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ON VARIOUS RELATIVE PROPER HOMOTOPY GROUPS

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Dedicated to Mother Young

1. Introduction.

A large number of problems in geometric topology can be reduced to questions about homotopy properties near infinity of non-compact locally compact spaces or proper maps between them. These properties were studied most successfully under the following four notions of a homotopy for proper maps: a proper homotopy ([2], [10], and [22]), a germ proper homotopy ([2] and [3]), a weak proper homotopy ([7]), and a homotopy at ∞ ([4] and [5]).

The present paper is an introduction into a systematic study of the above four types of homotopy for proper maps using the techniques of algebraic topology. The key idea in this paper is to associate to every pair (K, L) of connected, (non-compact) locally compact, separable metric ANR spaces and a paper map a of the ray $[1, \infty)$ into L certain groups which will correspond to the relative homotopy groups in the ordinary homotopy theory.

The description of such groups for the germ proper homotopy was given by Brown in [2]. Here we define those groups anew in such a way that their analogues in the other three homotopy theories are apparent. In fact, we introduce eleven different groups for the triplet (K, L, a) and study their most elementary properties and investigate how do they relate to various shape invariant groups (inward, approaching, and fundamental) of compacta [20], [21].

In order to sketch our main results, in this introduction, we shall consider only relative proper homotopy groups. The map a will play the role of a base point and triplets $(\underline{\underline{D}}^n, \underline{\underline{S}}^{n-1}, \underline{\underline{*}}) = (D^n \times [1, \infty), S^{n-1}\times[1, \infty), \{*\}\times[1, \infty))$ and $(\underline{D}^n, \underline{S}^{n-1}, \underline{*}) = ((\bigcup_{k=1}^{\infty} D^n\times\{k\})\cup\underline{*}, \bigcup_{k=1}^{\infty} S^{n-1}\times\{k\})\cup\underline{*}, \underline{*})$ are two possible choices for a space which will play the role of the relative n-cell $(D^n, S^{n-1}, *)$ in the definition of relative homotopy groups. We identify $\{*\}\times[1, \infty)$ and $[1, \infty)$ and denote by $P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ the collection of all proper maps $f: (\underline{D}^n, \underline{S}^{n-1})\to(K, L)$

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which agree with a on $\underline{*}$. Let $\underline{\pi}_n(K, L, a)$ be the set of all equivalence classes under the relation of proper homotopy rel $\underline{*}$ on $P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$. For $n \geqq 2$, it is possible to define a group operation on $\underline{\pi}_n(K, L, a)$ by multiplying restrictions of representatives on each relative n-cell $(D^n, S^{n-1}, *) \times \{k\}$ in $(\underline{D}^n, \underline{S}^{n-1}, \underline{*})$. By replacing $(\underline{D}^n, \underline{S}^{n-1}, \underline{*})$ with $({}^n\underline{\underline{D}}, \underline{\underline{S}}^{n-1}, \underline{*})$ we similarly introduce groups $\underline{\underline{\pi}}_n(K, L, a)$ $(n>1)$ and a pointed set $\underline{\underline{\pi}}_1(K, L, a)$. The group $\underline{\pi}_n(K, L, a)$ $(\underline{\underline{\pi}}_n(K, L, a))$ is called the n-th relative discrete (continuous) proper homotopy group of (K, L) at a.

Motivated by Hu's paper [12], we first show in § 2 that groups $\underline{\pi}_n(K, L, a)$ and $\underline{\underline{\pi}}_n(K, L, a)$ are naturally isomorphic to the n-th relative homotopy group of pointed pairs $S(K, L, a)$ and $I(K, L, a)$, respectively, associated functorially to (K, L, a). From this observation we see that every statement about relative homotopy groups has an analogue about both relative discrete and relative continuous proper homotopy groups. Then we show how to compute $\underline{\pi}_n(K, L, a)$ from relative homotopy groups of complements of an exhausting sequence of compacta in K.

In § 3 we prove that for every pointed pair (X, A, x) of compact metric spaces there is a naturally associated pair of contractibe Q-manifolds $(M(X), M(A))$ and a proper map $a: \underline{*} \to M(A)$ such that Quigley's inward group $I_n(X, A, x)$ and approaching group $J_n(X, A, x)$ [21] of (X, A, x) are naturally isomorphic to $\underline{\pi}_n(M(X), M(A), a)$ and $\underline{\underline{\pi}}_n(M(X), M(A), a)$, respectively. Borsuk's fundamental group $F_n(X, A, x)$ of (X, A, x) [21] is naturally isomorphic to the relative limit group $\overleftarrow{\pi}_n(M(X), M(A), a)$ of $(M(X), M(A))$ at a defined as the inverse limit of an inverse sequence of relative homotopy groups of complements of an increasing sequence of compacta in $M(X)$ (see § 2).

By combining results in sections 2 and 3 it follows immediately that every statement about relative homotopy groups has an analogue about both inward and approaching relative groups of pointed pairs of compacta.

In § 4 we show that for $n \geqq 3$, any pair (K, L) of connected, locally compact, separable metric ANR spaces, and a proper map $a: \underline{*} \to L$ there is an exact sequence

$$0 \longrightarrow \overleftarrow{\pi}_n(K, L, a) \xrightarrow{j_\pi^n} \underline{\pi}_n(K, L, a) \xrightarrow{Id\text{-}sh_\pi^n} \underline{\pi}_n(K, L, a) \xrightarrow{\delta_\pi^n} \underline{\underline{\pi}}_{n-1}(K, L, a)$$
$$\xrightarrow{q_\pi^p} \overleftarrow{\pi}_{n-1}(K, L, a) \longrightarrow 0$$

connecting the n-th and the $(n-1)$-st relative limit group of (K, L) at a.

Finally, in § 5 we introduce the notion of a $\{(D^n, S^{n-1}, *)\}$-movable at ∞ triplet (K, L, a) and show that on such triplets the homomorphism $Id\text{-}sh_\pi^n$ in the above sequence is an epimorphism.

In view of § 3 our theorems in §§ 4 and 5 generalize the principal results of [21].

2. Various relative homotopy groups.

We shall consider only triplets (M, M_1, M_0) of connected, locally compact, separable metric ANR spaces such that $M_0 \subset M_1 \subset M$ and M_0 and M_1 are closed subsets of M.

A map is called *proper* if the counter-image of each compact set is compact. On the collection $P(M, M_1, M_0; N, N_1, N_0)$ of all proper maps between triples (M, M_1, M_0) and (N, N_1, N_0) (i.e., proper maps $f: M \to N$ with $f(M_1) \subset N_1$ and $f(M_0) \subset N_0$) that agree on M_0 we can introduce eight equivalence relations $=$, Π, ρ, π, $=^\infty$, Π^∞, ρ^∞, and π^∞ as follows. For $f, g \in P(M, M_1, M_0; N, N_1, N_0)$,

$f = g$ iff maps f and g are identical,

$f \Pi g$ iff maps f and g are *homotopic rel* M_0 (i.e., there is a map $H: (M \times I, M_1 \times I, M_0 \times I) \to (N, N_1, N_0)$, where $I = [0, 1]$ is the unit interval, such that $H_0 = f$, $H_1 = g$, and $H_t | M_0 = f | M_0 = g | M_0$ for all $t \in I$),

$f \rho g$ iff maps f and g are *weakly proper homotopic rel* M_0 [7] (i.e., for every compactum B in N there is a compactum A in M and a map $H: (M \times I, M_1 \times I, M_0 \times I) \to (N, N_1, N_0)$ with $H_0 = f$, $H_1 = g$, $H_t(M - A) \subset N - B$, and $H_t | M_0 = f | M_0 = g | M_0$ for all $t \in I$),

$f \pi g$ iff maps f and g are *properly homotopic rel* M_0 (i.e., there is a proper map $H: (M \times I, M_1 \times I, M_0 \times I) \to (N, N_1, N_0)$ such that $H_0 = f$, $H_1 = g$, and $H_t | M_0 = f | M_0 = g | M_0$ for all $t \in I$),

$f =^\infty g$ iff maps f and g have the same *germ* [2] (i.e., there is a compactum A in M such that $f | M - A = g | M - A$),

$f \Pi^\infty g$ iff maps f and g are *germ homotopic rel* M_0 [2] (i.e., there is a compactum A in M such that $(f | M - A) \Pi (g | M - A)$),

$f \rho^\infty g$ iff maps f and g are *homotopic at* ∞ *rel* M_0 [4] (i.e., for every compactum B in N there is a compactum A in M and a map $H: (M - A, M_1 - A, M_0 - A) \times I \to (N - B, N_1 - B, N_0 - B)$ with $H_0 = f | M - A$, $H_1 = g | M - A$, and $H_t | M_0 - A = f | M_0 - A = g | M_0 - A$ for all $t \in I$), and

$f \pi^\infty g$ iff maps f and g are *germ proper homotopic rel* M_0 [2], [3] (i.e., there is a compactum A in M such that $(f | M - A) \pi (g | M - A)$).

The obvious inclusions (denoted by arrows) among these relations are indicated in the diagram below. In general none of the arrows can be reversed.

$$\begin{array}{ccc}
\Pi & \longrightarrow & \Pi^{\infty} \\
\uparrow & & \uparrow \\
\rho & \longrightarrow & \rho^{\infty} \\
\uparrow & & \uparrow \\
\pi & \longrightarrow & \pi^{\infty} \\
\uparrow & & \uparrow \\
= & \longrightarrow & =^{\infty}
\end{array}$$

The following simple proposition shows that the first three arrows in the above diagram can be reversed when N and N_1 are contractible.

PROPOSITION. 2.1. *Let f, $g \in P(M, M_1, M_0; N, N_1, N_0)$ where N and N_1 are contractible locally compact separable metric ANR spaces and let $R \in \{\Pi, \rho, \pi\}$. Then $f\, R\, g$ iff $f\, R^{\infty}\, g$.*

The equivalence class of $f \in P(M, M_1, M_0; N, N_1, N_0)$ with respect to the relations $=^{\infty}$, Π, Π^{∞}, ρ, ρ^{∞}, π, and π^{∞} are denoted by f, $\Pi(f)$, $\Pi^{\infty}(f)$, $\rho(f)$, $\rho^{\infty}(f)$, $\pi(f)$, and $\pi^{\infty}(f)$, respectively.

In order to define our relative homotopy groups we need a triple that will play the role of a triple $(D^n, S^{n-1}, *)$, consisting of the unit n-disc D^n, its boundary $(n-1)$-sphere S^{n-1} and a point $* \in S^{n-1}$, in the ordinary homotopy theory.

Two most useful choices for such a triple are $(\underline{D}^n, \underline{S}^{n-1}, \underline{*}) = (D^n \times [1, \infty), S^{n-1} \times [1, \infty), \{*\} \times [1, \infty))$ and its subtriple $(\underline{\underline{D}}^n, \underline{\underline{S}}^{n-1}, \underline{*}) = ((\bigcup_{k=1}^{\infty} D^n \times \{k\}) \cup \underline{*}, (\bigcup_{k=1}^{\infty} S^{n-1} \times \{k\}) \cup \underline{*}, \underline{*})$.

