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FIXED POINTS OF ELEMENTARY SUBGROUPS OF CHEVALLEY GROUPS ACTING ON TREES

By

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0. Introduction

Consider the following condition on a group G:

(FA) *For any tree X and any action without inversions of G on X, the set X^{G} of fixed points is non-empty.*

Jean-Pierre Serre has shown that every group with this property has many interesting group theoretical properties (cf. [5]). He has also shown that the special linear group $SL(2, \boldsymbol{Z})$ of degree 2 over the ring $\boldsymbol{Z}$ of rational integers does not satisfy (FA), but $SL(3, \boldsymbol{Z})$ does. In this paper we shall generalize this result to the elementary subgroup $E_{\rho}(\Phi, R)$ (See Section 2 below.) of a Chevalley group of type Φ over a commutative ring R with an identity under the assumption that Φ is irreducible of rank $\geqq 2$ and the additive group R^{+} of R is finitely generated. For any group G, we use $[G, G]$ to denote the commutator subgroup of G generated by all $[x, y]=xyx^{-1}y^{-1}$, $x, y\in G$.

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1. The action of a group on a tree

We begin with the definition of graphs. A *graph* $X=(S(X), Ar(X))$ consists of a non-empty set $S(X)$ and a subset $Ar(X)$ of $S(X)\times S(X)$ such that $(s, s)\notin Ar(X)$ for any $s\in S(X)$ and $Ar(X)=Ar(X)^{t}$, where

$$Ar(X)^{t}=\{(s, s')|(s', s)\in Ar(X)\}.$$

Each element of $S(X)$ (resp. $Ar(X)$) is called a *vertex* (resp. an *edge*). We shall sometimes identify a graph X with the set $S(X)$ of vertices and an edge (s, s') with (s', s). A series of finitely many edges $(s_0, s_1), (s_1, s_2),\cdots, (s_{n-1}, s_n)$ is called a *path of length n connecting s_0 and s_n*, and we shall denote it by $(s_0, s_1,\cdots, s_n)$. In particular, the path $(s_0, s_1,\cdots, s_n)$ is called *geodesic* if the vertices $s_0, s_1,\cdots, s_n$ are all distinct. Any path connecting two distinct vertices can be reduced to a geodesic

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path. On the other hand, the path $(s_0, s_1, \cdots, s_n)$ is said to be a *loop* when $n \geqq 3$, $s_0 = s_n$ and the path $(s_1, s_2, \cdots, s_n)$ is geodesic. A graph is defined to be *connected* if for any two distinct vertices there is a path connecting these two vertices. A connected graph is called a *tree* when it has no loops. It is easy to see that any two distinct vertices in a tree are connected by one and only one geodesic path.

An *automorphism* f of a graph X is a set theoretical bijection from $S(X)$ onto itself such that a pair of vertices (s, s') is in $Ar(X)$ if and only if $(f(s), f(s'))$ is in $Ar(X)$. The set of all automorphisms of a graph X forms naturally a group $Aut(X)$ under the composition of maps. We say that *a group G acts on a graph X* when there is a group homomorphism σ of G into $Aut(X)$. Then every element g of G induces an automorphism of X defined by

$$x \longmapsto \sigma(g)x = gx$$

Now consider an action of a group G on a tree $X = (S(X), Ar(X))$. It sometimes happens that there exist $(s, s') \in Ar(X)$ and $g \in G$ such that $(gs, gs') = (s', s)$. In this case we say that this action has an *inversion*. It is known that an action with inversions can be reduced to the case without inversions by the following method of barycentric subdivision. Assume that $(s, s') \in Ar(X)$ and $g \in G$ is an inversion. Take a tree $X' = (S(X'), Ar(X'))$ in place of X, where

$$S(X') = S(X) \cup \{s''\}, \quad s'' \notin S(X) \quad \text{and}$$

$$Ar(X') = (Ar(X) - \{(s, s')\}) \cup \{(s, s''), (s'', s')\}.$$

The action of G on X can be naturally extended to X' by defining $gs'' = s''$ for all $g \in G$. In this paper we assume that *no action of a group on a tree has any inversions.*

