i \cup (\bigcup_{n=1}^{\infty} \Lambda_n)$ and R_n the disjoint sum of K_i and Λ_n. That is,

$$R_n^* = R_n \cup \{r_n\}, \qquad R_n = K_i \cup \Lambda_n .$$

Define a transformation $q_n : T_i \to R_n^*$ as follows:

$$q_n(p) = p \qquad \text{if } p \in K_i ,$$

$$q_n(p) = \lambda \qquad \text{if } p \in H_\lambda,\ \lambda \in \Lambda_n ,$$

$$q_n(p) = r_n \qquad \text{if } p \in Z_i - \bigcup \{H_\lambda : \lambda \in \Lambda_n\} .$$

We introduce into R_n^* the quotient topology with respect to q_n. Define a transformation $q : T_i \to \prod_{n=1}^{\infty} R_n^*$ as follows:

$$q(p) = (q_n(p))_{n \in N},\ p \in T_i .$$

As seen in the preceding proof, q is an embedding. Thus we shall show that each R_n^* is an almost metric space plus one point, that is, R_n is an almost metric space. First, we shall show that R_n is an M_1-space. To see that R_n is regular, suppose $\lambda\in V$ for an open set V and a point $\lambda\in\Lambda_n$. There exists a $\delta\in\Delta(H'_\lambda)$ such that

$$S(L(H'_\lambda, \delta), \mathcal{W}_i)\subset f_i^{-1}(q_n^{-1}(V))\cap H'_\lambda,$$

where H'_λ is an open set of Z such that

$$H'_\lambda\cap Z_i=H_\lambda, \quad \bar{H}'^Z_\lambda\cap H_\mu=\emptyset \quad \text{for every } \mu\neq\lambda,\ \mu\in\Lambda_n.$$

Set

$$V_0=\{\lambda\}\cup(\cup\{St(\alpha): \alpha\in\delta\}).$$

Then V_0 is an open neighborhood of λ such that $\bar{V}_0^{R_n}\subset V$. Thus R_n is regular. Since

$$Z(n)=K_i\cup(\cup\{H_\lambda: \lambda\in\Lambda_n\})$$

is paracompact and $\{H_\lambda: \lambda\in\Lambda_n\}$ is a discrete closed collection in $Z(n)$, there exists an open collection $\{H'_\lambda: \lambda\in\Lambda_n\}$ of Z such that $H_\lambda=H'_\lambda\cap Z_i$ for every $\lambda\in\Lambda_n$ and $\{\bar{H}'^{Z(n)}_\lambda: \lambda\in\Lambda_n\}$ is discrete in $Z(n)$. Take a $\delta_\lambda\in\Delta(H')$ such that $L(H'_\lambda, \delta_\lambda)\subset H'_\lambda$. Let $\mathcal{V}=\{V_\xi: \xi\in\Xi\}$ be a locally finite (in K_i) open over of K_i refining $\{St(\alpha): \alpha\in A_i\}$. For each $\lambda\in\Lambda_n$, let $\Delta_0(\lambda)$ be the collection of all subsets δ of Ξ such that

$$H(\lambda, \delta)=H_\lambda\cup(\cup\{V_\xi: \xi\in\delta\})$$

is an open set of T_i and $H(\lambda, \delta)\subset M(H'_\lambda, \delta_\lambda)$. We repeat the essential part of the proof of part 4. This completes the proof.

K. Nagami proved recently that the statements (1), (2), (3) and (4) in the next theorem are equivalent for a free L-space X [7, Theorem 2.3]. In this section we shall show that these statements are equivalent for every special μ-space X.

LEMMA. *Let X be a paracompact σ-space with a closed network $\mathcal{F}=\bigcup_{i=1}^{\infty}\mathcal{F}_i$, where each $\mathcal{F}_i$ is locally finite in X, and each $\mathcal{F}_i$ has a locally finite open collection $\{V(F): F\in\mathcal{F}_i\}$ of X such that $F\subset V(F)$ for every $F\in\mathcal{F}_i$. Assume that if $p\in G$ for an open set G and a point $p\in X$, then there exists an $F_p\in\mathcal{F}$ and an open set V_p such that*

$$p\in F_p\cap V_p\subset G\cap V(F_p),$$

$$\operatorname{Ind} B(V_p)\leq n-1.$$

Then $\operatorname{Ind} X\leq n$.