These two possibilities imply that we can introduce twelve types of relative homotopy groups for a pair (K, L) of connected, locally compact, separable metric ANR spaces and a proper map $a: \underline{*} \to L$ as follows. Let $P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ denote the collection of all proper maps $f: (\underline{D}^n, \underline{S}^{n-1}, \underline{*}) \to (K, L, a(\underline{*}))$ which agree with a on $\underline{*}$. The set $P_a(\underline{\underline{D}}^n, \underline{\underline{S}}^{n-1}; K, L)$ is defined similarly. Now, for any $R \in \{\Pi, \Pi^{\infty}, \rho, \rho^{\infty}, \pi, \pi^{\infty}\}$, let $\underline{R}_n(K, L, a)$ be the set of all R-classes of maps in $P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ and let $\underline{\underline{R}}_n(K, L, a)$ denote the set of all R-classes of maps in $P_a(\underline{\underline{D}}^n, \underline{\underline{S}}^{n-1}; K, L)$.

When $n \geqq 2$, we can define group operations in $\underline{R}_n(K, L, a)$ and $\underline{\underline{R}}_n(K, L, a)$ coordinatewise. More precisely, let $\nu^n: (D^n, S^{n-1}, *) \to (D^n, S^{n-1}, *) \vee (D^n, S^{n-1}, *)$ be a fixed continuous comultiplication mapping of the homotopy cogroup $(D^n, S^{n-1}, *)$. Then the element $R(f) + R(g)$ in $\underline{\underline{R}}_n(K, L, a)$ is represented by a proper map $h: (\underline{\underline{D}}^n, \underline{\underline{S}}^{n-1}, \underline{*}) \to (K, L, a(\underline{*}))$ where $h | \underline{*} = a$ and for each $k \in N$ (N

denotes the natural numbers), $h|(D^n, S^{n-1}, *)\times\{k\}=(f|(D^n, S^{n-1}, *)\times\{k\}$, $g|(D^n, S^{n-1}, *)\times\{k\})\circ(\nu^n\times id_{\{k\}})\colon (D^n, S^{n-1}, *)\times\{k\}\to(K, L, a(k))$. The element $R(f)+R(g)$ in $\underline{\underline{R}}_n(K, L, a)$ is similarly represented by a proper map $H\colon(\underline{\underline{D}}^n, \underline{\underline{S}}^{n-1}, \underline{*}) \to(K, L, a(\underline{*}))$ where $H|\underline{*}=a$ and for each $t\in[1, \infty)$, $H|(D^n, S^{n-1}, *)\times\{t\}=(f|(D^n, S^{n-1}, *)\times\{t\}, g|(D^n, S^{n-1}, *)\times\{t\})\circ(\nu^n\times id_{\{t\}})$. It is easy to check that in these definitions $R(f)+R(g)$ depends only on $R(f)$ and $R(g)$ and not on the choice of proper maps f and g and that with this multiplications $\underline{R}_n(K, L, a)$ and $\underline{\underline{R}}_n(K, L, a)$ are groups (and abelian for $n\geqq 3$).

Note that groups $\underline{R}_n(K, L, a)$ and $\underline{\underline{R}}_n(K, L, a)$ for $R\in\{\Pi^\infty, \rho^\infty, \pi^\infty\}$ depend up to an isomorphism only on the germ $\underline{a}$ of a. Moreover, for those R, $\underline{R}_n(K, L, a)$ depends up to an isomorphism only on the end of L determined by the map a [1]. On the other hand, groups $\underline{\underline{R}}_n(K, L, a)$ and $\underline{\underline{R}}_n(K, L, b)$ (and, similarly, groups $\overleftarrow{\pi}_n(K, L, a)$ and $\overleftarrow{\pi}_n(K, L, b)$ defined below) need not be isomorphic even though a and b might represent the same end of L (the complement in the Hilbert cube Q of a Z-set copy of a space X constructed by Keesling [15] gives an example of this phenomenon).

We have defined twelve groups but there are only at most ten different groups among them because by (2.2) below groups $\underline{\pi}_n(K, L, a)$ and $\underline{\pi}_n^\infty(K, L, a)$, are identical with groups $\underline{\rho}_n(K, L, a)$ and $\underline{\rho}_n^\infty(K, L, a)$, respectively.

PROPOSITION 2.2. *Proper maps $f, g\in P_a(\underline{\underline{D}}^n, \underline{\underline{S}}^{n-1}; K, L)$ are weakly proper homotopic (homotopic at ∞) rel $\underline{*}$ iff f and g are proper homotopic (germ proper homotopic) rel $\underline{*}$.*

PROOF. Let $C_0=\emptyset\subset C_1\subset C_2\subset\cdots$ be a sequence of compact subsets of K such that $\bigcup_{i>0} C_i=K$ and let $f\,\rho\, g$. Then there is a sequence $n(1)=1<n(2)<\cdots$ of natural numbers such that for every m, $n(i)\leqq m<n(i+1)$, there is a homotopy $H^m\colon((D^n, S^{n-1}, *)\times\{m\})\times I\to(K-C_{i-1}, L-C_{i-1})$ rel $(*, m)$ between $f|(D^n, S^{n-1}, *)\times\{m\}$ and $g|(D^n, S^{n-1}, *)\times\{m\}$ $(i=1, 2, \cdots)$. The map $H\colon(\underline{\underline{D}}^n, \underline{\underline{S}}^{n-1}, \underline{*})\times I\to(K, L, a(\underline{*}))$ which agrees with H^m on $(D^n\times\{m\})\times I$ and on each $\underline{*}\times\{t\}$ $(t\in I)$ with a is a proper homotopy rel $\underline{*}$ joining f and g. Hence, $f\,\rho\, g$ iff $f\,\pi\, g$ because we already observed that $f\,\pi\, g$ implies $f\,\rho\, g$. The proof for the other case is the same. The only difference is that in general we can not get $n(1)=1$.

REMARK 2.3. The groups $\underline{\pi}_n^\infty(K, L, a)$ and $\underline{\Pi}_n^\infty(K, L, a)$ were introduced by Brown ([2] and [3]) who called them the n-th relative proper homotopy group $\underline{\pi}_n(K, L, \underline{a})$ and the n-th relative repeated homotopy group $\pi_n(K, L, \underline{a})$ of (K, L) based at $\underline{a}$, respectively. His description of these groups is easily seen to be equivalent with ours using the homotopy extension theorem (HET) [14, p. 117]. In order to have consistent terminology, we shall call $\underline{\pi}_n^\infty(K, L, a)$ the *(n-th relative) discrete germ proper homotopy group (of (K, L) at* $\underline{a}$*),* $\underline{\Pi}_n^\infty(K, L, a)$ *the discrete germ homotopy group,* $\underline{\underline{\pi}}_n^\infty(K, L, a)$ the *(n-th relative) continuous germ proper homotopy group (of (K, L)* at *a),* $\underline{\underline{\Pi}}_n^\infty(K, L, a)$ the *continuous germ homotopy group,* and so on. However, for obvious reasons, we shall try not to use those names too often.

Motivated by Hu's methods in [12] we shall now give another description of groups $\underline{\pi}_n(K, L, a)$ and $\underline{\underline{\pi}}_n(K, L, a)$ which allows to consider these groups as (singular) relative homotopy groups of certain pointed pairs functorially associated to (K, L, a).

Let K be a connected, locally compact, separable metric ANR space and let $a : \underline{*} \to K$ be a proper map representing the end $e \in EK$ of K and let α denote the sequence $(a(*, i))_{i=1}^\infty$ in K. Let S be the subspace $\{0\} \cup \{1/n \mid n=1, 2, 3, \cdots\}$ of $I=[0, 1]$. The space of all continuous maps $s : (S, 0) \to (FK, e)$ (FK is the Freudenthal end-point compactification of K) such that $s^{-1}(EK)=0$ and the sequence $(s(1/i))_{i=1}^\infty$ is equivalent to α [1] (i.e., both sequences converge in FK to e), with the compact-open topology, will be denoted by $S(K, a)$ and called the *tangent S-space* of K at a. Similarly, the space of all continuous maps $p : (I, 0) \to (K \cup \infty, \infty)$ ($K \cup \infty$ is the one-point compactification of K) such that $p^{-1}(\infty)=0$, with the compact-open topology, will be denoted by $I(K, \infty)$ and called the *tangent I-space* of K at ∞ (see [12, p. 174]).

Let (K, L) be a pair of connected, locally compact, separable metric ANR spaces and let $a : \underline{*} \to L$ be a proper map. Then $S(L, a)$ is a subspace of $S(K, a)$ and $I(L, \infty)$ is a subspace of $I(K, \infty)$. The pointed pairs (K^*, L^*, s_a) and (K^{**}, L^{**}, p_a), where $K^*=S(K, a)$, $L^*=S(L, a)$, $s_a : (S, 0) \to (FK, e)$ is a map given by $s_a(0)=e$ and $s_a(1/i)=a(*, i)$ for $i=1, 2, 3, \cdots$, $K^{**}=I(K, \infty)$, $L^{**}=I(L, \infty)$, and $p_a : (I, 0) \to (K \cup \infty, \infty)$ is a map given by $p_a(0)=\infty$ and $p_a(t)=a(*, 1/t)$ for $t \in (0, 1]$, will be called the *tangent S-pair* and the *tangent I-pair* of (K, L, a) and denoted by $S(K, L, a)$ and $I(K, L, a)$, respectively.

Lemma 2.4. *The associations of the tangent S-pair* $S(K, L, a)$ *and the tangent I-pair* $I(K, L, a)$ *can be considered as functors from the category of (pairs of) connected, locally compact, separable metric ANR spaces with a fixed proper map of* $\underline{*}$ *and proper maps respecting those maps to the category of pointed (pairs of) topological spaces and continuous maps preserving base points.*

Proof. Obvious.

THEOREM 2.5. *The n-th (relative) discrete proper homotopy group functor* $\underline{\pi}_n$ *is naturally isomorphic to the functor* $\pi_n \circ S$ *and the n-th (relative) continuous proper homotopy group functor* $\underline{\underline{\pi}}_n$ *is naturally isomorphic to the functor* $\pi_n \circ I$.

PROOF. We shall prove that functors $\underline{\pi}_n$ and $\pi_n \circ S$ are naturally isomorpic. The proof for functors $\underline{\underline{\pi}}_n$ and $\pi_n \circ I$ is similar.

Let K, L, a, and α be as above. Let a proper map $f \in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ represent an element $\pi(f)$ of $\underline{\pi}_n(K, L, a)$. For each point $d \in D^n$ the sequence $(f(d, i))_{i=1}^{\infty}$ is an admissible sequence in K (in L if $d \in S^{n-1}$) equivalent to α [1]. Hence, in a natural way, f determines a continuous map $\varphi: (D^n, S^{n-1}, *) \to S(K, L, a)$. We associate $[\varphi] \in \pi_n S(K, L, a)$ to $\pi(f)$ and denote this correspondence by $t_{(K,L,a)}$. We shall show that $t = t_{(K,L,a)}$ is an isomorphism and that for each proper map $p: (K, L, a) \to (K', L', a')$ (i.e., $p \circ a = a'$) the diagram

(2.6)
$$\begin{tikzcd}
\underline{\pi}_n(K, L, a) \arrow[r, "p_*"] \arrow[d, "t_{(K,L,a)}"'] & \underline{\pi}_n(K', L', a') \arrow[d, "t_{(K',L',a')}"] \\
\pi_n S(K, L, a) \arrow[r, "S(p)_*"'] & \pi_n S(K', L', a')
\end{tikzcd}$$

commutes.

