TSUKUBA J. MATH.
Vol. 3, No. 2 (1979), 53–58

# ON A PROBLEM ABOUT SKOLEM'S PARADOX OF TAKEUTI'S VERSION

By

Masazumi HANAZAWA

## §1. Introduction.

In his paper [2], Takeuti gave a theorem as a proof-theoretical interpretation of so-called Skolem's paradox. He showed as a corollary that Gödel-Bernays set theory with the additional axioms

$$\forall X \exists x \in \omega (f(x)=X)$$

is consistent, where $f$ is a newly introduced function symbol.

Concerning it, we are interested here in the question whether we can add consistently the axiom schema

$$\vec{\forall}(\mathfrak{F}(0) \wedge \forall x(\mathfrak{F}(x) \rightarrow \mathfrak{F}(x+1)) \rightarrow \forall x \in \mathfrak{F}(x))$$ [1),2)]

bisides with an axiom $\forall x \exists y \in \omega (f(y)=x)$ to any consistent set theory.

We show in this paper that the above question is affirmative for any con sistent extension of $ZF$.[3)]

## §2. Results.

2.1. CONVENTION. In the first order predicate calculus with equality, we regard the equality axioms as logical axioms. It means that equility axioms for any language are provided always tacitly.

$f, g, h, \ldots$ denote function symbols.

$\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \ldots$ denote terms.

$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ denote formulae.

$\boldsymbol{s}$ is a unary function symbol and $\boldsymbol{s}(\mathfrak{t})$ is abbreviated by $\mathfrak{t}'$.

**seq** is a binary function symbol and **seq**$(\mathfrak{s}, \mathfrak{t})$ is abbreviated by $\mathfrak{s}^{\backprime}\mathfrak{t}$.

$P, Q, R, \ldots$ denote predicate symbols.

---

1) $\vec{\forall}\mathfrak{A}$ denotes the universal closure of a formula $\mathfrak{A}$.
2) The formula $\mathfrak{F}(0)$ contains possibly the newly introduced function symbol $f$.
3) See [1], where the rather folklore result that the question is affirmative for any consistent extension of $ZFC$ is shown.

Received May 1, 1979.

$\boldsymbol{N}$ is a unary predicate symbol.

$\boldsymbol{Iq}$ is a binary predicate symbol and $\boldsymbol{Iq}(\mathfrak{s}, \mathfrak{t})$ is abbreviated by $\mathfrak{s}<\mathfrak{t}$.

2.2. THEOREM 1. Let $\Gamma$ be a consistent theory in which the sentences of the following forms are provable:

$\boldsymbol{N}(1)\wedge\forall x(\boldsymbol{N}(x)\rightarrow\boldsymbol{N}(x'))\wedge\forall x(x'\neq 1)\wedge\forall x\forall y(x'=y'\rightarrow x=y)$.

$\vec{\forall}(\mathfrak{A}(1)\wedge\forall x(\mathfrak{A}(x)\rightarrow\mathfrak{A}(x'))\rightarrow\forall x(\boldsymbol{N}(x)\rightarrow\mathfrak{A}(x)))$ for any formula $\mathfrak{A}(x)$,

$\forall x(\boldsymbol{N}(x)\rightarrow\neg x<1\wedge 1<x'\wedge\forall y(\boldsymbol{N}(y)\rightarrow x'<y'\leftrightarrow x<y))$,

$\forall x\forall y\forall z(\boldsymbol{N}(z)\rightarrow\exists w(\forall v(\boldsymbol{N}(v)\wedge v<z\rightarrow x`v=w`v)\wedge w`z=y))$.

Then the theory $\varDelta$ which is obtained from $\Gamma$ by adjoining the following axioms is consistent:

$\forall x\exists y(\boldsymbol{N}(y)\wedge\boldsymbol{f}(y)=x)$,

$\vec{\forall}(\mathfrak{F}(1)\wedge\forall x(\mathfrak{F}(x)\rightarrow\mathfrak{F}(x'))\rightarrow\forall x(\boldsymbol{N}(x)\rightarrow\mathfrak{F}(x)))$,

where $\boldsymbol{f}$ is a function symbol not in $\Gamma$ and $\mathfrak{F}(x)$ is any formula containing the function symbol $\boldsymbol{f}$.