PROOF. Let H, K be a pair of disjoint closed sets of X and H_1, K_1 be a pair of open sets of X such that

$$H\subset H_1, \quad K\subset K_i, \quad \bar{H}_1\cap\bar{K}_1=\emptyset.$$

Take for each point $p\in X$ an $F_p\in\mathcal{F}$ an open set V_p such that

$$p\in F_p\subset V_p\subset(X-\bar{H}_1 \text{ or } X-\bar{K}_1)\cap V(F_p)\,,$$

$$\operatorname{Ind} B(V_p)\leqq n-1\,.$$

Choose a subset $X_0\subset X$ such that

$$\mathcal{F}_0=\{F_p : p\in X\}=\{F_p : p\in X_0\}\,,$$

$$F_p\neq F_{p'} \qquad \text{if } p,\ p'\in X_0,\ p\neq p'\,.$$

Since $\mathcal{F}_0$ covers X, $\mathcal{V}=\{V_p : p\in X_0\}$ covers X. It is easily seen that $\mathcal{V}$ is a σ-locally finite open collection such that $\bar{\mathcal{V}}<\{X-H,\ X-K\}$. Thus by [3, Theorem 11. 12] H, K are separated by a closed set P with $\operatorname{Ind} P\leqq n-1$ and we have $\operatorname{Ind} X\leqq n$.

In the proof of the next theorem we use the following term: An anti-cover $\mathcal{U}$ of F in X is said to be *approaching* (*uniformly approaching*) *to* F *with respect to* an open collection $\mathcal{V}$ if for every open set $V\in\mathcal{V}$ with $F\subset V$ (for every open set $V\in\mathcal{V}$) $\overline{S(X-V,\ \mathcal{U})}\cap F=\emptyset$ ($\overline{S(X-V,\ \mathcal{U})}\cap V\cap F=\emptyset$).

THEOREM 4. *For a special μ-space X the following are equivalent*:

(1) $\dim X\leqq n$.

(2) *There exists a closed mapping f of a special μ-space Z with* $\dim Z\leqq 0$ *onto X such that* $\operatorname{ord} f\leqq n+1$.

(3) $X=\bigcup_{i=1}^{n+1} Z$, *where* $\dim Z_i\leqq 0$ *for each i.*

(4) $\operatorname{Ind} X\leqq n$.

PROOF. The implications (2)→(3)→(4)→(1) are already known. Thus it remains to prove the implication (1)→(2). Suppose $X\subset\prod_{i=1}^{\infty} X_i$ and $\dim X\leqq n$, where each X_i is a σ-metric space with a special scale $\{X_{im} : m\in N\}$. Let $\mathcal{W}_{im}$ be a uniformly approaching anti-cover of X_{im}. Let $\mathcal{F}_{im}=\bigcup_{j=1}^{\infty}\mathcal{F}_{imj}$ be a network of the subspace X_{im}, where each $\mathcal{F}_{imj}=\{F_\lambda : \lambda\in\Lambda_{imj}\}$ is a discrete closed collection of X_{im}. By [4, Theorem 2] there exists a replica $\rho_i : X_i\to\hat{X}_i$ such that

(1) if G is an open set of X_i with $p\in G\cap X_{im}$, then there exists an open set W of X_i such that $p\in W\cap X_{im}\subset G$ and $\rho_i(W)$ is open in $\hat{X}_i$.

Let $\mathcal{V}_{imj}=\{V_\lambda : \lambda\in\Lambda_{imj}\}$ be a discrete open collection of X_i such that $F_\lambda\subset V_\lambda$ for each $\lambda\in\Lambda_{imj}$. Set

$$F_{imj}=\cup\{F_\lambda : \lambda\in\Lambda_{imj}\}\,.$$

Since $\rho_i(F_{imj})$ is a closed set of a metric space $\hat{X}_i$, there exists an approaching anti-cover $\mathcal{U}_{imj}$ of F_{imj} in X_i with respect to $\{\rho_i^{-1}(V): V$ open in $\hat{X}_i\}$. Let $\pi_i: X\to X_i$ be the restriction to X of the projection and set

$$X'_{im}=\pi_i^{-1}(X_{im}),$$

$$\mathcal{F}'_{imj}=\pi_i^{-1}(\mathcal{F}_{imj})=\{F'_\lambda=\pi_i^{-1}(F_\lambda): \lambda\in\Lambda_{imj}\},$$

$$\mathcal{U}'_{imj}=\pi_i^{-1}(\mathcal{U}_{imj}),$$

$$\mathcal{V}'_{imj}=\pi_i^{-1}(\mathcal{V}_{imj})=\{V'_\lambda: \lambda\in\Lambda_{imj}\},$$