CLAIM 1. The function t is onto.

PROOF. Let $[\varphi] \in \pi_n S(K, L, a)$ be represented by a continuous map $\varphi: (D^n, S^{n-1}, *) \to S(K, L, a)$. Define a map $f': (D^n, S^{n-1}) \times N \to (K, L)$ by $f'(d, k) = \varphi(d)(1/k)$, for $d \in D^n$ and $k \in N$, and extend f' to a map $f: (\underline{D}^n, \underline{S}^{n-1}, \underline{*}) \to (K, L, a(\underline{*}))$ such that f and a agree on $\underline{*}$. Clearly, $t(\pi(f)) = [\varphi]$.

CLAIM 2. The function t is one-to-one.

PROOF. Let $f, g \in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ and assume that $[\varphi] = t(\pi(f))$ and $[\psi] = t(\pi(g))$ coincide. Let $\chi: (D^n, S^{n-1}, *) \times I \to S(K, L, a)$ be a homotopy joining φ and ψ. Define $H': [(D^n, S^{n-1}) \times N] \times I \to (K, L)$ by $H'(d, k, s) = \chi(d, s)(1/k)$, for $d \in D^n$, $k \in N$, and $s \in I$, and extend H' to a proper map $H: (\underline{D}^n, \underline{S}^{n-1}, \underline{*}) \times I \to (K, L, a)$ so that H_s agrees with a on $\underline{*}$, for each $s \in I$. Hence, f and g are properly homotopic rel $\underline{*}$, i.e., $\pi(f) = \pi(g)$.

CLAIM 3. The function t is a homomorphism of groups for $n \geqq 2$.

PROOF. Consider $f, g \in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$. Recall that the element $\pi(f) + \pi(g)$ in $\underline{\pi}_n(K, L, a)$ is represented by a proper map $h: (\underline{D}^n, \underline{S}^{n-1}, \underline{*}) \to (K, L, a(\underline{*}))$ where

$h|*=a$ and for each $k\in N$, $h|(D^n, S^{n-1}, *)\times\{k\}=(f|(D^n, S^{n-1}, *)\times\{k\}, g|(D^n, S^{n-1}, *)\times\{k\})\circ(\nu^n\times id_{\{k\}})$ ($\nu^n:(D^n, S^{n-1}, *)\to(D^n, S^{n-1}, *)\vee(D^n, S^{n-1}, *)$ is a fixed continuous comultiplication mapping of the homotopy cogroup $(D^n, S^{n-1}, *)$). But it is clear that a map $\chi:(D^n, S^{n-1}, *)\to S(K, L, a)$ associated by the above construction to h is just $(\varphi, \psi)\circ\nu^n:(D, S^{n-1}, *)\to S(K, L, a)$ where φ and ψ are associated to f and g, respectively. Thus $t(\pi(f)+\pi(g))=t(\pi(f))+t(\pi(g))$ and t is a homomorphism.

CLAIM 4. The diagram (2.6) is commutative.

PROOF. If $f\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$, then $p_\#(\pi(f))$ is represented by a proper map $p\circ f\in P_{a'}(\underline{D}^n, \underline{S}^{n-1}; K', L')$. Now, $t_{(K', L', a')}\circ p_\#(\pi(f))$ is represented by a continuous map $\psi:(D^n, S^{n-1}, *)\to S(K', L', a)$ defined as $\psi(d)(1/k)=p\circ f(d, k)$, $d\in D^n$, $k\in N$. We can view ψ as $S(p)_\#(\varphi)$, where $\varphi:(D^n, S^{n-1}, *)\to S(K, L, a)$ is defined by $\varphi(d)(1/k)=f(d, k)$. But, clearly, $[\varphi]=t_{(K, L, a)}(\pi(f))$. Hence, $S(p)_\# t_{(K, L, a)} = t_{(K', L', a')}\circ p_\#$.

REMARK 2.7. By the similar method one can prove that the diagram

$$\begin{array}{ccc}
\underline{\pi}_n(K, L, a) & \xrightarrow{\underline{\delta}} & \underline{\pi}_{n-1}(L, a) \\
{\scriptstyle t_{(K, L, a)}}\downarrow & & \downarrow{\scriptstyle t_{(L, a)}} \\
\pi_n S(K, L, a) & \xrightarrow{\delta} & \pi_{n-1} S(L, a)
\end{array}$$

commutes for every triplet (K, L, a) (the boundary homomorphism $\underline{\delta}$ is defined coordinatewise). Hence, homotopy systems $\mathcal{H}_S=(\underline{\pi}, \underline{\delta}, \#)$ and $\mathcal{H}'_S=(\pi\circ S, \pi\circ S, \#)$ are equivalent in the sense of [13, p. 121]. Similarly, homotopy systems $\mathcal{H}_I=(\underline{\underline{\pi}}, \underline{\underline{\delta}}, \#)$ and $\mathcal{H}'_I=(\pi\circ I, \pi\circ I, \#)$ are also equivalent. It follows from these equivalences that for every statement concerning (relative) homotopy groups there is a corresponding statement about (relative) discrete and continuous proper homotopy groups. In fact, for every homotopy invariant functor $\mathcal{F}$ on the category of (pointed) topological pairs we can define proper homotopy invariant functors $\underline{\mathcal{F}}$ and $\underline{\underline{\mathcal{F}}}$ by $\underline{\mathcal{F}}=\mathcal{F}\circ S$ and $\underline{\underline{\mathcal{F}}}=\mathcal{F}\circ I$. In particular, there are discrete proper homology and cohomology groups $\underline{H}_n$ and $\underline{H}^n$ and continuous proper homology and cohomology groups $\underline{\underline{H}}_n$ and $\underline{\underline{H}}^n$.

REMARK 2.8. Observe that the groups $\pi_n I(K, L, a)$, $H_n I(K, L, a)$, and $H^n I(K, L, a)$ are precisely Hu's (relative) local homotopy group $\lambda_n(K\cup\infty, L\cup\infty, \infty; p_a)$ of $(K\cup\infty, L\cup\infty)$ at ∞ based at the path p_a, (relative) local homology group $L_n(K\cup\infty, L\cup\infty, \infty)$ of $(K\cup\infty, L\cup\infty)$ at ∞, and (relative) local cohomology group $L^n(K\cup\infty, L\cup\infty, \infty)$ of $(K\cup\infty, L\cup\infty)$ at ∞, respectively (see [12, p. 175 and p. 199]).

We continue with a description (similar to Brown's description of groups $\underline{\pi}_n^\infty(K, L, a)$ involving the functor $\mathscr{P}$ [2], [11]) of a method for computing discrete groups of a triple (K, L, a).

Let (G_{ij}) be an infinite diagram

$$\begin{tikzcd}
G_{11} & G_{21} & G_{31} \cdots \\
G_{12} \arrow[u] & G_{22} \arrow[u] & G_{32} \cdots \arrow[u] \\
G_{13} \arrow[u] & G_{23} \arrow[u] & G_{33} \cdots \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

of groups and homomorphisms (indicated by arrows). Let $\sum(G_{ij})$ denote all sequences $(g_{i,k(i)})_{i=1}^\infty$ of elements of the above groups, $g_{i,k(i)}\in G_{i,k(i)}$, such that $k(i)\to\infty$ as $i\to\infty$. If $g=(g_{i,k(i)})$ and $g'=(g'_{i,k'(i)})$ are two sequences in $\sum(G_{ij})$, we write

$g\,\widehat{\Pi}\,g'$ iff for every $i\in N$ there is $l(i)\in N$ such that $g_{i,k(i)}^{l(i)}=(g'_{i,k'(i)})^{l(i)}$, where $g_{i,k(i)}^{l(i)}$ is the image of $g_{i,k(i)}$ in the group $G_{i,l(i)}$ under the composed vertical homomorphisms in the diagram (and, of course, $l(i)\leqq k(i)$, $k'(i)$).

$g\,\widehat{\Pi}^\infty g'$ iff there is $i_0\in N$ such that for every $i\geqq i_0$ there is an $l(i)$ satisfying the above condition.

$g\,\hat{\pi}\,g'$ iff there is a sequence $(l(i))_{i=1}^\infty$ in N converging to ∞ such that $g_{i,k(i)}^{l(i)}=(g'_{i,k'(i)})^{l(i)}$ for each $i\in N$.

$g\,\hat{\pi}^\infty g'$ iff there is an $i_0\in N$ such that for every $i\geqq i_0$ there is $l(i)\in N$ with $\lim_{i\geqq i_0} l(i)=\infty$ and $g_{i,k(i)}^{l(i)}=(g'_{i,k'(i)})^{l(i)}$.

It is easily seen that these are four equivalence relations on $\sum(G_{ij})$. The equivalence class of $g=(g_{i,k(i)})$ under $\widehat{\Pi}$ is denoted by $\widehat{\Pi}(g)$ and the set of all equivalence classes by $\widehat{\Pi}(G_{ij})$. If $\widehat{\Pi}(g)$, $\widehat{\Pi}(g')\in\widehat{\Pi}(G_{ij})$, select representatives $g=(g_{i,k(i)})$, $g'=(g'_{i,k'(i)})$ so that $k(i)=k'(i)$ for each i. Then define $\widehat{\Pi}(g)\cdot\widehat{\Pi}(g')=\widehat{\Pi}((g_{i,k(i)}\cdot g'_{i,k(i)}))$. This gives a well defined group operation on $\widehat{\Pi}(G_{ij})$. Clearly, $\widehat{\Pi}$ can be considered as a functor from the category of diagrams (G_{ij}) into the category of groups (abelian groups if all groups G_{ij} are abelian). The analogous definitions and statements hold also for the remaining three relations (functors).

(2.9) Let (K, L) be a pair of connected, locally compact, separable metric

ANR spaces, let $\{C_k\}$ be a sequence of compact subsets of K such that $C_1=\emptyset$, $C_k \subset \operatorname{Int}(C_{k+1})$, and $\bigcup_{k=1}^{\infty} C_k = K$, and let $a:\underline{*}\to L$ be a proper map. For each k, let B_k be the component of $L-(L\cap C_k)$ which contains $a(*, t)$ for all sufficiently large $t\in[1, \infty)$ and let A_k be the component of $K-C_k$ containing B_k. Choose an increasing sequence $\nu=\{1=\nu_1<\nu_2<\nu_3<\cdots\}$ such that $a(*, t)\in B_k$ for all $t\geqq \nu_{k+1}$. Let $(\pi_n)_\nu$ denote the following diagram of groups $(n\geqq 2)$.

$$
\begin{array}{cccccc}
\pi_n(K, L, a(1))\cdots & \cdots & \pi_n(K, L, a(\nu_2))\cdots & & \cdots\pi_n(K, L, a(\nu_3))\cdots \\
\uparrow & & \uparrow \quad\quad \uparrow & & \uparrow \\
* & \cdots & * \quad \pi_n(A_1, B_1, a(\nu_2))\cdots & & \cdots\pi_n(A_1, B_1, a(\nu_3))\cdots \\
\uparrow & & \uparrow \quad\quad \uparrow & & \uparrow \\
* & \cdots & * \quad\quad * \cdots & & \cdots\pi_n(A_2, B_2, a(\nu_3))\cdots \\
\uparrow & & \uparrow \quad\quad \uparrow & & \uparrow \\
\vdots & \cdots & \vdots \quad\quad \vdots \cdots & & \vdots
\end{array}
$$

with vertical homomorphisms induced by inclusions and with $*$ replacing a group which agrees with the one above it.