2.3. PROOF of Theorem 1. Let $L_\Gamma$ and $L_\varDelta$ be the languages of $\Gamma$ and $\varDelta$ respectively. $\vdash_\Gamma \mathfrak{A}$ means the formula $\mathfrak{A}$ is provable in $\Gamma$. $\vdash\mathfrak{A}$ means the formula $\mathfrak{A}$ is provable from only logical axioms. We shall use the following abbreviation:

**Cond**$(x)$ is the $L_\Gamma$-formula $\boldsymbol{N}(x`1)$,

$x[y]$ is the $L_\Gamma$-term $(x`2)`y$ [4],

$x\subset y$ is the $L_\Gamma$-formula

$$\mathbf{Cond}(x)\wedge\mathbf{Cond}(y)\wedge x`1\leqq y`1\wedge\forall z(\boldsymbol{N}(z)\wedge z<x`1\rightarrow x[z]=y[z]).$$

The letters $p$, $q$, $r$ are used as variables that range over **Cond**$(*)$: e.g. $\exists p\mathfrak{A}(p)$ denotes the formula $\exists x(\mathbf{Cond}(x)\wedge\mathfrak{A}(x))$. In the same sense, the letters $i, j, k, l, m, n$ are used as variables ranging over $\boldsymbol{N}(*)$.

For each $L_\varDelta$-formula $\mathfrak{A}$ (resp. $L_\varDelta$-term $\mathfrak{t}$) we define its rank $\rho(\mathfrak{A})$ (resp. $\rho(\mathfrak{t})$) inductively as follows:

1) $\rho(x)=0$;
2) $\rho(g(\mathfrak{t}_1, \cdots, \mathfrak{t}_\mu))=\rho(\mathfrak{t}_1)+\cdots+\rho(\mathfrak{t}_\mu)+1$;
3) $\rho(\mathfrak{s}=\mathfrak{t})=\rho(\mathfrak{s})+2\cdot\rho(\mathfrak{t})$;
4) For a predicate symbol $P$ different from $=$,
$\rho(P(\mathfrak{t}_1, \cdots, \mathfrak{t}_\mu))=\rho(\mathfrak{t}_1)+\cdots+\rho(\mathfrak{t}_\mu)+1$;
5) $\rho(\neg\mathfrak{A})=\rho(\mathfrak{A})+1$;
6) $\rho(\mathfrak{A}\wedge\mathfrak{B})=\rho(\mathfrak{A})+\rho(\mathfrak{B})+1$;
7) $\rho(\exists x\mathfrak{A})=\rho(\mathfrak{A})+1$.

For every pair of $L_\varDelta$-formula $\mathfrak{A}$ and an individual variable $u$, we define an

4) "2" stands for "1′".

$L_\Gamma$-formula denoted by $[v\Vdash\mathfrak{A}]$ by induction on the rank $\rho(\mathfrak{A})$ as follows:

1) $[v\Vdash x=y]$ is $x=y$;
2) $[v\Vdash \boldsymbol{f}(x)=y]$ is $(\boldsymbol{N}(x)\wedge x<v`1\wedge v[x]=y)\vee(\neg\boldsymbol{N}(x)\wedge y=1)$;
3) For a function symbol $g$ different from $\boldsymbol{f}$,
$[v\Vdash g(x_1,\cdots,x_\mu)=y]$ is $g(x_1,\cdots,x_\mu)=y$;
4) If $\rho(\mathfrak{t}_1)+\cdots+\rho(\mathfrak{t}_\mu)>0$, then $[v\Vdash g(\mathfrak{t}_1,\cdots,\mathfrak{t}_\mu)=x]$ is
$\exists y_1\cdots\exists y_\mu([v\Vdash g(y_1,\cdots,y_\mu)=x]\wedge[v\Vdash\mathfrak{t}_1=y_1]\wedge\cdots\wedge[v\Vdash\mathfrak{t}_\mu=y_\mu])$;
5) If $\rho(\mathfrak{t})>0$, then
$[v\Vdash\mathfrak{s}=\mathfrak{t}]$ is $\exists x([v\Vdash\mathfrak{s}=x]\wedge[v\Vdash\mathfrak{t}=x[)$;
6) For a predicate symbol $P$ different from $=$, $[v\Vdash P(\mathfrak{t}_1,\cdots,\mathfrak{t}_\mu)]$ is
$\exists x_1\cdots\exists x_\mu([v\Vdash\mathfrak{t}_1=x_1]\wedge\cdots\wedge[v\Vdash\mathfrak{t}_\mu=x_\mu]\wedge P(x_1,\cdots,x_\mu))$;
7) $[v\Vdash\neg\mathfrak{A}]$ is $\neg(\exists p\supset v)[p\Vdash\mathfrak{A}]$;
8) $[v\Vdash\mathfrak{A}\wedge\mathfrak{B}]$ is $[v\Vdash\mathfrak{A}]\wedge[v\Vdash\mathfrak{B}]$;
9) $[v\Vdash\exists x\mathfrak{A}(x)]$ is $\exists y[v\Vdash\mathfrak{A}(y)]$[5].