$$\mathcal{W}'_{im}=\pi_i^{-1}(\mathcal{W}_{im}),$$

$$H_{imj}=\pi_i^{-1}(F_{imj})=\cup\{F'_\lambda: \lambda\in\Lambda_{imj}\},$$

$$V_{imj}=\cup\{V'_\lambda: \lambda\in\Lambda_{imj}\}.$$

Then $\mathcal{U}'_{imj}$ is an approaching anti-cover of H_{imj} with respect to $\{\pi_i^{-1}(\rho_i^{-1}(V)): V$ open in $\hat{X}_i\}$. Let $\bigcup_{k=1}^{\infty}\mathcal{U}_{imjk}$ is an anti-cover of H_{imj} in X refining $\mathcal{U}'_{imj}$, where each $\mathcal{U}_{imjk}=\{U_{imjk\alpha}: \alpha\in A_{imjk}\}$ is a discrete open collection of $X-H_{imj}$. Set

$$U_{imjk}=\cup\{U_{imjk\alpha}: \alpha\in A_{imjk}\}.$$

$$U_{imjk}=\bigcup_{t=1}^{\infty}K_{imjkt},$$

where each K_{imjkt} is a closed set of X. Similarly since $\mathcal{W}'_{im}$ is a uniformly approaching anti-cover of X'_{im} with respect to $\{\pi_i^{-1}(V): V$ open in $X_i\}$, we can get an anti-cover $\bigcup_{j=1}^{\infty}\mathcal{W}_{imj}$ of X'_{im} in X refining $\mathcal{W}'_{im}$, where each $\mathcal{W}_{imj}=\{W_{imj\beta}: \beta\in B_{imj}\}$ is a discrete open collection of $X-X'_{im}$. Set

$$W_{imj}=\cup\{W_{imj\beta}: \beta\in B_{imj}\}.$$

$$W_{imj}=\bigcup_{k=1}^{\infty}L_{imjk},$$

where each L_{imjk} is a closed set of X. Write the countable collection of disjoint pairs of closed sets as follows:

$$\{(K_{imjkt}, X-U_{imjk}): i, m, j, k, t\in N\}$$

$$\cup\{(L_{imjk}, X-W_{imj}): i, m, j, k\in N\}$$

$$=\{(P_i, Q_i): i\in N\}.$$

By [6, Lemma 3.6] there exists a contraction ρ of X onto a metric space $\hat{X}$ with $\dim\hat{X}\leq n$ such that each pair $(\rho(P_i), \rho(Q_i))$ is a disjoint pair of closed sets of $\hat{X}$. Let $\mathcal{G}_i=\{G_{i\mu}: \mu\in M_i\}$, $i\in N$ be a sequence of locally finite open covers of $\hat{X}$ satisfying the following: For each $i\in N$,

1) mesh $\mathcal{G}_i \leqq 1/i$,

2) $\bar{\mathcal{G}}_{i+1} < \mathcal{G}_i$,

3) $\mathcal{G}_i < \{\hat{X}-\rho(P_i),\ \hat{X}-\rho(Q_i)\}$,

4) ord $\mathcal{G}_i \leqq n+1$.

Let $p^{i+1}{}_i: M_{i+1} \to M_i$ be a transformation such that $p^{i+1}{}_i(\lambda)=\mu$ yields $\bar{G}_{i+1\lambda} \subset G_{i\mu}$. Set

$$Y=\{\langle\mu_i\rangle \in \varprojlim \{M_i,\ p^{i+1}{}_i\} : \bigcap_{i=1}^{\infty} G_{i\mu} \neq \emptyset\}.$$

$g: Y \to \hat{X}$ is defined as follows:

$$g(\langle\mu_i\rangle)=\bigcap_{i=1}^{\infty} G_{i\mu_i}, \qquad \langle\mu_i\rangle \in Y.$$

Then as seen in the proof of [7, Theorem 2.3], g is a closed mapping of Y with $\dim Y \leqq 0$ onto $\hat{X}$ such that ord $g \leqq n+1$. Construct $Z \subset X \times Y$ as follows:

$$Z=\{(x,\ y) \in X \times Y:\ \rho(x)=g(y)\}.$$

Let f, σ be the restrictions of the projections of $X \times Y$ onto X, Y, respectively. Then f is a closed mapping of Z onto X with ord $f \leqq n+1$ and σ is a contraction of Z onto Y. The following statement (2) follows from Assertion 1 of the proof of [7, Theorem 2.3]:

(2) For each $i \in N$, $f^{-1}(P_i)$, $f^{-1}(Q_i)$ can be separated in Z by the empty set. Since X is a special μ-space, Z is also a special μ-space. Therefore it remains to prove $\dim Z \leqq 0$. To prove this, we shall show the following statements (3), (4) and (5).