When a group G acts on a graph X, we denote

$$X^G = \{s \in S(X) \mid gs = s \text{ for all } g \in G\} \text{ and}$$

$$X^g = \{s \in S(X) \mid gs = s\}, \text{ where } g \text{ is a fixed element of } G.$$

PROPOSITION 1. *Let X be a tree.*

(i) *If X^G is non-empty, then X^G is a tree.*

(ii) *Let X_i $(1 \leqq i \leqq n)$ be subsets of X. If each X_i is a tree and $X_i \cap X_j$ is non-empty for all pairs (i, j), then $\bigcap_{1 \leqq i \leqq n} X_i$ is non-empty and connected.*

PROOF (i) We shall show that X^G is connected. For distinct vertices $s, s' \in X^G$, there is a unique geodesic path $(s, s_1, \cdots, s_n, s')$ in X. So $(gs, gs_1, \cdots, gs_n, gs') = (s, gs_1, \cdots, gs_n, s')$ is the same path for all $g \in G$. Therefore $s_1, s_2, \cdots, s_n \in X^G$.
(ii) Using induction on n, it is enough to show (ii) when $n=3$. Choose any $s_{ij} \in X_i \cap X_j$ for $i, j = 1, 2, 3$. We may assume $s_{12} \notin X_3$, $s_{23} \notin X_1$ and $s_{13} \notin X_2$, otherwise the proof is completed. So the three vertices s_{12}, s_{23} and s_{13} are all distinct. Since s_{12} and s_{23} (resp. s_{23} and s_{13}) are connected by a path in X_2 (resp. in X_3), there is a path connecting s_{12} and s_{13}. Reducing the path to be geodesic, we get a geodesic path connecting s_{12} and s_{13} which runs through $X_2 \cap X_3$. But by the uniqueness of geodesic paths, this path is contained in X_1. So we have $X_1 \cap X_2 \cap X_3 \neq \phi$. The connectedness of $X_1 \cap X_2 \cap X_3$ is obvious. q. e. d.

2. Elementary subgroups of Chevalley groups and some of their properties

In this section we recall the definition of elementary subgroups of Chevalley groups and give some of their properties. Let Φ be a (*reduced*) *irreducible root system* (cf. [2], Chap. 6), and V be the real Euclidean space spanned by Φ. When we choose a *base* Δ of the root system Φ, the set of positive (resp. negative) roots with respect to Δ is determined and we shall denote it by Φ^+ (resp. Φ^-). Let Δ' be a non-empty subset of $\Delta = \{\alpha_i\}$. Then

$$\{\Sigma n_i \alpha_i \in \Phi \mid n_i = 0 \text{ if } \alpha_i \notin \Delta'\}$$

is a (not necessarily irreducible) root system with a base Δ', and we shall denote it by

$$<\alpha_i \mid \alpha_i \in \Delta'>.$$

Each root $\alpha \in \Phi$ defines a reflection r_α of the space V, which sends α to $-\alpha$ and leaves pointwise fixed the hyperplane orthogonal to α. All reflections determined by the roots of Φ generate a group W called the *Weyl group of* Φ.

Each irreducible root system Φ determines uniquely (up to isomorphism) a finite dimensional simple Lie algebra $\mathfrak{g}(\Phi)$ over the field of complex numbers. Let ρ be a faithful representation of the Lie algebra $\mathfrak{g}(\Phi)$ on a finite dimensional vector space over the field of complex numbers, then we can construct the *Chevalley-Demazure group scheme* $G_\rho(\Phi, \ )$ *associated with* Φ *and* ρ (cf. [1], [3] and [7]). Since $G_\rho(\Phi, \ )$ is a covariant functor from the category of commutative rings to the category of groups, we get a group $G_\rho(\Phi, R)$ of the points of $G_\rho(\Phi, \ )$ in a commutative ring R with an identity. In particular, if $R = \boldsymbol{C}$ is the field of complex numbers, $G_\rho(\Phi, \boldsymbol{C})$ has the structure of a Lie group. $G_\rho(\Phi, \ )$ is called *simply connected* when the Lie group $G_\rho(\Phi, \boldsymbol{C})$ is simply connected, or equivalently when the set of fundamental weights is a base of the lattice generated by the set of all weights of the representation ρ. We shall give an example. Assume that Φ is of type A_l and $G_\rho(\Phi,\ \ )$ is simply connected. Then $G_\rho(\Phi, R)$ is isomorphic to the special linear group $SL(l+1, R)$ of degree $l+1$ over a commutative ring R with an identity. In general, when Φ is of type A_l, $G_\rho(\Phi, R)$ is isomorphic to a quotient group of $SL(l+1, R)$ by a central subgroup.