Lemmata 1, 2, 3 and 4 below follow from routine arguments.

LEMMA 1.

1) Except for $v$, all variables occurring free in $[v\Vdash\mathfrak{A}]$ occur free in $\mathfrak{A}$;
2) If both $x$ and $y$ are different from $v$, then
$\mathrm{Sub}_y^x[v\Vdash\mathfrak{A}]$ is $[v\Vdash\mathrm{Sub}_y^x\mathfrak{A}]$[6];
3) $\vdash_\Gamma x=y\wedge[p\Vdash\mathfrak{A}(x)]\rightarrow[p\Vdash\mathfrak{A}(y)]$;
4) $\vdash_\Gamma x_1=y_1\wedge\cdots\wedge x_\mu=y_\mu\wedge[p\Vdash\mathfrak{A}(x_1,\cdots,x_\mu)]\rightarrow[p\Vdash\mathfrak{A}(y_1,\cdots,y_\mu)]$;
5) $\vdash_\Gamma[p\Vdash\mathfrak{t}=x]\wedge[p\Vdash\mathfrak{t}=y]\rightarrow x=y$;
6) $\vdash_\Gamma[p\Vdash g(\mathfrak{t}_1,\cdots,\mathfrak{t}_\mu)=\mathfrak{s}]\longleftrightarrow$
$\exists x_1\cdots\exists x_\mu\exists y([p\Vdash\mathfrak{t}_1=x_1]\wedge\cdots\wedge[p\Vdash\mathfrak{t}_\mu=x_\mu]\wedge[p\Vdash\mathfrak{s}=y]\wedge[p\Vdash g(x_1,\cdots,x_\mu)=y])$;
7) $\vdash_\Gamma q\supset p\wedge[p\Vdash\mathfrak{A}]\rightarrow[q\Vdash\mathfrak{A}]$;
8) $\vdash_\Gamma\forall p\exists q\supset p\exists w[q\Vdash\mathfrak{t}=w]$, for each term $\mathfrak{t}$;
9) $\vdash_\Gamma[p\Vdash\mathfrak{t}=w]\rightarrow([p\Vdash\mathfrak{A}(\mathfrak{t})]\longleftrightarrow[p\Vdash\mathfrak{A}(w)])$;
10) If $\mathfrak{A}$ is an $L_\Gamma$-formula, then $\vdash\mathfrak{A}\longleftrightarrow[p\Vdash\mathfrak{A}]$.

For an $L_\Delta$-formula $\mathfrak{A}$, we shall denote by $\mathfrak{A}^*$ the $L_\Delta$-formula $\forall p\exists q\supset p\ [q\Vdash\mathfrak{A}]$.

LEMMA 2. If $\mathfrak{A}$ is an $L_\Gamma$-formula, then $\vdash_\Gamma\mathfrak{A}\longleftrightarrow\mathfrak{A}^*$.

LEMMA 3. If $\vdash\mathfrak{A}$, then $\vdash_\Gamma\mathfrak{A}^*$.

LEMMA 4. If $\vdash_\Gamma\mathfrak{A}^*$ and $\vdash_\Gamma(\mathfrak{A}\rightarrow\mathfrak{B})^*$, then $\vdash_\Gamma\mathfrak{B}^*$.

LEMMA 5. $\vdash_\Gamma(\forall x\exists y(\boldsymbol{f}(y)=x))^*$.

---

5) The variable $y$ is any variable not occurring free in $[v\Vdash\exists x\mathfrak{A}(x)]$,
6) $\mathrm{Sub}_y^x\mathfrak{A}$ means the substitution of $y$ for $x$ (occurring free) in $\mathfrak{A}$ (tacitly assuming that $y$ is free for $x$ in $\mathfrak{A}$).

PROOF. Since $\vdash_\Gamma \forall x \forall y \forall i \exists z(\forall j<i(x`j=z`j) \wedge z`i=y)$, it follows that $\vdash_\Gamma \forall x \forall p \exists q \supset p \exists i(q[i]=x)$ which implies $\vdash_\Gamma \forall p[p \Vdash \forall x \exists i(\boldsymbol{f}(i)=x)]$.