(3) If D is a semi-canonical neighborhood of H_{imj} with respect to $\mathcal{U}'_{imj}$, then there exists a closed and open set V of Z such that

$$f^{-1}(H_{imj}) \subset V \subset f^{-1}(D).$$

PROOF. Set $E=S(X-D,\ \mathcal{U}'_{imj})$. Then $\mathcal{D}=\{f^{-1}(E),\ f^{-1}(D)-f^{-1}(H_{imj})\}$ is an open cover of $Z-f^{-1}(H_{imj})$. By (2) for each i, m, j, k, $t \in N$ there exsits a closed and open set R_{imjkt} of Z such that

$$f^{-1}(K_{imjkt}) \subset R_{imjkt} \subset f^{-1}(U_{imjk}).$$

Set

$$R_{imjkt\alpha}=R_{imjkt} \cap f^{-1}(U_{imjk\alpha}),$$

$$\mathcal{R}_{imjkt}=\{R_{imjkt\alpha}:\ \alpha \in A_{imjk}\},$$

$$\mathcal{R}_{imj}=\cup\{\mathcal{R}_{imjkt}:\ k,\ t \in N\}.$$

Then $\mathcal{R}_{imj}$ is a σ-discrete cover of $Z-f^{-1}(H_{imj})$ consisting of closed and open sets of Z refining $\mathcal{D}$. Thus by [3, Theorem 11. 12] there exists a closed and open set V' of $Z-f^{-1}(H_{imj})$ such that

$$Z-(f^{-1}(E)\cup f^{-1}(H_{imj}))\subset V'\subset f^{-1}(D)-f^{-1}(H_{imj}).$$

Since D is semi-canonical, $V=V'\cup f^{-1}(H_{imj})$ is the desired set.

(4) If D is an open set of X_i such that

$$F_{imj}\subset D\subset\cup\{V_\lambda:\ \lambda\in\Lambda_{imj}\},$$

then there exists an open set V of Z such that

$$V\cap f^{-1}(X'_{im})=f^{-1}(\pi_i^{-1}(D))\cap f^{-1}(X'_{im}),$$

$$V\subset f^{-1}(\pi_i^{-1}(D))$$

and such that $V\cap(Z-f^{-1}(X'_{im}))$ is closed in $Z-f^{-1}(X'_{im})$.

PROOF. Set $E=S(X-\pi_i^{-1}(D),\ \mathcal{W}'_{im})$. Then

$$\mathcal{D}=\{f^{-1}(E),\ f^{-1}(\pi_i^{-1}(D))\cap(Z-f^{-1}(X'_{im}))\}$$

is an open cover of the subspace $Z-f^{-1}(X'_{im})$. By (2) for each $i,\ m,\ j,\ k\in N$ there exists a closed and open set T_{imjk} of Z such that

$$f^{-1}(L_{imjk})\subset T_{imjk}\subset f^{-1}(W_{imj}).$$

Set

$$T_{imjk\beta}=f^{-1}(W_{imj\beta}(\cap T_{imjk},$$

$$\mathcal{T}_{imjk}=\{T_{imjk\beta}:\ \beta\in B_{imj}\},$$

$$\mathcal{T}_{im}=\cup\{\mathcal{T}_{imjk}:\ j,\ k\in N\}.$$

Then $\mathcal{T}_{im}$ is a σ-discrete cover of $Z-f^{-1}(X'_{im})$ consisting of closed and open sets of Z and refining $\mathcal{D}$. Thus by [3, Theorem 11. 12] again, there exists a closed and open set V' of $Z-f^{-1}(X'_{im})$ such that

$$Z-(f^{-1}(E)\cup f^{-1}(X'_{im}))\subset V'\subset f^{-1}(\pi_i^{-1}(D))-f^{-1}(X'_{im}).$$

Then $V=V'\cup(f^{-1}(\pi_i^{-1}(D))\cap f^{-1}(X'_{im}))$ is the desired set.

(5) If U is an open set of X with $x\in U$, then there exist a finite set $\{\lambda_1,\cdots,\lambda_k\}$ of indices with $\lambda_t\in\Lambda_{i(t)m(t)j(t)}$, $t=1,\cdots,k$, and a closed and open set O of Z such that

$$f^{-1}(x)\subset\bigcap_{t=1}^{k}f^{-1}(F'_{\lambda_t})\subset O\subset f^{-1}(U)\cap(\bigcap_{t=1}^{k}f^{-1}(V'_{\lambda_t})).$$

