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A NOTE ON THE HEREDITARY PROPERTIES IN THE PRODUCT SPACE

By

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In this note we shall investigate some hereditary properties of a subspace of a product space.

Let X_{α} be a topological space for each $\alpha\in I$ and A be a subset of I. p_{A} is the projection: $\prod_{\alpha\in I}X_{\alpha}\rightarrow\prod_{\alpha\in A}X_{\alpha}$, *i.e.* $p_{A}(x)$ is the restricted function of x whose domain is A. A is co-countable if $I-A$ is countable.

The family of sets is linked if each pair of its members has non-empty intersection. The space has (K)-property (precaliber $\aleph_{1}$) (caliber $\aleph_{1}$), if any uncountable family of non-empty open subsets of X includes an uncountable subfamily which is linked (has the finite intersection property) (has non-empty intersection). [2]

THEOREM. *Let X_{α} be second-countable for $\alpha\in I$ and X be a subspace of $\prod_{\alpha\in I}X_{\alpha}$ and ϕ be one of the properties*:

1) *the countable chain condition,* 2) *(K)-property,* 3) *precaliber $\aleph_{1}$,* 4) *caliber $\aleph_{1}$,* 5) *the seperability,* 6) *the Lindelöf property.*

Then, X satisfies the hereditarily ϕ if and only if for any subspace Y of X, there exists co-countable subset A of I such that $p_{A}''Y$ satisfies ϕ.

LEMMA 1. (N. A. Sanin) [1] *Let Γ be an uncountable set of finite sets, then Γ includes an uncountable subfamily Δ which is quasi-disjoint i.e. $x\cap y\subseteq\bigcap\Delta$ for each different x and y of Δ.*

See [1] for the proof.

LEMMA 2. *Let f be a continuous function whose domain is X and X satisfies ϕ in the theorem. Then, the range of f also satisfies ϕ.*

Proof. Easy to check.

Let $\{V_{n}^{\alpha};\ n<\omega\}$ be a base of X_{α}. Then, $\{p_{A}^{-1}V_{n_{1}}^{\alpha_{1}}\times\cdots\times V_{n_{m}}^{\alpha_{m}};\ A=\{\alpha_{1},\cdots,\alpha_{m}\},\ m<\omega\}=\mathcal{V}$ is a base of $\prod_{\alpha\in I}X_{\alpha}$. The domain of the basic open set $V(=p_{A}^{-1}V_{n_{1}}^{\alpha_{1}}\times$

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$\cdots\times V^{\alpha_m}_{n_m})$ is A, which is finite, and is denoted by dom V.

LEMMA 3. *Let θ be an unconntable subfamily of $\mathcal{V}$. Then, θ includes an uncountable subfamily Φ which has the following properties:*

a) $\{\text{dom } V;\ V\in\Phi\}$ *is quasi-disjoint,*

b) $p''_A V=p''_A W$ *for each* $V,\ W\in\Phi$, *where* $A=\bigcap\{\text{dom } V;\ V\in\Phi\}$.

PROOF. By Lemma 1, θ includes an uncountable subfamily θ' such that $\{\text{dom } V;\ V\in\theta'\}$ is quasi-disjoint.

Let $A=\bigcap\{\text{dom } V;\ V\in\theta'\}$. Then, $\{p''_A V;\ V\in\theta'\}$ is countable. Hence, some uncountable subfamily Φ of θ' has the properties in the lemma.

PROOF OF THEOREM. The necessity is clear and so we shall prove the sufficiency. Suppose that X does not satisfy the hereditarily ϕ. Then, there exists a subset $\{x_\alpha;\ \alpha<\omega_1\}$ and a family $\{O_\alpha;\ \alpha<\omega_1\}$ of open subsets of X such that $x_\alpha\in O_\alpha$ for each $\alpha<\omega_1$ and

i) $x_\alpha\notin O_\beta$ for any $\beta\neq\alpha$,

ii) for any uncountable subset S of ω_1, there exists a pair $\alpha,\ \beta$ of S; $O_\alpha\cap O_\beta\cap\{x_\alpha;\ \alpha<\omega_1\}=\phi$,

iii) for any uncountable subset S of ω_1, there exists a finite subset F of S; $\bigcap_{\alpha\in F} O_\alpha\cap\{x_\alpha;\ \alpha<\omega_1\}=\phi$,

iv) for any uncountable subset S of ω_1, $\bigcap_{\alpha\in S} O_\alpha\cap\{x_\alpha;\ \alpha<\omega_1\}=\phi$,

v) $x_\alpha\notin O_\beta$ for any $\beta>\alpha$, or

vi) $x_\alpha\notin O_\beta$ for any $\beta<\alpha$, according to ϕ is 1), 2), 3), 4), 5) or 6), respectively.

We may take the above $O_\alpha(\alpha<\omega_1)$ from $\mathcal{V}$. By Lemma 3, without a loss of generality we can assume that $\{O_\alpha;\ \alpha<\omega_1\}$ satisfies the conditions a) and b) of Lemma 3.

Now, we apply the assumption and Lemma 2 to $\{x_\alpha;\ \alpha<\omega_1\}$. Then, there exists a co-countable subset A of I such that $p''_A\{x_\alpha;\ \alpha<\omega_1\}$ satisfies ϕ and $\bigcap\{\text{dom } O_\alpha;\ \alpha<\omega_1\}\cap A$ is empty. Since $\{\text{dom } O_\alpha;\ \alpha<\omega_1\}$ is quasi-disjoint, we may assume dom $O_\alpha-\bigcap\{\text{dom } O_\alpha;\ \alpha<\omega_1\}\subseteq A$ for $\alpha<\omega_1$. There exists(s)

i)′ α such that $p_A(x_\alpha)\in p''_A O_\beta\cap p''_A O_\gamma$ for some $\beta\neq\gamma$,

ii)′ an uncountable subset S of ω_1 such that $p''_A O_\alpha\cap p''_A O_\beta\cap p''_A\{x_\alpha;\ \alpha<\omega_1\}\neq\phi$ for each distinct $\alpha,\ \beta\in S$,

iii)′ an uncountable subset S of ω_1 such that $\bigcap_{\alpha\in F} p''_A O_\alpha\cap p''_A\{x_\alpha;\ \alpha<\omega_1\}\neq\phi$ for any finite $F\subseteq S$, iv)′ α and an uncountable subset S of ω_1 such that $p_A(x_\alpha)\in\bigcap_{\alpha\in S} p_A''O_\alpha$,

v)′ α such that $p_A(x_\alpha)\in p_A''O_\beta$ for some $\beta>\alpha$, or

vi)′ α such that $p_A(x_\alpha)\in p_A''O_\beta$ for some $\beta<\alpha$,

according that ϕ is 1), 2), 3), 4), 5) or 6), respectively.

By the assumption of A and the fact $x_\alpha\in O_\alpha$, $p_A(x_\alpha)\in p_A''O_\beta$ holds if and only if $x_\alpha\in O_\beta$ holds, for each α, β. So, i)′, …, or vi)′ contradicts to i), …, or vi) respectively.

Now, the proof is complete.

Since the hereditary separability is equivalent to the hereditary caliber-$\aleph_1$-property, it is a little interesting to compare the two cases 4) and 5) in the theorem.

References

[1] Juhasz, I., Cardinal Functions in Topology, Amsterdam (1971).

[2] Kunen, K. and Tall, F.D., Between Martin's Axiom and Souslin's Hypothesis, Fundamenta Mathematicae CII. 3 (1979).

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