LEMMA 6. $\vdash_\Gamma (\vec{\forall}(\mathfrak{F}(1) \wedge \forall x(\mathfrak{F}(x) \to \mathfrak{F}(x')) \to \forall x(\boldsymbol{N}(x) \to \mathfrak{F}(x)))^*$, for any $L_\Delta$-formula $\mathfrak{F}(x)$.

PROOF. It suffices to show that for any $L_\Delta$-formula $\mathfrak{F}(x)$,

$$\vdash_\Gamma \vec{\forall}(\exists i \mathfrak{F}(i) \to \exists i(\mathfrak{F}(i) \wedge \forall j<i \neg \mathfrak{F}(j)))^*.$$

We explain the proof informally to save the space. In general, it holds that $\vdash_\Gamma (\forall x \mathfrak{A})^* \longleftrightarrow \forall x \mathfrak{A}^*$ and $\vdash_\Gamma (\mathfrak{A} \to \mathfrak{B})^* \longleftrightarrow \forall p([p \Vdash \mathfrak{A}] \to \exists q \supset p[q \Vdash \mathfrak{B}])$. So it is sufficient to prove $\exists q \supset p[q \Vdash \exists i(\mathfrak{F}(i) \wedge \forall j<i \neg \mathfrak{F}(j))]$ assuming $[p \Vdash \exists i \mathfrak{F}(i)]$. Suppose $[p \Vdash \exists i \mathfrak{F}(i)]$. Note that "$\exists q \supset p[q \Vdash \mathfrak{F}(i)]$" is an $L_\Gamma$-formula. Hence by means of the mathematical induction in $\Gamma$, we obtain the following in $\Gamma$:

$$\exists i(\exists q \supset p[q \Vdash \mathfrak{F}(i)]) \to \exists i(\exists q \supset p[q \Vdash \mathfrak{F}(i)] \wedge \forall j<i \neg (\exists q \supset p[q \Vdash \mathfrak{F}(j)])).$$

Since $[p \Vdash \exists i \mathfrak{F}(i)]$ implies $\exists i(\exists q \supset p[q \Vdash \mathfrak{F}(i)])$, there exist $i$ and $q$ such that $q \supset p \wedge [q \Vdash \mathfrak{F}(i)] \wedge \forall j<i \forall q \supset p \neg [q \Vdash \mathfrak{F}(j)]$. Then $\forall j<i[q \Vdash \neg \mathfrak{F}(j)]$, which implies $[q \Vdash \forall j<i(\neg \mathfrak{F}(j))]$.
Hence $\exists q \supset p[q \Vdash \exists i(\mathfrak{F}(i) \wedge \forall j<i \neg \mathfrak{F}(j))]$.

LEMMA 7. If an $L_\Delta$-formula $\mathfrak{A}$ is provable in $\Delta$, then $\vdash_\Gamma \mathfrak{A}^*$.

PROOF. Besides Lemmata 5 and 6, it holds that $\vdash_\Gamma \mathfrak{A}^*$ for every theorem $\mathfrak{A}$ of $\Gamma$ because of Lemma 2. Thus, $\vdash_\Gamma \mathfrak{A}^*$ for every axiom $\mathfrak{A}$ in $\Delta$. So, by Lemmata 3 and 4, $\vdash_\Gamma \mathfrak{A}^*$ for every theorem $\mathfrak{A}$ in $\Delta$.

CONCLUSION. $\Delta$ is consistent.

PROOF. If not, $\Gamma$ should be inconsistent by Lemmata 2 and 7.

2.4. Remarks.

2.4.1. COROLLARY 1. If $\Gamma$ is any consistent extension of Zermelo's set theory $Z$ but its language is the same as $Z$, then the theory which is obtained from $\Gamma$ by adjoining the following axiom is consistent:

$$\forall x \exists y \in \omega(f(y)=x)$$

and all the sentences of the form

$$\vec{\forall}(\mathfrak{F}(0) \wedge \forall x(\mathfrak{F}(x) \to \mathfrak{F}(x+1)) \to \forall x \in \omega \mathfrak{F}(x)),$$

where $f$ is a newly introduced function symbol and $\mathfrak{F}(x)$ is any formula of the language $L_\Gamma \cup \{f\}$.