PROOF. Let $x=\langle x_i\rangle\in U$. Then there exist an integer k and open sets U_t of $X_{i(t)}$, $t=1,\cdots,k$ such that

$$x\in\bigcap_{t=1}^{k}\pi_{i(t)}^{-1}(U_t)\subset U.$$

For each $t=1, \cdots, k$, there exists an $m(t)\in N$ with $x_{i(t)}\in X_{i(t)m(t)}$ because $\{X_{i(t)m}: m\in N\}$ is a scale of $X_{i(t)}$. Since $\mathcal{F}_{i(t)m(t)}$ forms a network of $X_{i(t)m(t)}$, we can choose a $\lambda_t\in\Lambda_{i(t)m(t)j(t)}$ with $x_{i(t)}\in F_{\lambda_t}\subset U_t$. By (1) there exists an open set G_t of $X_{i(t)}$ such that

$$F_{\lambda_t}\subset G_t\cap X_{i(t)m(t)}\subset V_{\lambda_t}\cap U_t$$

and such that $\rho_{i(t)}(G_t)$ is open in $\hat{X}_{i(t)}$. For each $\lambda\neq\lambda_t$, $\lambda\in\Lambda_{i(t)m(t)j(t)}$ choose by (1) again an open set G_λ of $X_{i(t)}$ such that

$$F_\lambda\subset G_\lambda\cap X_{i(t)m(t)}\subset V_\lambda$$

and such that $\rho_{i(t)}(G_\lambda)$ is open in $\hat{X}_{i(t)}$. Set

$$G(t)=G_t\cup(\cup\{G_\lambda: \lambda\in\Lambda_{i(t)m(t)j(t)}, \lambda\neq\lambda_t\}).$$

Then $\pi_{i(t)}^{-1}(G(t))$ is a semi-canonical neighborhood of $H_{i(t)m(t)j(t)}$ with respect to $\mathcal{U}'_{i(t)m(t)j(t)}$. By (3) there exists a closed and open set $V(t)$ of Z such that

$$f^{-1}(H_{i(t)m(t)j(t)})\subset V(t)\subset f^{-1}(\pi_{i(t)}^{-1}(G(t))).$$

Set

$$D(t)=(U_t\cap V_{\lambda_t})\cup(\cup\{V_\lambda: \lambda\in\Lambda_{i(t)m(t)j(t)}, \lambda\neq\lambda_t\}).$$

Then $D(t)$ is an open set of $X_{i(t)}$ such that

$$F_{i(t)m(t)j(t)}\subset D(t)\subset\cup\{\dot{V}_\lambda: \lambda\in\Lambda_{i(t)m(t)j(t)}\}.$$

Therefore by (4) there exists an open set $W(t)$ of Z such that

$$W(t)\cap f^{-1}(X'_{i(t)m(t)})=f^{-1}(\pi_{i(t)}^{-1}(D(t)))\cap f^{-1}(X'_{i(t)m(t)}),$$

$$W(t)\subset f^{-1}(\pi_{i(t)}^{-1}(D(t)))$$

and such that $W(t)\cap(Z-f^{-1}(X'_{i(t)m(t)}))$ is closed in the subspace $Z-f^{-1}(X'_{i(t)m(t)})$. Set

$$O(t)=W(t)\cap f^{-1}(V'_{\lambda_t})\cap V(t),$$

$$O=\bigcap_{t=1}^{k}O(t).$$

Then O is a closed and open set of Z with the required property. Set

$$\mathcal{P}_{imjk}=f^{-1}(\mathcal{F}'_{imj})\wedge\{\sigma^{-1}(p_k^{-1}(\mu)): \mu\in M_k\}$$

$$=\{P_\xi: \xi\in\Xi_{imjk}\},$$

$$\mathcal{Q}_{imjk}=f^{-1}(\mathcal{V}'_{imj})\wedge\{\sigma^{-1}(p_k^{-1}(\mu)): \mu\in M_k\}$$

$$=\{Q_\xi: \xi\in\Xi_{imjk}\},$$

where each $p_k: Y \to M_k$ is the restriction to Y of the projection. Then each $\mathcal{Q}_{imjk}$ is a discrete open collection of Z and $\mathcal{P}_{imjk}$ is a discrete closed collection of Z such that $P_\xi \subset Q_\xi$ for each $\xi \in \Xi_{imjk}$. By (5) the following statement is easily shown to be true.

(6) If $z \in U$ for an open set U of Z and a point $z \in Z$, there exist a finite subset $\{\xi_1, \cdots, \xi_k\} \subset \cup \{\Xi_{imjk} : i, m, j, k \in N\}$ and a closed and open set O such that

$$z \in \bigcap_{j=1}^{k} P_{\xi_j} \subset O \subset \bigcap_{j=1}^{k} Q_{\xi_j} \cap U .$$

Thus by the above lemma, we have $\operatorname{Ind} Z \leqq 0$. This completes the proof.