For each root $\alpha\in\Phi$, there is a group isomorphism

$$t \longmapsto x_\alpha(t)$$

of the additive group R^+ of R onto a subgroup X_α of $G_\rho(\Phi, R)$. X_α is called the *root subgroup* corresponding to the root α. The *elementary subgroup* $E_\rho(\Phi, R)$ is defined to be the subgroup of $G_\rho(\Phi, R)$ generated by all X_α for $\alpha\in\Phi$. When $G_\rho(\Phi,\ \ )$ is simply connected, $E_\rho(\Phi, R)$ is equal to $G_\rho(\Phi, R)$ if R is a local ring (cf. [1], Proposition 1.6) or R is a Euclidean domain (cf. [7], §8). But, in general, $E_\rho(\Phi, R)$ is a proper subgroup of $G_\rho(\Phi, R)$. For a base Δ of Φ, let $U_\rho(\Phi, R, \Delta)$ be the subgroup of $E_\rho(\Phi, R)$ generated by all X_α for $\alpha\in\Phi^+$. Then $U_\rho(\Phi, R, \Delta)$ is unipotent and hence nilpotent (cf. [7], p. 26).

Now we shall make a list of some relations between generators in the elementary subgroup $E_\rho(\Phi, R)$ (cf. [3], [6] and [7]).

(*RI*) *For any* $s, t\in R$ *and* $\alpha\in\Phi$,

$$x_\alpha(s)x_\alpha(t)=x_\alpha(s+t).$$

(*RII*) *Let rank* $\Phi=l\geqq 2$. *For any* $s, t\in R$ *and* $\alpha, \beta\in\Phi$ *such that* $\alpha+\beta\neq 0$,

$$[x_\alpha(s), x(t)]=\Pi x_{i\alpha+j\beta}\ (N_{\alpha,\beta,i,j}\, s^i t^j)$$

where the product on the right is taken over all roots of the form $i\alpha+j\beta$, *for positive integers* i *and* j *arranged in some fixed order, and* $N_{\alpha,\beta,i,j}$ *are integers depending only on* α, β *and the chosen ordering.*

(*RIII*) *For any* $t\in R$ *and* $\alpha, \beta\in\Phi$

$$w_\alpha x_\beta(t)w_\alpha^{-1}=x_{r_\alpha(\beta)}(\pm t),$$

where $w_\alpha=x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$.

PROPOSITION 2. *Let* R *be a commutative ring with an identity and* Φ *be an irreducible root system of rank* $\geqq 2$. *For each* $x_\alpha(t)\in E_\rho(\Phi, R)$, $\alpha\in\Phi$, $t\in R$, *there exist a positive integer* n *and a base* Δ *of* Φ *such that*

(i) $x_\alpha(t)\in U$

(ii) $x_\alpha(t)^n=x_\alpha(nt)\in[U, U]$, *where* $U=U_\rho(\Phi, R, \Delta)$

PROOF. Choosing a suitable base Δ, we may assume that the given root α is positive and not simple. Furthermore we may assume that Φ is an irreducible root system of rank 2, that is, $\Phi=A_2$, or B_2, or G_2. Since the proof of the case A_2 or B_2 is easy, we shall prove the most complicated case $\Phi=G_2$. Set $\Delta=\{\alpha_1, \alpha_2\}$ and $\Phi=\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2), \pm(3\alpha_1+\alpha_2), \pm(3\alpha_1+2\alpha_2)\}$, then $\alpha=\alpha_1+\alpha_2$, or $2\alpha_1+\alpha_2$, or $3\alpha_1+\alpha_2$, or $3\alpha_1+2\alpha_2$. As special relations of (RII), we have