THEOREM 2.10. *For each $R\in\{\Pi, \Pi^\infty, \pi, \pi^\infty\}$, $\hat{R}((\pi_n)_\nu)$ is naturally isomorphic to $\underline{R}_n(K, L, a)$.*

PROOF. We shall consider only the case $R=\pi$. The proof for the other groups is similar. We shall also not discuss the meaning of the word natural in the statement of the theorem (see [2] for some indications as to what it should mean) in order to keep the paper at reasonable lenght.

Let a proper map $f\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ represent an element $\pi(f)\in\underline{\pi}_n(K, L, a)$. For each integer i, let $k_i=k_i^f$ be the maximum of numbers k such that $(f(D^n\times\{i\}), f(S^{n-1}\times\{i\})\subset(A_k, B_k)$ and $a([i-1, \infty))\subset B_k$. Then the map $f_i= f|(D^n, S^{n-1}, *)\times\{i\} : (D^n, S^{n-1}, *)\times\{i\}\to(A_{k_i}, B_{k_i}, a(i))$ determines an element g_{i,k_i} of $\pi_n(A_{k_i}, B_{k_i}, a(i))$. Since f is a proper map, $\{(g_{i,k_i})\}\in\sum((\pi_n)_\nu)$. Let α denote a map which associates $\hat{\pi}((g_{i,k_i}))\in\hat{\pi}((\pi_n)_\nu)$ to $\pi(f)$. It is easy to check that α is an isomorphism.

There is a one more group (called the *limit group*) associated to a triple (K, L, a) that we shall need. It is defined (with the above notation) to be the inverse limit of the inverse sequence

$$\pi_n(K, L, a(1)) \longleftarrow \pi_n(A_1, B_1, a(\nu_2)) \longleftarrow \pi_n(A_2, B_2, a(\nu_3)) \longleftarrow \cdots$$

(where bonding homomorphisms are induced by inclusions followed by the change of a base point along the path $a|[\nu_{n-1}, \nu_n]$) and it is denoted $\bar{\pi}_n(K, L, a)$. The group $\bar{\pi}_n(K, L, a)$ does not depend (up to an isomorphism) on the choice of sequences $\{C_k\}$ and ν (see [2], [23], and (4.4) below).

There are some interesting homomorphisms between groups that we defined which we describe now.

With the notation from (2.9) and the proof of (2.10), for a proper map $f\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ and every $i\geqq 2$, let $f_{i-1}^*: (D^n, S^{n-1}, *)\times\{i-1\}\to(A_{k_i}, B_{k_i}, a(i-1))$ be a map representing $(a_{\#}^i)^{-1}([f_i])\in\pi_n(A_{k_i}, B_{k_i}, a(i-1))$, where $[f_i]\in\pi_n(A_{k_i}, B_{k_i}, a(i))$ is a homotopy class determined by a map f_i and $a_{\#}^i: \pi_n(A_{k_i}, B_{k_i}, a(i-1))\to\pi_n(A_{k_i}, B_{k_i}, a(i))$ is the isomorphism induced by the path $a^i=a|[i-1, i]$. The maps $\{f_i^*\}_{i=1}^{\infty}$ determine a proper map $f^*\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$. It is easy to check that for every $R\in\{\Pi, \Pi^{\infty}, \rho, \rho^{\infty}, \pi, \pi^{\infty}\}$ the map $sh_R^n: \underline{R}_n(K, L, a)\to\underline{R}_n(K, L, a)$ which maps $R(f)$ into $R(f^*)$ is well defined and is a homomorphism of groups (called the *shift operator* [2]) ($n\geqq 2$).

The second homomorphism δ_R^n (the *boundary operator*) maps $\underline{R}_n(K, L, a)$ into $\underline{R}_{n-1}(K, L, a)$ and is defined as follows. Let $f\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ represent an element $R(f)\in\underline{R}_n(K, L, a)$. Then $\delta_R^n(R(f))=R(g)\in\underline{R}_{n-1}(K, L, a)$ is represented by a map $g\in P_a(\underline{D}^{n-1}, \underline{S}^{n-2}; K, L)$ constructed in the following way. We regard each relative n-cell $(D^n, S^{n-1}, *)_i=(D^n, S^{n-1}, *)\times\{i\}$ in $(\underline{D}^n, \underline{S}^{n-1}, \underline{*})$ as the cone $C(D^{n-1})_i$ over the $(n-1)$-cell such that the intersection of $C(D^{n-1})_i$ with the ray $\underline{*}$ is precisely its vertex v_i. Let f_i' $(i\geqq 1)$ be a map of $(C(D^{n-1})_i, Bd(C(D^{n-1})_i), v_i)$ into $(K, L, a(i))$ homotopic (in (A_{k_i}, B_{k_i})) to $f_i=f|C(D^{n-1})_i$ such that f_i' maps the segment $[v_i, (*, 1)]$, where $*\in S^{n-2}\subset D^{n-1}$, linearly onto the segment $a([i, i+1])$, and it maps the base of $C(D^{n-1})_i$ into the point $a(i+1)$. Let the map $g=B(\{f_i'\})$ map the horizontal t-section of $D^{n-1}\times[i, i+1]$ in the same way as f_i' maps the t-section of $C(D^{n-1})_i$, for every $i>0$ and $t\in I$.

Finally, the homomorphism $r_R^n: \underline{R}_n(K, L, a)\to\underline{R}_n(K, L, a)$ (called the *restriction operator*) takes $R(f)$, $f\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$, into $R(f|(\underline{D}^n, \underline{S}^{n-1}, \underline{*}))$.

3. Connections with inward, approaching, and fundamental groups.

Now we shall relate Quigley's inward groups $I_n(X, A, x)$ [21] of a pointed pair of compact metric spaces (X, A, x) with discrete proper homotopy groups of a certain pair of contractible Q-manifolds and show that his approaching groups $\underline{\pi}_n(X, A, x)$ and Borsuk's fundamental groups $\underline{\pi}_n(X, A, x)$ are in a similar way related to the continuous proper homotopy groups and the limit homotopy groups, respectively, of the same pair of Q-manifolds.

Consider the pointed pair (X, A, x) of compact metric spaces as a subset of the Hilbert cube Q and let (N_i, M_i) be a decreasing sequence of compact Q-manifold neighborhoods of (X, A) in Q such that $(N_1, M_1)=(Q, Q)$ and $\bigcap_{i>0}(N_i, M_i)=(X, A)$. Let $H(X)=N_1\times\{1\}\cup\{(q, t)\mid q\in N_i,\ 1/i\leqq t\leqq 1/(i-1),\ i>0\}\cup X\times\{0\}$, and $H(A)=M_1\times\{1\}\cup\{(q, t)\mid q\in M_i,\ 1/i\leqq t\leqq 1/(i-1),\ i>0\}\cup A\times\{0\}$. Let $H(X, A)$ be the pair $(H(X), H(A))$. We shall identify (X, A, x) with the triplet $(X\times\{0\}, A\times\{0\}, \{x\}\times\{0\})$ in $H(X, A)$. Let $a:\underline{*}\rightarrow H(A)$ be defined by $a(t)=(x, 1/t)$, for $t\in\underline{*}=[1, \infty)$. Finally, denote the pair $(M(X), M(A))$ by $M(X, A)$, where $M(X)=H(X)-X$ and $M(A)=H(A)-A$. Observe that $H(X, A)$ is a pair of Hilbert cubes, $M(X, A)$ is a pair of contractible Q-manifolds, X is a Z-set in $H(X)$, A is a Z-set in $H(A)$ [8], and a is a proper map of $\underline{*}$ into $M(A)$.

REMARK 3.1. The reader familiar with the description of the strong shape category in [9] can easily check that M can be considered as a functor from the strong shape category (of pointed pairs of compacta in Q) into the proper homotopy category (of pairs of ANR's pointed by proper maps of $\underline{*}$). It can also be considered as a functor from the category Sh_{FE} (of pointed pairs of compacta in Q) constructed by Kodama and Ono [16] (which is equivalent to Mardešić-Segal shape category [19]) into the weak proper homotopy category (of pairs of ANR's pointed by proper maps of $\underline{*}$).

THEOREM 3.2. *The Quigley's relative inward group functor I_n [21], the functor $\underline{\pi}_n\circ M$, and the functor $\underline{\pi}_n^\infty\circ M$ are naturally isomorphic.*

PROOF. We shall prove that I_n and $\underline{\pi}_n\circ M$ are naturally isomorphic. The isomorphism of $\underline{\pi}_n\circ M$ and $\underline{\pi}_n^\infty\circ M$ follows from (2.1).

Consider the triple (X, A, x) as being embedded in $H(X, A)$. If $\xi:(D^n, S^{n-1}, *)\times N\rightarrow(H(X), H(A), x)$ is an inward mapping [21], let $\xi^*:(D^n, S^{n-1})\times N\rightarrow(M(X), M(A))$ be defined as follows. For a point $d\in D^n$ and a natural number $k\in N$, $\xi^*(d, k)$ is a point of $M(X)$ whose Q-coordinate is equal to the Q-coordinate of $\xi(d, k)$ and whose I-coordinate is the maximum between $1/k$ and the I-coordinate of $\xi(d, k)$. Then ξ^* extends to a proper map $\xi^{**}\in P_a(\underline{D}^n, \underline{S}^{n-1}; M(X), M(A))$. Let $\underline{t}_{(X,A,x)}([\xi])=\pi(\xi^{**})$, where $[\xi]$ is the element of $I_n(X, A, x)$ represented by ξ and $\pi(\xi^{**})$ is the element of $\underline{\pi}_n(M(X), M(A), a)=\underline{\pi}_n(M(X, A, x))$.

CLAIM 1. The function $\underline{t}=\underline{t}_{(X,A,x)}$ is onto.

PROOF. Let $f\in P_a(\underline{D}^n, \underline{S}^{n-1}; M(X), M(A))$. The map f restricted to $D^n\times N$ is an inward mapping from $(D^n, S^{n-1}, *)$ into (X, A, x) except that base points are not preserved. Hence we must modify f by shrinking $f(\underline{*})=a(\underline{*})$ to a point.

In order to do this, let $\{U_i\}_{i=1}^{\infty}$ be a decreasing sequence of open neighborhoods of x in Q such that $U_i \subset N_i$, $U_{i+1} \subset \bar{U}_{i+1} \subset U_i$ and assume also that $N_{i+1} \subset \mathrm{Int}\, N_i \subset N_i$, for each $i>0$. Now, construct a map $\lambda: H(X) \to I$ such that λ is 1 on $(N_1 - U_1) \times \{1\}$, 1/2 on $Bd(U_2) \times [1/2, 1] \cup (N_2 - U_2) \times \{1/2\}$, 1/3 on $Bd(U_3) \times [1/3, 1] \cup (N_3 - U_3) \times \{1/3\}$, and so on and is interpolated continuously in between. Observe that $\lambda^{-1}(0) = \{x\} \times I \subset H(X)$

Let a map $\tilde{f}: (D^n, S^{n-1}, *) \times N \to (H(X), H(A), x)$ has at a point $(d, k) \in D^n \times N$ as the value the point of $H(X)$ whose Q-coordinate is the Q-coordinate of $f(d, k)$ and whose I-coordinate is the minimum between $\lambda(f(d, k))$ and the I-coordinate of $f(d, k)$.