The author proved that the characterization (A) of $\dim X$ stated in the next theorem is possible for a special σ-metric space X [2, Theorem 1] and for a free L-space X [2, Theorem 2]. Since by Theorem 2 every free L-space is a special μ-space, these two results can be regarded to be the corollaries to the next theorem.

THEOREM 5. *Let X be a special μ-space. Then*

(A) $\dim X \leqq n$ *if and only if there exists a σ-closure-preserving open base $\mathcal{W}$ for X such that* $\dim B(W) \leqq n-1$ *for every* $W \in \mathcal{W}$.

PROOF. The if part of (A) follows from [8, Lemma 7] because every special μ-space is M_1. The only if part: If we can show the validity of (A) for the case $n=0$, then the only if part of (A) for the general case follows from [2, Lemma 1] and Theorem 1, (1)↔(2). Suppose $A \subset \prod_{i=1}^{\infty} X_i$ and $\dim X \leqq 0$, where each X_i is a special σ-metric space with a scale $\{X_{im} : m \in N\}$ such that each X_{im} has a uniformly approaching anti-cover $\mathcal{W}_{im}$. Let

$$\mathcal{U}_{imj} = \{U_{imj\alpha} : \alpha \in A_{imj}\}, \qquad \mathcal{F}_{imj} = \{F_{imj\alpha} : \alpha \in A_{imj}\},$$
$$j \in N,$$

be sequences of locally finite open covers of X_i and of locally finite closed covers of X_{im} such that

$$F_{imj\alpha} \subset U_{imj\alpha} \qquad \text{for every } \alpha \in A_{imj}$$

and such that

(1) if $p \in G$ for an open set G of X_i and a point $p \in X_{im}$, then there exists an $\alpha \in A_{imj}$ for some $j \in N$ such that

$$p \in F_{imj\alpha} \subset X_{im} \cap U_{imj\alpha} \subset G .$$

Set

$$\mathcal{U}'_{imj}=\{U'_{imj\alpha}=\pi_i^{-1}(U_{imj\alpha}):\ \alpha\in A_{imj}\},$$

$$\mathcal{F}'_{imj}=\{F'_{imj\alpha}=\pi_i^{-1}(F_{imj\alpha}):\ \alpha\in A_{imj}\},$$

$$\mathcal{W}'_{im}=\{W'_{im\beta}:\ \beta\in B_{im}\}=\pi_i^{-1}(\mathcal{W}_{im}),$$

$$X'_{im}=\pi_i^{-1}(X_{im}),$$

where $\pi_i: X\to X_i$ is the restriction of the projection. Since X is hereditariry paracompact, we can assume that each $\mathcal{W}'_i$ is locally finite in the subspace $X-X'_{im}$. Then $\mathcal{W}'_{im}$ admits its closed shrinking $\{H_{im\beta}:\ \beta\in B_{im}\}$ with $H_{im\beta}\subset W'_{im\beta}$ for every $\beta\in B_{im}$. Since $\operatorname{Ind}(X-X'_{im})\leqq 0$, for every $\beta\in B_{im}$ there exists a closed and open set $P_{im\beta}$ of $X-X'_{im}$ such that

$$H_{im\beta}\subset P_{im\beta}\subset W'_{im\beta}.$$

For each $\alpha\in A_{imj}$, let $\Delta_{imj}(\alpha)$ be the totality of subsets δ of B_{im} such that

$$P_{imj\alpha}(\delta)=(U'_{imj\alpha}\cap X'_{im})\cup(\cup\{P_{im\beta}:\ \beta\in\delta\}) \tag{2}$$

is an open set of X such that

$$F'_{imj\alpha}\subset P_{imj\alpha}(\delta)\subset U'_{imj\alpha}.$$

Set

$$\mathcal{N}=\{N_k:\ k\in N\}=\{A\subset N\times N\times N:\ |A|<\aleph_0\}.$$

Take an arbitrary $N_k\in\mathcal{N}$ such that

$$N_k=\{(i(t),\ m(t),\ j(t)):\ t=1,\cdots,s\}$$

with $i(t),\ m(t),\ j(t)\in N,\ t=1,\cdots,s$. For each $\alpha=(\alpha_1,\cdots,\alpha_s)\in\prod_{t=1}^{s}A_{i(t)m(t)j(t)}$ $\bigcap_{t=1}^{s}F'_{i(t)m(t)j(t)\alpha_t}$ is closed and $\bigcap_{t=1}^{s}U'_{i(t)m(t)j(t)\alpha_t}$ is open in X such that

$$\bigcap_{t=1}^{s}F'_{i(t)m(t)j(t)\alpha_t}\subset\bigcap_{t=1}^{s}U'_{i(t)m(t)j(t)\alpha_t}.$$