$$[x_{\alpha_2}(s), x_{3\alpha_2+\alpha_2}(t)]=x_{3\alpha_1+2\alpha_2}(\pm st),$$

$$[x_{2\alpha_1+\alpha_2}(s), x_{\alpha_1}(t)]=x_{3\alpha_1+\alpha_2}(\pm 3st),$$

$$[x_{\alpha_1+\alpha_2}(s), x_{\alpha_1}(t)]=x_{2\alpha_1+\alpha_2}(\pm 2st)x_{3\alpha_1+\alpha_2}(\pm 3st^2)x_{3\alpha_1+2\alpha_2}(\pm 3s^2t) \quad \text{and}$$

$$[x_{\alpha_1}(s), x_{\alpha_2}(t)]=x_{\alpha_1+\alpha_2}(\pm st)x_{2\alpha_1+\alpha_2}(\pm s^2t)x_{3\alpha_1+\alpha_2}(\pm s^3t)x_{3\alpha_1+2\alpha_2}(\pm 2s^3t^2).$$

So we can choose $n=1$ (resp. $n=2$, $n=3$, $n=6$), when $\alpha=3\alpha_1+2\alpha_2$ (resp. $\alpha=2\alpha_1+\alpha_2$, $\alpha=3\alpha_1+\alpha_2$, $\alpha=\alpha_1+\alpha_2$).
q. e. d.

For an irreducible root system Φ the *highest root* exists uniquely with respect to some fixed base of Φ (cf. [2], Chap. 6).

PROPOSITION 3. *Let Φ be an irreducible root system of rank $\geqq 3$ with a base $\Delta=\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$. Then we can choose α_1 such that the following are satisfied.*

(i) *$\Phi'=\langle\alpha_2, \cdots, \alpha_l\rangle$ is an irreducible root system of rank $l-1$ with a base $\Delta'=\{\alpha_2, \cdots, \alpha_l\}$.*

(ii) *Let α_0 and β_0 be the highest roots of Φ and Φ' with respect to the base Δ and Δ' respectively. If Φ is not of type C_l or F_4 (resp. Φ is of type C_l or F_4), put $\gamma_1=\alpha_0-\beta_0$ (resp. $2\gamma_1=\alpha_0-\beta_0$). Then $\gamma_1\in\Phi^+$ and $\langle\beta_0, \gamma_1\rangle$ is of type A_2 (resp. of type B_2) with highest root α_0.*

(iii) *Put $\gamma_2=\gamma_1-\alpha_1$. Then if Φ is of type A_l or C_l, $\gamma_2=0$. Otherwise, $\gamma_2\in\Phi^+$.*

(iv) *If Φ is not of type A_l or C_l or F_4, then $\langle\alpha_1, \gamma_2\rangle$ is of type A_2 and the highest root is γ_1.*

(v) *If Φ is of type F_4, then $\langle\alpha_1, \gamma_2\rangle$ is of type B_2 and the highest root is $\alpha_1+2\gamma_2$.*

PROOF. The proposition can be proved by the classification of irreducible root systems (cf. [2] Chap. 6). For each system, we give the Dynkin diagram having α_1 as a terminal node, the type of Φ' and the expressions of $\alpha_0, \beta_0, \gamma_1$ and γ_2 by the base of Φ.

$A_l(l\geqq 3)$

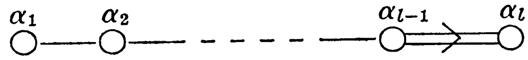
Φ' is of type A_{l-1}.

$\alpha_0=\alpha_1+\alpha_2+\cdots\cdots+\alpha_l$,

$\beta_0=\alpha_2+\alpha_3+\cdots\cdots+\alpha_l$,

$\gamma_1=\alpha_1$ and $\gamma_2=0$.

$B_l(l\geqq 3)$



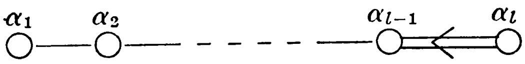
Φ' is of type B_{l-1}.