It is easy to check that $\xi = \tilde{f}$ is an inward mapping from $(D^n, S^{n-1}, *)$ into (X, A, x) and that ξ^{**} and f are proper homotopic rel $\underline{*}$. Hence, $\underline{t}([\xi]) = \pi(f)$.

CLAIM 2. The function $\underline{t}$ is one-to-one.

PROOF. Take inward maps $\xi, \eta: (D^n, S^{n-1}, *) \times N \to (H(X), H(A), x)$ and assume that ξ^{**} and η^{**} are properly homotopic rel $\underline{*}$ via a proper homotopy $H: (\underline{D}^n, \underline{S}^{n-1}, \underline{*}) \times I \to (M(X), M(A), a)$. Let $\tilde{H}$ be obtained from H by the above construction. Then $\tilde{H}_0(d, k)$, for $(d, k) \in D^n \times N$, has the Q-coordinate equal to the Q-coordinate $\nu(d, k)$ of $\xi(d, k)$ while its I-coordinate is $\min\{\max(\mu(d, k), 1/k), \lambda(\nu(d, k), \max(\mu(d, k), 1/k))\}$, where $\mu(d, k)$ is the I-coordinate of $\xi(d, k)$. Just as in the proof of Claim 1, it follows that $\tilde{H}_0$ is inward homotopic to ξ. Similarly, $\tilde{H}_1$ is inward homotopic to η. Since $\tilde{H}$ is an inward homotopy between $\tilde{H}_0$ and $\tilde{H}_1$, we get that ξ and η are inward homotopic.

CLAIM 3. For $n \geqq 2$, the function $\underline{t}$ is a homomorphism of groups.

PROOF. Let $\xi, \eta: (D^n, S^{n-1}, *) \times N \to (H(X), H(A), x)$ be inward maps from $(D^n, S^{n-1}, *)$ into (X, A, x) and suppose that $\zeta: (D^n, S^{n-1}, *) \times N \to (H(X), H(A), x)$ represents $[\xi] + [\eta]$ in $I_n(X, A, x)$. Then, for every $k \in N$, $\zeta_k = \zeta | (D^n, S^{n-1}, *) \times \{k\}$ is equal to the composition $(\xi_k, \eta_k) \circ \nu^n$. If we now construct ξ^{**}, η^{**}, and ζ^{**} we easily see that $\pi(\zeta^{**})$ represents $\pi(\xi^{**}) + \pi(\eta^{**})$ in $\underline{\pi}_n(M(X, A, x))$.

CLAIM 4. Let $f: (X, A, x) \to (Y, B, y)$ be a continuous map and let $F: (H(X), H(A), x) \to (H(Y), H(B), y)$ be an extension of f such that $F^{-1}(Y) = X$. If $[\xi] \in I_n(X, A, x)$, let $f_*([\xi])$ be the element of $I_n(Y, B, y)$ represented by $F \circ \xi$. Then the diagram

$$\begin{array}{ccc} I_n(X, A, x) & \xrightarrow{\quad f_* \quad} & I_n(Y, B, y) \\ {\scriptstyle \underline{t}(X, A, x)} \downarrow & & \downarrow {\scriptstyle \underline{t}(Y, B, y)} \\ \underline{\pi}_n(M(X), M(A), a) & \xrightarrow{(F|M(X))_*} & \underline{\pi}_n(M(Y), M(B), b) \end{array}$$

commutes.

PROOF. A simple proof is left to the reader. Observe that Claim 4 holds also when the map f is replaced by an arbitrary morphism of the strong shape category of pairs [9] (or, what is equivalent, by a morphism of the fine shape category of pairs [17]) but the description of f_* and a morphism corresponding to $F|M(X)$ above are more complicated.

REMARK 3.3. Even though Quigley did not define explicitely the boundary homomorphism $\delta: I_n(X, A, x) \to I_{n-1}(A, x)$ it is obvious how to do this. By the similar method one can prove that the diagram

$$\begin{array}{ccc} I_n(X, A, x) & \xrightarrow{\delta} & I_{n-1}(A, x) \\ {\scriptstyle \underline{t}(X,A,x)} \downarrow & & \downarrow {\scriptstyle \underline{t}(A,x)} \\ \underline{\pi}_n(M(X), M(A), a) & \xrightarrow{\underline{\delta}} & \underline{\pi}_{n-1}(M(A), a) \end{array}$$

commutes for every triple (X, A, x) and $n>0$. This implies that homotopy systems $(I, \delta, *)$ and $(\underline{\pi} \circ M, \underline{\delta}, *)$ are equivalent [13].

In particular, since the long sequences of discrete proper homotopy groups of the pair $(M(X), M(A), a)$ and the triplet $(M(X), M(A), M(B), a)$, where $x \in B$ and $B \subset A \subset X$, are exact (see (2.7)), we immediately get.

COROLLARY 3.4. *The long inward sequences*

$$\cdots \to I_{n+1}(X, A, x) \xrightarrow{\delta} I_n(A, x) \xrightarrow{i} I_n(X, x) \xrightarrow{j} I_n(X, A, x) \xrightarrow{\delta} I_{n-1}(A, x) \to \cdots$$

and

$$\cdots \to I_{n+1}(X, A) \xrightarrow{\delta'} I_n(A, B) \xrightarrow{i'} I_n(X, B) \xrightarrow{j'} I_n(X, A) \xrightarrow{\delta'} I_{n-1}(A, B) \to \cdots$$

are exact.

REMARK 3.5. By combining Theorems (2.4) and (3.2) we see that the inward group $I_n(X, A, x)$ can be in a natural way considered as the (singular) relative homotopy group of $SM(X, A, x)$, the tangent S-pair of the triplet $(M(X), M(A), a)$.

Let $J_n(X, A, x)$ denote Quigley's n-th relative approaching group of (X, A, x) [21] (his notation for this group was $\underline{\pi}_n(X, A, x)$ and thus is not convenient here).

THEOREM 3.6. *The functors J_n, $\underline{\pi}_n \circ M$, and $\underline{\pi}_n^\infty \circ M$ are naturally isomorphic. Moreover, homotopy systems $(J, \delta, *)$ and $(\underline{\pi} \circ M, \underline{\delta}, *)$ are equivalent in the sense of* [13, *p.* 121].

PROOF. It follows from (2.1) that $\underline{\pi}_n M$ and $\underline{\pi}_n^\infty M$ are naturally isomorphic. We shall now only indicate the proof that J_n and $\underline{\pi}_n \circ M$ are naturally isomorphic because the details are either easy to check or are very similar to the ones used in the proof of (3.2).

Let $K(X)$ be the space obtained from $H(X)$ by shrinking X to a point and let $p: H(X) \to K(X)$ be the natural projection. Let $K(X, A, x)$ denote the quadruple $(K(X), K(A), x_0; p_a)$ where $K(A) = p(H(A))$, $x_0 = p(X)$, and $p_a: I \to K(A)$ is a path defined by $p_a(0) = x_0$ and $p_a(t) = p \circ a(1/t)$ for $t \in (0, 1]$. Recall (see (2.8)) that the group $\underline{\pi}_n(M(X), M(A), a)$ is isomorphic to Hu's relative local homotopy group $\lambda_n(K(X), K(A), x_0; p_a)$ of $(K(X), K(A))$ at x_0 based at the path p_a [12].

Let $\xi: (\underline{D}^n, \underline{S}^{n-1}, \underline{*}) \to (H(X), H(X), x)$ be an approaching n-map [21]. By definition and the fact that $H(A)$ is an ANR, we can assume that there is an $r \in \underline{*}$ such that $\xi(S^{n-1} \times [r, \infty)) \subset H(A)$. Define a new approaching n-map ξ' by $\xi'(d, t) = \xi(d, t+r)$, for each $(d, t) \in D^n \times [1, \infty)$. Then ξ' is approaching homotopic to ξ (via the approaching homotopy which sends $(d, t, s) \in D^n \times [1, \infty) \times I$ into $\xi(d, t+sr)$). Hence, we could assume from the begining that ξ maps $\underline{S}^{n-1}$ into $H(A)$.

From ξ we construct a map $\xi^* \in P_a(\underline{D}^n, \underline{S}^{n-1}; M(X), M(A))$ as follows. For $d \in D^n$ and $t \in [1, \infty)$, $\xi(d, t)$ is a point in $H(X)$ with the Q-coordinate $\nu(d, t)$ and the I-coordinate $\mu(d, t)$. Let $\xi^*(d, t)$ be a point of $H(X)$ whose Q-coordinate is $\nu(d, t)$ and whose I-coordinate is $\max(\mu(d, t), 1/t)$.

The composition $p \circ \xi^* \in P_{p \circ a}(\underline{D}^n, \underline{S}^{n-1}; K(X) - \{x_0\}, K(A) - \{x_0\})$ induces an admissible map [12] $\xi^{**}: ([D^n \times (0, 1]] \cup \infty, [S^{n-1} \times (0, 1]] \cup \infty, \infty) \to (K(X), K(A), x_0)$ such that $\xi^{**}(*, t) = p \circ a(*, 1/t) = p_a(t)$ for each $t \in (0, 1]$. We define $\underline{t}([\xi]) = \underline{t}_{(X, A, x)}([\xi])$ to be $[\xi^{**}]$, where $[\xi]$ is the element of $J_n(X, A, x)$ represented by ξ and $[\xi^{**}]$ is the element of $\lambda_n(K(X), K(A), x_0; p_a)$ represented by ξ^{**} [12].

The necessary properties of $\underline{t}$ are now verified in much the same way as this was done in the proof of (3.2) for t. The only differences are that instead of natural numbers we use the ray $[1, \infty)$ and that from $H(X)$ we must pass to

$K(X)$ by composing with the projection p.

REMARK 3.7. Since the groups $\underline{\underline{\pi}}_n(M(X), M(A), a)$ and $\pi_n(I(M(X), M(A), a))$ are naturally isomorphic (by (2.4)), it follows that the relative approaching group $J_n(X, A, x)$ can also be in a natural way considered as the (singular) relative homotopy group of $IM(X, A, x)$, the tangent I-pair of the triplet $(M(X), M(A), a)$. Hence, for every statement about (relative) homotopy groups there is an analogous statement about (relative) approaching groups. The corollaries below ((3.8) was first proved in [21]) are trivial applications of this correspondence.