Since $\operatorname{Ind}X\leqq 0$, there exists a closed and open set $V_{k\alpha}$ of X such that

$$\bigcap_{t=1}^{s}F'_{i(t)m(t)j(t)\alpha_t}\subset V_{k\alpha}\subset\bigcap_{t=1}^{s}U'_{i(t)m(t)j(t)\alpha_t}.$$

Set

$$W_{k\alpha}(\delta)=(\bigcap_{t=1}^{s}P_{i(t)m(t)j(t)\alpha_t}(\delta_t))\cap V_{k\alpha}, \tag{3}$$

$$\delta=(\delta_1,\cdots,\delta_s)\in\prod_{t=1}^{s}\Delta_{i(t)m(t)j(t)}(\alpha_t),$$

$$\mathcal{W}_k(\alpha)=\{W_{k\alpha}(\delta):\ \delta\in\prod_{t=1}^{s}\Delta_{i(t)m(t)j(t)}(\alpha_t)\},$$

$$\mathcal{W}_k=\cup\{\mathcal{W}_k(\alpha):\ \alpha\in\prod_{t=1}^{s}A_{i(t)m(t)j(t)}\},$$

$$\mathcal{W}=\cup\{\mathcal{W}_k:\ k\in N\}.$$

Then we shall show that $\mathcal{W}$ has the required property. To see this, we shall establish the following statements (4), (5) and (7).

(4) $\mathcal{W}$ consists of closed and open sets of X.

This follows (3) and from the fact that each $V_{k\alpha}$, $P_{im\beta}$, $\beta\in B_{im}$ are closed and open in X, $X-X'_{im}$, respectively.

(5) $\mathcal{W}$ is a σ-closure-preserving collection in X.

Note that each $\{V_{k\alpha}:\ \alpha\in\prod_{t=1}^{s}A_{i(t)m(t)j(t)}\}$ is locally finite in X in the sense that every point of X has a neighborhood in X which intersects $V_{k\alpha}$ for finitely many different $\alpha\in\prod_{t=1}^{s}A_{i(t)m(t)j(t)}$. Since by (3) $W_{k\alpha}(\delta)\subset V_{k\alpha}$ for every $\delta\in\prod_{t=1}^{s}\Delta_{i(t)m(t)j(t)}(\alpha_t)$, it follows that it suffices to show that each $\mathcal{W}_k(\alpha)$ is closure-preserving in X in order to show (5). Let $N_k\in\mathfrak{N}$ with $|N_k|=s$ and α be as follows:

$$N_k=\{(i(t),\ m(t),\ j(t)):\ t=1,\ \cdots,\ s\},$$

$$\alpha=(\alpha_1,\ \cdots,\ \alpha_t)\in\prod_{t=1}^{s}A_{i(t)m(t)j(t)}\ .$$

To show that $\mathcal{W}_k(\alpha)$ is closure-preserving in X, we shall show by induction on $n=1,\ \cdots,\ s$ the following proposition (P_n):

(P_n) For every subset $M\subset\{1,\ \cdots,\ s\}$ with $|M|\leqq n$

$$\mathcal{P}(M)=\{W_M(\delta)=(\bigcap_{t\in M}P_{i(t)m(t)j(t)\alpha_t}(\delta_t))\cap V_{k\alpha}:$$
$$\delta=(\delta_t)_{t\in M}\in\prod_{t\in M}\Delta_{i(t)m(t)j(t)}(\alpha_t)\}$$

is closure-preserving in X.

Take $M\subset\{1,\ \cdots,\ s\}$ with $|M|=1$. If $M=\{(i,\ m,\ j)\}$ with $(i,\ m,\ j)\in N_k$, then it is easily seen that $\mathcal{P}(M)$ is closure-preserving in X because $V_{k\alpha}$ is closed and open in X and $\{P_{im\beta}\cap V_{k\alpha}:\ \beta\in B_{im}\}$ is locally finite in $X-X'_{im}$. Assume that (P_m) is true for every $m=1,\ \cdots,\ n-1$. Let $M\subset\{1,\ \cdots,\ s\}$ with $|M|=n$, and let Δ_0 be an arbitrary subset of $\prod_{t\in M}\Delta_{i(t)m(t)j(t)}(\alpha_t)$. Suppose that