$\alpha_0=\alpha_1+2\ (\alpha_2+\alpha_3+\cdots\cdots+\alpha_l)$,

$\beta_0=\alpha_2+2\ (\alpha_3+\cdots\cdots+\alpha_l)$,

$\gamma_1=\alpha_1+\alpha_2$ and $\gamma_2=\alpha_2$.

$C_l(l\geqq 3)$



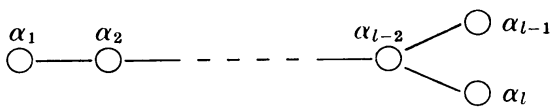
Φ' is of type C_{l-1}.

$\alpha_0=2\ (\alpha_1+\cdots\cdots+\alpha_{l-1})+\alpha_l$,

$\beta_0=2\ (\alpha_2+\cdots\cdots+\alpha_{l-1})+\alpha_l$,

$\gamma_1=\alpha_1$ and $\gamma_2=0$.

$D_l(l\geqq 4)$



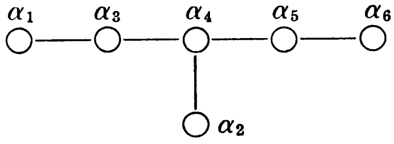
Φ' is of type A_3 (if $l=4$) or D_{l-1} (if $l\geqq 5$).

$\alpha_0=\alpha_1+2\ (\alpha_2+\cdots\cdots+\alpha_{l-2})+\alpha_{l-1}+\alpha_l$,

$\beta_0=\alpha_2+2\ (\alpha_3+\cdots\cdots+\alpha_{l-2})+\alpha_{l-1}+\alpha_l$,

$\gamma_1=\alpha_1+\alpha_2$ and $\gamma_2=\alpha_2$.

E_6



Φ' is of type D_4.

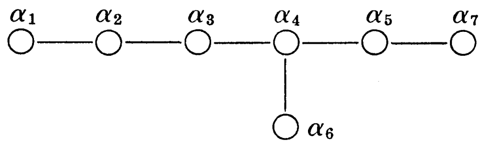
$\alpha_0=\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$,

$\beta_0=\alpha_2+\alpha_3+2\ (\alpha_4+\alpha_5)+\alpha_6$,

$\gamma_1=\alpha_1+\alpha_2+\alpha_3+\alpha_4$ and

$\gamma_2=\alpha_2+\alpha_3+\alpha_4$.

E_7



Φ' is of type E_6 .

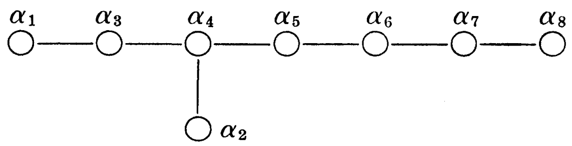
$\alpha_0=\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+2\alpha_7$,

$\beta_0=\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7$,

$\gamma_1=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_7$ and

$\gamma_2=\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_7$.

E_8



Φ' is of type D_7 .

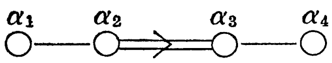
$\alpha_0=2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+2\alpha_8$,

$\beta_0=\alpha_2+\alpha_3+2\ (\alpha_4+\alpha_5+\alpha_6+\alpha_7)+\alpha_8$,

$\gamma_1=2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8$ and

$\gamma_2=\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8$.

F_4



Φ' is of type C_3 .

$\alpha_0=2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$,

$\beta_0=\alpha_2+2\alpha_3+2\alpha_4$,

$\gamma_1=\alpha_1+\alpha_2+\alpha_3$ and

$\gamma_2=\alpha_2+\alpha_3$. q. e. d.

PROPOSITION 4. *Let Φ be an irreducible root system of rank $\geqq 2$ and Δ be any fixed base of Φ. Then the elementary subgroup $E_\rho(\Phi, R)$ is generated by $\{X_\alpha | \alpha \in \Psi\}$, where*

$$\Psi=(\Phi^+-\{\alpha_0\})\cup\{-\alpha_0\}$$

and α_0 *is the highest root of* Φ *with respect to* Δ.