COROLLARY 3.8. *The long approaching sequence*

$$\cdots \to J_{n+1}(X, A, x) \xrightarrow{\delta} J_n(A, x) \xrightarrow{i} J_n(X, x) \xrightarrow{j} J_n(X, A, x) \xrightarrow{\delta} J_{n-1}(A, x) \to \cdots$$

of the pointed pair (X, A, x) *of compacta is exact.*

COROLLARY 3.9. *The long approaching sequence*

$$\cdots \to J_{n+1}(X, A) \xrightarrow{\delta'} J_n(A, B) \xrightarrow{i'} J_n(X, B) \xrightarrow{j'} J_n(X, A) \xrightarrow{\delta'} J_{n-1}(A, B) \to \cdots$$

of the triple (X, A, B) *is exact.*

Let $F_n(X, A, x)$ denote the n-th relative fundamental group of (X, A, x) [21] or, equivalently, the inverse limit group of the inverse system

$$\pi_n(N_1, M_1, x) \longleftarrow \pi_n(N_2, M_2, x) \longleftarrow \pi_n(N_3, M_3, x) \longleftarrow \cdots$$

with bonding homomorphisms induced by inclusions. This group is usually denoted $\underline{\pi}_n(X, A, x)$ and also called the n-th relative shape group of (X, A, x) but in our paper that notation might be confusing.

THEOREM 3.10. *The functors* F_n *and* $\overleftarrow{\pi}_n \circ M$ *are naturally isomorphic. Moreover, homotopy systems* $(F, \delta, *)$ *and* $(\overleftarrow{\pi} \circ M, \overleftarrow{\delta}, *)$ *are equivalent.*

PROOF. A routine proof is left to the reader.

4. The exact sequence.

In this section we shall show that for $R=\pi$ and $R=\pi^\infty$ and every triple (K, L, a), the five terms sequence of groups and homomorphisms $(n \geqq 3)$

$$0 \longrightarrow \ker(Id\text{-}sh_R^n) \xrightarrow{i_R^n} \underline{R}_n(K, L, a) \xrightarrow{Id\text{-}sh_R^n} \underline{R}_n(K, L, a)$$
$$\xrightarrow{\delta_R^n} \underline{R}_{n-1}(K, L, a) \xrightarrow{r_R^{n-1}} \operatorname{Im} r_R^{n-1} \longrightarrow 0$$

is exact. Then we shall prove that groups $\operatorname{Im} r_R^n$, $\ker(Id\text{-}sh_R^n)$, and $\tilde{\pi}_n(K, L, a)$ are isomorphic for all $n \geqq 3$. In view of the results in the previous section, it follows that Quigley's exact sequence from the n-th to the $(n-1)$-st relative fundamental group of a pointed pair of compacta [21] can be regarded as the special case of the above sequence.

Let Id denote the identity homomorphism on the group $\underline{R}_n(K, L, a)$ and let i_R^n be the inclusion of the kernel of $Id\text{-}sh_R^n$ into $\underline{R}_n(K, L, a)$.

THEOREM 4.1. *For $R \in \{\pi, \pi^\infty\}$, every pair (K, L) of connected, locally compact, separable metric ANR spaces, every proper map $a: \underline{*} \to L$, and every $n \geqq 3$, the above sequence of groups and homomorphisms is exact.*

PROOF. We must show that the sequence is exact at the third and at the fourth group. The exactness at other terms is obvious.

CLAIM 1. $\ker(\delta_R^n) = \operatorname{Im}(Id\text{-}sh_R^n)$.

PROOF. Let $R(f) \in \underline{R}_n(K, L, a)$. We shall show that if $R(g) = R(f)\text{-}sh_R^n(R(f)) = R(f) - R(f^*)$ then $\delta_R^n(R(g)) = 0 \in \underline{R}_{n-1}(K, L, a)$. We can assume that $R(g)$ is represented by a proper map $g' \in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ such that the restriction g'_m of g' onto $C(D^{n-1})_m$ has the property that g'_m restricted to the lower half of $C(D^{n-1})_m$ is in the homotopy class of f_m, g'_m restricted to the upper half of $C(D^{n-1})_m$ is in the homotopy class $-[f_{m+1}]$ (with the base point moved to $a(m)$), g'_m maps the middle section of $C(D^{n-1})_m$ into the point $a(m+(1/2))$, and g'_m maps the segment $[v_m, (*, 1)]$ linearly onto the segment $a([m, m+1])$, for every $m>0$. Observe that for each $m>1$, $B(\{g'_i\})|D^{n-1}\times[m-(1/2), m+(1/2)]$ is null-homotopic in the complement of any compact set in K provided m is large enough. By applying the HET trice we see that $B(\{g'_i\})|D^{n-1}\times[m-(1/2), m+(1/2)]$ is homotopic rel $a([m-(1/2), m+(1/2)])$ to a map of $D^{n-1}\times[m-(1/2), m+(1/2)]$ which maps a pair $(x, t) \in D^{n-1}\times[m-(1/2), m+(1/2)]$ into the point $a(t)$, and that these homotopies for $B(\{g'_i\})|D^{n-1}\times[m-(1/2), m+(1/2)]$ and $B(\{g'_i\})|D^{n-1}\times[m+(1/2), m+(3/2)]$ agree on $D^{n-1}\times\{m+(1/2)\}$, for every $m>1$. Piecing these homotopies together we see that $\delta_\pi^n(\pi(g)) = \pi(B(\{g'_i\})) = 0 \in \underline{\pi}_{n-1}(K, L, a)$. But then $\delta_R^n(R(g)) = R(B(\{g'_i\})) = 0$ also for $R = \pi^\infty$.

Conversely, let $R(f) \in \ker(\delta_R^n)$. Then $R(B(\{f'_i\})) = R(c)$, where $c \in P_a(\underline{\underline{D}}^{n-1}, \underline{\underline{S}}^{n-2}; K, L)$ maps $(d, t) \in D^{n-1}\times[1, \infty)$ into $a(t)$. Hence, there is a natural number $\lambda = \lambda_R$ and a proper map $H: (D^{n-1}, S^{n-2}, *)\times[\lambda, \infty)\times I \to (K, L)$ such that $\lambda_\pi = 1$, $H_0 = B(\{f'_i\})|D^{n-1}\times[\lambda, \infty)$, $H_1 = c|D^{n-1}\times[\lambda, \infty)$, and $H_t(*, s) = a(*, s)$ for all $t \in I$ and $s \geqq \lambda$. For every integer $m \geqq \lambda$, the restriction $H|D^{n-1}\times[m, m+1]\times\{0\}\cup$

$D^{n-1}\times\{m, m+1\}\times I$ can be considered as representing the sum $-[g_m]+[f_m]+i_{m+1,m}([g_{m+1}])$, where $g_m:(D^n, S^{n-1}, *)\to(K, L, a(m))$ and $g_{m+1}:(D^n, S^{n-1}, *)\to(K, L, a(m+1))$ have images outside larger and larger compact subsets of K as m increases, while $i_{m+1,m}:\pi_n(K, L, a(m+1))\to\pi_n(K, L, a(m))$ is an isomorphism induced by the path a from $a(m+1)$ to $a(m)$ (in fact, we must consider these isomorphisms for complements of above compacta in K). Since $H|D^{n-1}\times[m, m+1]\times\{0\}\cup D^{n-1}\times\{m, m+1\}\times I$ extends to all of $D^{n-1}\times[m, m+1]\times I$ and this extension agrees with c on $D^{n-1}\times[m, m+1]\times\{1\}$, we conclude that $-[g_m]+[f_m]+i_{m+1,m}([g_{m+1}])=0$. Hence $[f_m]=[g_m]-i_{m+1,m}([g_{m+1}])$. From this it follows that $R(f)=(Id\text{-}sh_R^n)(R(g))$, where $g\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ is any proper map which agrees with g_m on $D^n\times\{m\}$ for every integer $m\geqq\lambda$.

CLAIM 2. $\ker(r_R^{n-1})=\operatorname{Im}(\delta_R^n)$.

PROOF. The inclusion $\operatorname{Im}(\delta_R^n)\subset\ker(r_R^{n-1})$ is clear from the construction of δ_R^n. On the other hand, if $R(g)\in\ker(r_R^{n-1})$, then $r_R^{n-1}(R(g))=0$, i.e., there is a representative $g\in P_a(\underline{D}^{n-1}, \underline{S}^{n-2}; K, L)$ of $R(g)$ and a natural number $\lambda=\lambda_{R,g}$ such that $\lambda_{\pi,g}=1$ and $g|(D^{n-1}\times[\lambda, \infty))\cap\underline{D}^{n-1}$ is proper homotopic rel $\{*\}\times[\lambda, \infty)$ to $c|(D^{n-1}\times[\lambda, \infty))\cap\underline{D}^{n-1}$. By applying the proper homotopy extension theorem [9] we see that $R(g)$ is represented also by a proper map $f\in P_a(\underline{D}^{n-1}, \underline{S}^{n-2}; K, L)$ such that $f(D^{n-1}\times\{m\})=a(m)$ for every $m\in N$. Clearly, $R(f)\in\operatorname{Im}(\delta_R^n)$ so that $R(g)\in\operatorname{Im}(\delta_R^n)$ because $R(f)=R(g)$.

REMARK 4.2. The above sequence is well defined also for $n=2$. In this case $Id\text{-}sh_R^2$, δ_R^2, and r_R^1 are merely functions of pointed sets. The proof of (4.1) can be applied to show that in this case the sequence is also exact.

THEOREM 4.3. *For $R\in\{\rho, \rho^\infty, \pi, \pi^\infty\}$, every pair (K, L) of connected, locally compact, separable metric ANR spaces, every proper map $a:\underline{*}\to L$, and every $n\geqq3$, the subgroups* $\ker(Id\text{-}sh_R^n)$ *and* $\operatorname{Im}(r_R^n)$ *of $\underline{R}_n((K, L, a)$ coincide and are naturally isomorphic to $\check{\pi}_n(K, L, a)$.*

PROOF. We shall prove that $\ker(Id\text{-}sh_R^n)=\operatorname{Im}(r_R^n)$ and that $\operatorname{Im}(r_R^n)$ is isomorphic to $\check{\pi}(K, L, a)$.

CLAIM 1. $\operatorname{Im}(r_R^n)\subset\ker(Id\text{-}sh_R^n)$.

PROOF. Let $g\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ represent an element $R(g)\in\operatorname{Im}(r_R^n)$. Then there is a proper map $F\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ such that $g\,R\,f$ where $f=F|\underline{D}^n$. Let us first consider the most complicated case when $R=\rho$. Let $C_0=\emptyset\subset C_1\subset C_2\subset\cdots$

be an increasing sequence of compacta in K such that $K=\bigcup_{i=0}^{\infty} C_i$. Then there is a sequence $m_0=1<m_1<m_2<\cdots$ of natural numbers such that for every m, $m_{i-1}\leqq m<m_i$, there is a homotopy $H^m:(D^n, S^{n-1}, *)\times\{m\}\to(K-C_{i-1}, L-C_{i-1}, a(m))$ between f_m and g_m $(i=1, 2, 3, \cdots)$. It is clear that the collection of all homotopies $H^1, H^2, \cdots$ defines a proper homotopy between f and g. By applying the proper homotopy extension theorem [9] it follows that there is a proper map $G\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ with $g=G|\underline{D}^n$. Hence, $R(g)\in\ker(Id\text{-}sh_\rho^n)$. In the case $R=\rho^\infty$ we don't have homotopies $H^1, \cdots, H^{m_1-1}$ so that we can only conclude that there is a proper map $G\in P_{a|[m_1,\infty)}(D^n\times[m_1, \infty), S^{n-1}\times[m_1, \infty); K, L)$ such that $g|(D^n\times[m_1, \infty))\cap\underline{D}^n=G|(D^n\times[m_1, \infty))\cap\underline{D}^n$. This is however sufficient to get $\rho^\infty(g)\in\ker(Id\text{-}sh_{\rho^\infty}^n)$. The remaining cases $R=\pi$ and $R=\pi^\infty$ are handled similarly.