(6) $$p\in\overline{\cup\{W_M(\delta):\ \delta\in\Delta_0\}}\,.$$

If $p\in X-\bigcup_{t\in M} X'_{i(t)m(t)}$, then $p\in W_M(\delta)$ for some $\delta\in\Delta_0$ because

$$\{(\bigcap_{t\in M} P_{i(t)m(t)\beta_t})\cap V_{k\alpha}:(\beta_t)_{t\in M}\in\prod_{t\in M} B_{i(t)m(t)}\}$$

is locally finite at p. If $p\in\bigcap_{t\in M} X'_{i(t)m(t)}$, then $p\in W_M(\delta)$ for every δ is obtained from the relation

$$\overline{\bigcup\{W_M(\delta):\delta\in\Delta_0\}}\cap(\bigcap_{t\in M} X'_{i(t)m(t)})$$

$$\subset V_{k\alpha}\cap(\bigcap_{t\in M} X'_{i(t)m(t)})\subset W_M(\delta).$$

As the final case, we consider the case $p\in\bigcup_{t\in M} X'_{i(t)m(t)}-\bigcap_{t\in M} X'_{i(t)m(t)}$. Set

$$M'=\{t\in M: p\notin X'_{i(t)m(t)}\}.$$

Then $1\leqq|M'|<n$. Obviously from (6)

$$p\in\overline{\bigcup\{W_{M'}(\delta):\delta=(\delta_t)_{t\in M}\in\Delta_0\}}.$$

By the induction assumption, $p\in W_{M'}(\delta)$ for some $\delta=(\delta_t)_{t\in M}\in\Delta_0$. Since $p\in W_{M-M'}(\delta)$, $p\in W_M(\delta)$. This shows that $\mathcal{P}(M)$ is closure-preserving in X. Therefore it follows from (P_s) that $\mathcal{P}(\{1,\cdots,s\})=\mathcal{W}_k(\alpha)$ is closure-preserving in X.

(7) $\mathcal{W}$ is an open base for X.

Suppose that $p\in G$ for an open set G and a point p of X. There exist an $s\in N$ and an open set U_t of $X_{i(t)}$, $t=1,\cdots,s$, such that

$$p\in\bigcap_{t=1}^{s}\pi_{i(t)}^{-1}(U_t)\subset G.$$

We have an $m(t)\in N$ such that $p_{i(t)}\in X_{i(t)m(t)}$. By (1) chose a $j(t)\in N$ and an $\alpha_t\in A_{i(t)m(t)j(t)}$ such that

$$p_{i(t)}\in F_{i(t)m(t)j(t)\alpha_t}\subset U_{i(t)m(t)j(t)\alpha_t}\cap X_{i(t)m(t)j(t)}\subset U_t.$$

Since $\mathcal{W}_{i(t)m(t)}$ is a uniformly approaching anti-cover of $X_{i(t)m(t)}$ in $X_{i(t)}$ and $\{P_{i(t)m(t)\beta}:\beta\in B_{i(t)m(t)}\}<\mathcal{W}'_{i(t)m(t)}$, there exists a $\delta_t\in\Delta_{i(t)m(t)j(t)}(\alpha_t)$ such that $P_{i(t)m(t)j(t)\alpha_t}(\delta_t)$ defined by (2) is an open set of X such that

$$F'_{i(t)m(t)j(t)\alpha_t}\subset P_{i(t)m(t)j(t)\alpha_t}(\delta_t)$$

$$\subset U'_{i(t)m(t)j(t)\alpha_t}\cap\pi_{i(t)}^{-1}(U_t).$$

Set

$$\delta=(\delta_1,\cdots,\delta_s)\in\prod_{t=1}^{s}\Delta_{i(t)m(t)j(t)}(\alpha_t),$$

$$\alpha=(\alpha_1,\cdots,\alpha_s)\in\prod A_{i(t)m(t)j(t)},$$

$$N_k = \{(i(t), m(t), j(t)) : t=1, \cdots, s\} \in \mathfrak{N}.$$

Then we have

$$p \in W_{k\alpha}(\delta) \subset G.$$

This completes the proof.

Finally we propose the problem:

PROBLEM. Is every special μ-space a free L-space?

If there exists a space that is a special μ-space which is not a free L-space, then from Theorem 3 it follows that the problem of K. Nagami [7, Problem 2.11] is answered negatively.

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Addendum

I am informed by the refree and S. Oka that Theorem 4 is generalized to the class of μ-spaces, which is strictly weaker than that of special μ-spaces. This is stated in S. Oka's paper "Free patched spaces and fundamental theorems of dimension theory", which is forthcoming in Bull. Acad. Polon.

Quite recently, in the letter to the author, S. Oka has pointed the following:

THEOREM. *If X is a paracompact σ-metric space with a scale $\{X_i : i \in N\}$ such that each X_i has an approaching anti-cover in X, then X is a free L-space.*

Therefore the problem stated in the final part is solved by him positively.