2.4.2. Following the proof of Theorem 1, we obtain the following model theoretical theorem which seems to have an independent interest:

THEOREM 2. Let $\mathfrak{M}=\langle \mathcal{M}, \mathcal{N}, 0, *', *+*, *\cdot*, *`*, \cdots\rangle$ be a countable model satisfying

1) $\mathfrak{M}\models(\text{Peano arithmetic})^{N}$ [7],
2) $\mathfrak{M}\models \vec{\forall}(\mathfrak{A}(0)\wedge\forall x(\mathfrak{A}(x)\rightarrow\mathfrak{A}(x'))\rightarrow\forall x(\boldsymbol{N}(x)\rightarrow\mathfrak{A}(x)))$ for all formulae $\mathfrak{A}$ of the language for $\mathfrak{M}$,
3) $\mathfrak{M}\models\forall a\forall x\forall i\exists y(\forall j<i(x`j=y`j)\wedge y`i=a)$.

Then there exists a function $\mathscr{f}$: $\mathcal{M}\rightarrow\mathcal{M}$ such that

1) $\mathscr{f}\upharpoonright\mathcal{N}: \mathcal{N}\rightarrow\mathcal{M}$ is surjective,
2) $(\mathfrak{M}, \mathscr{f})\models\vec{\forall}(\mathfrak{F}(0)\wedge\forall x(\mathfrak{F}(x)\rightarrow\mathfrak{F}(x'))\rightarrow\forall x(\boldsymbol{N}(x)\rightarrow\mathfrak{F}(x)))$ for all formulae $\mathfrak{F}(x)$ of the language for $(\mathfrak{M}, \mathscr{f})$.

PROOF. Call an ordered pair $(i, p)$ a forcing condition if $i\in\mathcal{N}$ and $p\in\mathcal{M}$. Define the relation $(i, p)\mathrel{-\!\!3}(j, q)$ by "$\mathfrak{M}\models\underline{i}<\underline{j}$ and $\forall k\in\mathcal{N}$ $(\mathfrak{M}\models(\underline{k}\leqq\underline{i}\rightarrow\underline{p}`\underline{k}=\underline{q}`\underline{k})$". Define the forcing condition $(i, p)\Vdash\mathfrak{A}$ between a forcing condition $(i, p)$ and a sentence $\mathfrak{A}$ of the language for $(\mathfrak{M}, \mathscr{f}, a)_{a\in\mathfrak{M}}$ as follows:

1) $(i, p)\Vdash\underline{v}=\underline{w}\Longleftrightarrow v=w$;
2) $(i, p)\Vdash\boldsymbol{f}(\mathrm{t})=\mathrm{u}\Longleftrightarrow(\exists v, w\in\mathcal{M})((i, p)\Vdash(\mathrm{t}=\underline{v}\wedge\mathrm{u}=\underline{w})$ and $((v\in\mathcal{N}$ and $\mathfrak{M}\models(\underline{v}<\underline{i}\wedge\underline{p}`\underline{v}=\underline{w}))$ or $(v\notin\mathcal{N}$ and $w=0)))$;
3) If $g$ is a function symbol different from $\boldsymbol{f}$, then $(i, p)\Vdash g(\mathrm{t}_1, \cdots, \mathrm{t}_\nu)=\mathrm{u}\Longleftrightarrow$ $(\exists v_1, \cdots, v_\nu, w\in\mathcal{M})(\mathfrak{M}\models g(\underline{v}_1, \cdots, \underline{v}_\nu)=\underline{w}$ and $(i, p)\Vdash(\mathrm{t}_1=\underline{v}_1\wedge\cdots\wedge\mathrm{t}_\nu=\underline{v}_\nu\wedge\mathrm{u}=\underline{w}))$;
4) $(i, p)\Vdash P(\mathrm{t}_1, \cdots, \mathrm{t}_\nu)\Longleftrightarrow$ $(\exists v_1, \cdots, v_\nu\in\mathcal{M})(\mathfrak{M}\models P(\underline{v}_1, \cdots, \underline{v}_\nu)\wedge(i, p)\Vdash(\mathrm{t}_1=\underline{v}_1\wedge\cdots\wedge\mathrm{t}_\nu=\underline{v}_\nu))$;
5) $(i, p)\Vdash\mathfrak{A}\wedge\mathfrak{B}\Longleftrightarrow(i, p)\Vdash\mathfrak{A}$ and $(i, p)\Vdash\mathfrak{B}$;
6) $(i, p)\Vdash\neg\mathfrak{A}\Longleftrightarrow\forall j\in\mathcal{N}\forall q\in\mathcal{M}((i, p)\mathrel{-\!\!3}(j, q)\Rightarrow\neg(j, q)\Vdash\mathfrak{A})$;
7) $(i, p)\Vdash\exists x\mathfrak{A}(x)\Longleftrightarrow\exists v\in\mathcal{M}((i, p)\Vdash\mathfrak{A}(\underline{v}))$.