CLAIM 2. $\ker(Id\text{-}sh_R^n)\subset\mathrm{Im}(r_R^n)$.

PROOF. We shall assume $R=\rho$ and leave the other easier cases to the reader. Let $\rho(f)\in\ker(Id\text{-}sh_\rho^n)$. Then $\rho(f)=sh_\rho^n(\rho(f))$. Using the method of the proof of Claim 1, we see that every representative f of $\rho(f)$ is proper homotopic to a proper map $f'\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ such that $[f'_i]=(a_\#^i)^{-1}([f_{i+1}])$ where $[f'_i]\in\pi_n(A_{k_i}, B_{k_i}, a(i))$, $[f_{i+1}]\in\pi_n(A_{k_i}, B_{k_i}, a(i+1))$, and $a_\#^i$ is the isomorphism induced by the path $a^i=a|[i, i+1]$ (the notation is explained in § 2) for $i=1, 2, 3, \cdots$. Hence, for every $i\in N$, there is a map $H^i:(D^n, S^{n-1}, *)\times[i, i+1]\to(K, L, a([i, i+1]))$ such that $H^i|D^n\times\{i\}=f_i$, $H^i|D^n\times\{i+1\}=f_{i+1}$, $H^i(*, t)=a(*, t)$ for all $t\in[i, i+1]$, and the image of H^i is in A_{n_i} where $n_i\to\infty$ as $i\to\infty$. By glueing all maps H^i together we shall get a proper map $F\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ satisfying $F|\underline{D}^n=f$. In other words, $\rho(f)\in\mathrm{Im}(r_\rho^n)$.

CLAIM 3. The groups $\mathrm{Im}(r_R^n)$ and $\bar{\pi}_n(K, L, a)$ are isomorphic.

PROOF. If $R(g)\in\mathrm{Im}(r_R^n)$ is represented by a proper map $g\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$, then there is a proper map $F\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ such that $g\,R\,f$ where $f=F|\underline{D}^n$. With the notation from (2.9) and the proof of (2.10), select a sequence $m_1=1<m_2<m_3<\cdots$ of natural numbers such that $(F(D^n\times[\nu_{m_i}, \nu_{m_{i+1}}]), F(S^{n-1}\times[\nu_{m_i}, \nu_{m_{i+1}}]))\subset(A_i, B_i)$ and $m_i\geqq i+1$ for each $i=1, 2, 3, \cdots$. Hence, the map $f_{\nu_{m_i}}$ when shifted into the point $a(\nu_{i+1})$ determines an element of $\pi_n(A_i, B_i, a(\nu_{i+1}))$ while the collection of all maps $f_{\nu_{m_i}}$ determines an element $\{f_{\nu_{m_i}}\}$ of $\bar{\pi}_n(K, L, a)$. The function α which associates $\{f_{\nu_{m_i}}\}$ to $R(g)$ is easily seen to be well defined and is an isomorphism of $\mathrm{Im}(r_R^n)$ onto $\bar{\pi}_n(K, L, a)$.

REMARK 4.4. As in (4.2) observe that versions of (4.3) for $n=1$ and $n=2$ also hold.

By combining (4.1) and (4.3) we get the following.

COROLLARY 4.5. *Under the assumptions of* (4.1), *the sequence of groups and homomorphisms*

$$0 \longrightarrow \overleftarrow{\pi}_n(K, L, a) \xrightarrow{j_R^n} \underline{R}_n(K, L, a) \xrightarrow{Id\text{-}sh_R^n} \underline{R}_n(K, L, a) \xrightarrow{\delta_R^n} \underline{\underline{R}}_{n-1}(K, L, a)$$
$$\xrightarrow{q_R^{n-1}} \overleftarrow{\pi}_{n-1}(K, L, a) \longrightarrow 0$$

is exact, where j_R^n *and* q_R^n *are obtained from* i_R^n *and* r_R^n, *respectively, composing with appropriate isomorphisms from* (4.3).

The routine application of techniques in § 3 implies that the following theorem holds. It provides an alternative proof of a main result in [20] and shows that Quigley's theorem is a consequence of Corollary (4.5).

THEOREM 4.6. *With the notation from the section* 3, *the ladder*

$$\begin{tikzcd}
0 \arrow[r] & F_n(X, A, x) \arrow[r, "i_n"] \arrow[d, "\overleftarrow{t}"] & I_n(X, A, x) \arrow[r, "Id_n - A_n"] \arrow[d, "\underline{t}"] & I_n(X, A, x) \arrow[r] \arrow[d, "\underline{t}"] & {} \\
0 \arrow[r] & \overleftarrow{\pi}_n(M(X), M(A), a) \arrow[r, "j_\pi^n"] & \underline{\pi}_n(M(X), M(A), a) \arrow[r, "Id - sh_\pi^n"] & \underline{\pi}_n(M(X), M(A), a) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\delta_n"] & J_{n-1}(X, A, x) \arrow[r, "\gamma_{n-1}"] \arrow[d, "\underline{\underline{t}}"] & F_{n-1}(X, A, x) \arrow[r] \arrow[d, "\overleftarrow{t}"] & 0 \\
{} \arrow[r, "\delta_\pi^n"] & \underline{\underline{\pi}}_{n-1}(M(X), M(A), a) \arrow[r, "q_\pi^{n-1}"] & \overleftarrow{\pi}_{n-1}(M(X), M(A), a) \arrow[r] & 0
\end{tikzcd}$$

commutes.

In the above ladder the upper horizontal sequence is the sequence (0.2) from [21] and the vertical arrows are (natural) isomorphisms from theorems (3.2), (3.6), and (3.10). The definition of the natural isomorphism $\overleftarrow{t}=\overleftarrow{t}_{(X,A,x)}$ in (3.10) is obvious provided we recall that X is a Z-set in the Hilbert cube $H(X)$ [8].

The homomorphisms appearing in the sequence in (4.5) are natural so that the reader will have no difficulty to prove the following.

THEOREM 4.7. *Let* (K, L) *and* (K', L') *be two pairs of connected, locally com-*

pact, separable metric ANR's and let $a:\underline{*}\to L$ *and* $a':\underline{*}\to L'$ *be proper maps. If* $f:(K, L)\to(K', L')$ *is a proper map satisfying* $f\circ a=a'$, $n\geqq 3$, *and* $R\in\{\pi, \pi^\infty\}$, *then the ladder*

$$\begin{array}{ccccccc}
0 \longrightarrow & \check{\pi}_n(K, L, a) & \xrightarrow{j_R^n} & \underline{R}_n(K, L, a) & \xrightarrow{Id-sh_R^n} & \underline{R}_n(K, L, a) & \longrightarrow \\
& \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\
0 \longrightarrow & \check{\pi}_n(K', L', a') & \xrightarrow{j_R^n} & \underline{R}_n(K', L', a') & \xrightarrow{Id-sh_R^n} & \underline{R}_n(K', L', a') & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\delta_R^n} & \underline{R}_{n-1}(K, L, a) & \xrightarrow{q_R^{n-1}} & \check{\pi}_{n-1}(K, L, a) & \longrightarrow 0 \\
& \downarrow f_* & & \downarrow f_* & \\
\xrightarrow{\delta_R^n} & \underline{R}_{n-1}(K', L', a') & \xrightarrow{q_R^{n-1}} & \check{\pi}_{n-1}(K', L', a') & \longrightarrow 0,
\end{array}$$

in which the vertical homomorphisms are induced by f, *commutes.*

REMARK 4.8. The ladders in (4.6) and (4.7) are well defined also for $n=2$. Using remarks (4.2) and (4.4) it can be proved that in this case too they commute.

5. $\{(D^n, S^{n-1}, *)\}$-movable at ∞ pairs.

This last section introduces the notion of a $\mathcal{C}_{p0}$-movable at ∞ triplet (K, L, a), where $\mathcal{C}_{p0}$ is a class of pointed pairs of topological spaces, (K, L) is a pair of connected, locally compact, separable metric spaces, and $a:\underline{*}\to L$ is a proper map. We prove that for a $\{(D^n, S^{n-1}, *)\}$-movable at ∞ triplet (K, L, a) (of ANR's) and $R\in\{\pi, \pi^\infty\}$ the function $Id\text{-}sh_R^n$: $\underline{R}_n(K, L, a)\to\underline{R}_n(K, L, a)$ is surjective for each $n\geqq 1$. In view of Theorem (4.6) this result extends the principal theorem in [21]. As a consequence we get that for a $\{(D^n, S^{n-1}, *)\}$-movable at ∞ triplet (K, L, a) the groups $\underline{R}_{n-1}(K, L, a)$ and $\check{\pi}_{n-1}(K, L, a)$ are isomorphic ($n\geqq 3$, $R\in\{\pi, \pi^\infty\}$). By invoking Theorems (4.6) and (5.3) below it follows that the approaching group $J_{n-1}(X, A, x)$ of a $\{(D^n, S^{n-1}, *)\}$-movable (as defined in (5.2)) pointed pair (X, A, x) of compact metric spaces and its fundamental group $F_{n-1}(X, A, x)$ are (naturally) isomorphic.

DEFINITION 5.1. Let $\mathcal{C}_{p0}$ be a class of pointed pairs $(Z, Z_0, *)$ of topological spaces, let (K, L) be a pair of locally compact separable metric spaces and let $a:\underline{*}\to L$ be a proper map. We call the triplet (K, L, a) $\mathcal{C}_{p0}$*-movable at* ∞ provided that for every compact set $B\subset K$ there is a compact set C containing B such that for every compact set D in K, $D\supset B$, and a map $f:(Z, Z_0, *)\to(K-C,$

$L-C$, $a(*)\cap(L-C)$) of $(Z, Z_0, *)$ in C_{p0} there is a homotopy $f_t:(Z, Z_0, *)\to(K-B, L-B, a(*)\cap(L-B))$ $(0\leqq t\leqq 1)$ with $f_0=f$ and $f_1(Z)\subset K-D$.

It is easy to verify that the above definition depends only on the germ of a map a.

DEFINITION 5.2. A pointed pair (X, A, x) of compact metric spaces is said to be *C_{p0}-movable*, where C_{p0} is a class of pointed pairs of topological spaces, if for some embedding of X into an ANR M the following holds. For each neighborhood (U, U') of (X, A) in M (i.e., U is a neighborhood of X in M, U' is a neighborhood of A in M, and $U'\subset U$) there is a smaller neighborhood (V, V') of (X, A) in M such that for any neighborhood (W, W') of (X, A) in M and a map $f:(Z, Z_0, *)\to(V, V', x)$ of $(Z, Z_0, *)$ in C_{p0} there is a homotopy $f_t:(Z, Z_0, *)\to(U, U', x)$ $(0\leqq t\leqq 1)$ satisfying $f_0=f$, $(f_1(Z), f_1(Z_1))\subset(W, W')$, and $f_t(*)=x$ for each $t\in I$.

It can be proved that a pointed pair $(X, A, x)$of compacta is C_{p0}-movable iff the above condition holds in every ANR space containing X.

THEOREM 5.3. *Let C_{p0} be a class of pointed pairs of compact spaces and let (X, A, x) be a pointed pair of compacta with $X\subset Q$. Then (X, A, x) is C_{p0}-movable iff the triplet $(M(X), M(A), a)$ is C_{p0}-movable at ∞.*

PROOF. Assume that $(X, A, x)\subset H(X), H(A), x)$ is C_{p0}-movable. If B is a compactum in $M(X)$, then $U=H(X)-B$ is an open neighborhood of X in $H(X)$. With the notation from § 3, select a natural number $m>1$ such that $H_m(A)=\{(q, t)\,|\,q\in M_i,\ 1/(i+1)\leqq t\leqq 1/i,\ i\geqq m\}\cup A\times\{0\}$ is contained in U. Since $H_m(A)$ is an ANR [8], using the HET it follows that there is a neighborhood U' of $H_m(A)$ in U and a homotopy $r_t: U\to U$ $(0\leqq t\leqq 1)$ such that $r_0=id_U$, $r_1(U')\subset H_m(A)$ and $r_t|H_m(A)=id$ for all $t\in I$. Then pick an open neighborhood pair (V, V') of (X, A) in $H(X)$ with respect to (U, U') an in Definition (5.2) and put $C=(H(X)-V)\cup(H(A)-V')$. Consider a map $f:(Z, Z_0, *)\to(M(X)-C, M(A)-C, a(*)\cap(M(A)-C))$ of $(Z, Z_0, *)\in C_{p0}$ and a compactum D, $D\supset B$, in $M(X)$. Let $W=H(X)-D$. Now, repeat the argument used in the construction of U' and r_t to get an open neighborhood W' of A in W and a homotopy $q_t: U\to U$ such that $q_0=id_U$, $q_t(W)\subset W$, $q_t(U')\subset U'$, and $q_t(W')\subset H_m(A)$ for all $t\in I$. Also, since $\alpha=a([a^{-1}(f(*)), \infty))\cup\{x\}$ is a Z-set arc in $M(A)-C$, the natural collapse of α onto the point x extends to a pseudo-isotopy $\lambda_t: H(X)\to H(X)$, $0\leqq t\leqq 1$, supported on V' such that $\lambda_t(\alpha)\subset\alpha$, for each $t\in I$ (see [6]). The choice of the pair (V, V') implies that there is a homotopy $f_t:(Z, Z_0, *)\to(U, U', x)$ such that $f_0=\lambda_1\circ f$ and $(f_1(Z), f_1(Z_0))\subset(W, W')$. By changing only the I-coordinate in $H(X)$, it is easy to see that there is a deformation $D_t:(H(X), H(A))\to(H(X), H(A))$ with $D_t(W)\subset W-X$, $D_t(U)\subset U-X$, $D_t(\alpha)\subset\alpha$, and $D_t=id$ on $f(Z)$, for all $0<t\leqq 1$. Let $g_t:(Z, Z_0, *)\to(U, U', \alpha)$ $(0\leqq t\leqq 1)$ denote the join of homotopies $\lambda_t\circ f$ and $r_t\circ q_t\circ f_t$. Then $D_1\circ g_t:(Z, Z_0, *)\to(M(X)-B, M(A)-B, \alpha-\{x\})$ connects f with a map $D_1\circ r_1\circ q_1\circ f_1:(Z, Z_0, *)\to(M(X)-D, M(A)-D, \alpha-\{x\})$.

Conversely, suppose that the triplet $(M(X), M(A), a)$ is $\mathcal{C}_{p0}$-movable at ∞. Let (U, U') be an open neighborhood of (X, A) in $H(X)$. For a compactum $B=(H(X)-U)\cup(H(A)-U')$ in $M(X)$, select a compactum C, $C\supset B$, as in Definition (5.1) and put $V=H(X)-C$. Then, in the same way we did it above, pick an open nighborhood V' of A in $U'\cap V$ and a homotopy $r_t: V\to V$ $(0\leqq t\leqq 1)$ such that $r_0=id$ and $r_1(V')\subset H(A)$. We claim that the pair (V, V') meets our needs. Indeed, take any open neighborhood (W, W') of (X, A) in $H(X)$ and a map $f:(Z, Z_0, *)\to(V, V', x)$ of a pointed pair $(Z, Z_0, *)$ in $\mathcal{C}_{p0}$. Let $D=(H(X)-W)\cup(H(A)-W')$ and let D_t be a deformation described above satisfying, instead of the last condition there, the conditions $D_t(H(X)-C)\subset M(X)-C$, and $D_t(V')\subset V'-A$, for all $0<t\leqq 1$. By assumption, there is a homotopy $f_t:(Z, Z_0, *)\to(M(X)-B, M(A)-B, a(*)\cap(M(A)-B))$ such that $f_0=D_1\circ r_1\circ f$ and $(f_1(Z), f_1(Z_0))\subset(M(X)-D, M(A)-D)\subset(W, W')$. The composition $\lambda_1\circ f_t$ is a homotopy in (U, U') connecting the map $\lambda_1\circ D_1\circ r_1\circ f$ with $\lambda_1\circ f_1:(Z, Z_0, *)\to(W, W', x)$. It is clear that f and $\lambda_1\circ D_1\circ r_1\circ f$ are homotopic in (U, U') rel $*$. The join of the last two homotopies shows that the pointed pair (X, A, x) is $\mathcal{C}_{p0}$-movable.

THEOREM 5.4. *Let (K, L) be a pair of connected, locally compact, separable metric ANR spaces, let $a:\underline{*}\to L$ be a proper map, let $R\in\{\rho, \rho^\infty, \pi, \pi^\infty\}$, and let $n>0$ be an integer. If the triplet (K, L, a) is $\{(D^n, S^{n-1}, *)\}$-movable at ∞, then the function $Id\text{-}sh_R^n: \underline{R}_n(K, L, a)\to\underline{R}_n(K, L, a)$ is surjective.*

PROOF. By Proposition (2.2) it suffices to consider cases $R=\pi$ and $R=\pi^\infty$. We shall prove the theorem only for $R=\pi$ because the proof for $R=\pi^\infty$ is similar.

Let $C_0=\emptyset\subset C_1\subset C_2\subset\cdots$ be an exhausting sequence of compact subsets of K such that C_{i+1} satisfies the condition from Definition (5.1) with respect to C_i, for each $i\geqq 0$. Let $\pi(f)\in\underline{\pi}_n(K, L, a)$ and take a representative $f\in P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ of $\pi(f)$. With the notation from (2.9) and the proof of (2.10), for a pair (j, i) of natural numbers, let $f_{(j,i),t}:(D^n, S^{n-1}, *)\to(A_{k_j-1}, B_{k_j-1}, a(*)\cap B_{k_j-1})$ be a homotopy with $f_{(j,i),0}=f_j=f|(D^n, S^{n-1}, *)\times\{j\}$ and $(f_{(j,i),1}(D^n), f_{(j,i),1}(S^{n-1}), f_{(j,i),1}(*))\subset(A_{k_i}, B_{k_i}, a(i))$ when $k_j>1$, and let $f_{(j,i),t}; (D^n, S^{n-1}, *)\to(K, L, a(*))$ be a homotopy with $f_{(j,i),0}=f_j$ and $f_{(j,i),1}(*)=a(i)$ when $k_j=1$.

We shall define, for each $i>0$, a continuous map $g_i:(D^n, S^{n-1}, *)\to(A_{k_i}, B_{k_i}, a(i))$

such that the obvious map $g:(\underline{D}^n, \underline{S}^{n-1}, \underline{*})\to(K, L, a)$ constructed from the sequence $\{g_i\}_{i=1}^{\infty}$ is in $P_a(\underline{D}^n, \underline{S}^{n-1}; K, L)$ and has the property that $(Id\text{-}sh_{\pi}^n)(\pi(g))=\pi(f)$.

First pick an integer $m>1$ such that $k_i>1$ for all $i\geqq m$. Let $g_m:(D^n, S^{n-1}, *)\to(K, L, a(m))$ be a constant map into $a(m)$. For $i>m$, let $g_i:(D^n, S^{n-1}, *)\to(A_{k_i}, B_{k_i}, a(i))$ represent the element $-([f_{(m,i),1}]+[f_{(m+1,i),1}]+\cdots+[f_{(i-1,i),1}])$ of $\pi_n(A_{k_i}, B_{k_i}, a(i))$, and for $1\leqq i<m$, let $g_i:(D^n, S^{n-1}, *)\to(K, L, a(i))$ represent the element $[f_{(m-1,i),1}]+[f_{(m-2,i),1}]+\cdots+[f_{(i,i),1}]$ of $\pi_n(K, L, a(i))$. The routine proof that a map g has the required properties and a necessary alternations for the case $R=\pi^{\infty}$ are left to the reader.

By combining Theorem (5.4) and Corollary (4.5), and Theorems (5.3), (5.4), and (4.6), respectively, we get the following two results. The later improves Theorem 2.18 in [21] because a pointed movable pair of compact metric spaces is $\{(D^n, S^{n-1}, *)\}$-movable, for all $n>0$, while there exist pointed pairs of compacta that are $\{(D^n, S^{n-1}, *)\}$-movable, for each $n>0$, but that are not movable (see [18]).

THEOREM 5.5. *Let (K, L) be a pair of connected, locally compact, separable metric ANR spaces, let $a:\underline{*}\to L$ be a proper map, let $R\in\{\pi, \pi^{\infty}\}$, and let $n\geqq 2$. If the triplet (K, L, a) is $\{(D^n, S^{n-1}, *)\}$-movable at ∞, then the function $q_R^n:\underline{R}_{n-1}(K, L, a)\to\check{\pi}_{n-1}(K, L, a)$ is an isomorphism of groups for $n>2$ and a bijection of pointed sets for $n=2$.*

THEOREM 5.6. *If (X, A, x) is a $\{(D^n, S^{n-1}, *)\}$-movable pointed pair of compacta, for each $n>0$, then $J_n(X, A, x)$ and $F_n(X, A, x)$ are naturally isomorphic, for all $n\geqq 0$.*

COROLLARY 5.7. *Let (K, L) be a pair of connected, locally compact, separable metric ANR spaces and let $a:\underline{*}\to L$ be a proper map. If the triplet (K, L, a) is $\{(D^n, S^{n-1}, *)\}$-movable at ∞, for each $n>0$, then the long homotopy sequence*

$$\cdots\to\check{\pi}_{n+1}(K, L, a)\xrightarrow{\delta}\check{\pi}_n(L, a)\xrightarrow{i}\check{\pi}_n(K, a)\xrightarrow{j}\check{\pi}_n(K, L, a)\xrightarrow{\delta}\check{\pi}_{n-1}(L, a)\cdots\to$$

is exact.

PROOF. Apply Theorem (5.5) and the fact that the above sequence is exact for the groups $\underline{\underline{\pi}}_n$ in the place of the groups $\check{\pi}_n$ (see (2.7)).

COROLLARY 5.8. *If (X, A, x) is a $\{(D^n, S^{n-1}, *)\}$-movable pointed pair of compacta, for each $n>0$, then the long fundamental sequence $F(X, A, x)$ of*

(X, A, x) is exact.

PROOF. Apply (3.8) and (5.6).

REMARK 5.9. Throughout the paper we treated only the relative groups. With minor changes our definitions and proofs apply also to absolute versions. We leave the necessary alternations in the statements and their proofs to the reader.

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