Now, take a complete sequence $(i_0, p_0)\mathrel{-\!\!3}(i_1, p_1)\mathrel{-\!\!3}\cdots\mathrel{-\!\!3}(i_\nu, p_\nu)\mathrel{-\!\!3}\cdots$ and define $\mathscr{f}:\mathfrak{M}\rightarrow\mathfrak{M}$ by "$\mathscr{f}(x)=y\Longleftrightarrow\exists\nu((i_\nu, p_\nu)\Vdash\boldsymbol{f}(\underline{x})=\underline{y})$". Then the function $\mathscr{f}$ has the desired property as easily seen from the proof of Theorem.

2.4.3. As a by-product of the result, we obtain the result below, which concerns the problem of elimination of $\varepsilon$-symbol in the first order predicate calculus.

In the following, the $\varepsilon$-predicate calculus means the system obtained from the first order predicate calculus by adjoining the $\varepsilon$-symol.

7) $(\ldots)^{N}$ means the relativization by $\boldsymbol{N}$.

THEOREM 3. Let $\Gamma$ be any theory satisfying the condition mentioned in Theorem 1. Let $\Gamma_\varepsilon$ be the theory in $\varepsilon$-predicate calculus which is obtained from $\Gamma$ by adjoining an axiom schema of mathematical induction of the strengthened form

$$\vec{\forall}(\mathfrak{G}(0) \wedge \forall x(\mathfrak{G}(x) \to \mathfrak{G}(x')) \to \forall x(\boldsymbol{N}(x) \to \mathfrak{G}(x)))$$

where $\mathfrak{G}(x)$ is any formula of $\varepsilon$-predicate calculus.

Then, if a sentence $\mathfrak{S}$ is provable in $\Gamma_\varepsilon$ and is $\varepsilon$-symbol free, $\mathfrak{S}$ is provable in $\Gamma$.

PROOF. It suffices to prove $\Gamma_\varepsilon$ is consistent, By Theorem 1, the theory $\Delta = \Gamma \cup \{\forall x \exists y(\boldsymbol{N}(y) \wedge \boldsymbol{f}(y)=x)\} \cup \{\vec{\forall}(\mathfrak{F}(0) \wedge \forall x(\mathfrak{F}(x) \to \mathfrak{F}(x')) \to \forall x(\boldsymbol{N}(x) \to \mathfrak{F}(x))) \mid \mathfrak{F}$: a formula of $L_\Gamma \cup \{\boldsymbol{f}\}\}$ is consistent. Bisides $\varepsilon x \mathfrak{A}(x)$ can be interpreted in $\Delta$ as follows:

$$\varepsilon x(\mathfrak{A}(x) =^{df} \iota x((\exists z \mathfrak{A}(z) \to \mathfrak{A}(x)) \wedge \forall y(\boldsymbol{N}(y) \wedge (\exists z \mathfrak{A}(z) \to \mathfrak{A}(\boldsymbol{f}(y))) \to \exists w(\boldsymbol{N}(w) \wedge w \leqq y \wedge \boldsymbol{f}(w)=x))).$$

The following corollary is put to assess the meaning of this assertion:

COROLLARY 2. $ZF + \neg AC + \{\vec{\forall}(\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \in \omega \varphi(x) \mid \varphi$: a fomula possibly containing $\varepsilon$-symbols$\}$ is consistent in $\varepsilon$-predicate calculus.[8]

This is an extension of the well-known fact that $ZF + \neg AC$ is consistent even in $\varepsilon$-predicate calculus.[9]

## References

[1] Hanazawa, M., An Interpretation of Skolem's Paradox in the Predicate Calculus with $\varepsilon$-symbol, Science Reports of the Saitama University, Series A Vol. 9, No. 1 (1979) 11–19.
[2] Takeuti, G., On Skolem's theorem, J. Math. Soc. Japan, 9 (1957) 71–76.

Department of Mathematics
Faculty of Science
Saitama University
Urawa, Japan

8) We assume here the consistency of $ZF$.
9) Of course, the axiom schema of replacement must be understood to be a set of axioms, i.e. that schema is not applied to formulae containing $\varepsilon$-symbols·