PROOF. First we shall prove this in case the rank of Φ is 2, that is, Φ is of type A_2, or B_2, or G_2. Then we shall treat the case when the rank of Φ is greater than 2. Let $G[\Psi]$ be the subgroup of $E_\rho(\Phi, R)$ generated by $\{X_\alpha | \alpha \in \Psi\}$. We have to show that $G[\Psi]$ is equal to $E_\rho(\Phi, R)$.

Supposing first that we are in case A_2, or B_2, or G_2, put $\Delta=\{\alpha, \beta\}$ such that α is a short root and β is a long root if Φ is of type B_2 or G_2. Then $\alpha_0=\alpha+\beta$ (resp. $\alpha_0=2\alpha+\beta$, $\alpha_0=3\alpha+2\beta$) when Φ is A_2 (resp. B_2, G_2). We claim that there exists a base $\Delta'=\{\alpha', \beta'\}$ such that root subgroups corresponding to the roots $\pm\alpha'$ and $\pm\beta'$ are contained in $G[\Phi]$. Since we have a relation (cf. [7], §3 and §4)

$$[x_\alpha(s),\ x_\beta(t)]=x_{\alpha+\beta}(\pm st)$$

$$(\text{resp. } [x_\alpha(s),\ x_\beta(t)]=x_{\alpha+\beta}(\pm st)x_{2\alpha+\beta}(\pm s^2t),\ [x_\beta(s),\ x_{3\alpha+\beta}(t)]=x_{3\alpha+2\beta}(\pm st))$$

when Φ is A_2 (resp. B_2, G_2), $X_{\alpha_0}\subseteqq G[\Psi]$ and hence $w_{\alpha_0}=x_{\alpha_0}(1)x_{-\alpha_0}(-1)x_{\alpha_0}(1)\in G[\Psi]$. By $(RIII)$, every root subgroup corresponding to the root conjugate to a positive root by w_{α_0} is in $G[\Psi]$ (cf. [4], §4). So we can choose $\Delta'=\Delta=\{\alpha, \beta\}$ (resp. $\Delta'=\{\alpha, -(\beta+2\alpha)$ $\Delta'=\{-(\beta+\alpha), \beta\}$) if Φ is A_2 (resp. B_2, G_2), and the subgroup generated by $\{w_\gamma | \gamma\in\Delta'\}$ is contained in $G[\Psi]$. Since the Weyl group W of Φ is generated by the reflections corresponding to the roots of Δ' (cf. [2], Chap. 6, §1, Th. 2), every root subgroup corresponding to the root conjugate to a root of Δ' under W is in $G[\Psi]$. On the other hand, for any two roots of the same length, there is an element of W which maps one to the other (cf. [2], Chap. 6, §1, Prop. 11). Therefore every root subgroup is in $G[\Psi]$, and this completes the proof of this case.

Supposing next that we are in case the rank of $\Phi\geqq 3$, we proceed by induction on the rank and use the notation of Proposition 3. $\langle\beta_0, \gamma_1\rangle$ is an irreducible root system of rank 2 with highest root α_0. By hypothesis, $X_\gamma\subseteqq G[\Phi]$, where γ is any positive root in $\langle\beta_0, \gamma_1\rangle$ or $\gamma=-\alpha_0$. Then by the cases of rank 2, $X_{-\beta_0}, X_{-\gamma_1}\subseteqq G[\Phi]$. Since β_0 is the highest root of $\langle\alpha_2, \cdots, \alpha_l\rangle$, $X_{-\alpha_i}\subseteqq G[\Phi]$ $(2\leqq i\leqq l)$ by induction. It remains only to show that $X_{-\alpha_1}\subseteqq G[\Phi]$. If $\Phi=A_l$ or C_l, $\gamma_1=\alpha_1$, hence we have $X_{-\alpha_1}\subseteqq G[\Phi]$. If Φ is not A_l or C_l or F_4, then $\langle\alpha_1, \gamma_2\rangle$ is an irreducible root system of rank 2 with highest root γ_1. By an argument similar to the above, we have $X_{-\alpha_1}\subseteqq G[\Phi]$. Finally if Φ is of type F_4, $\langle -\gamma_1, \alpha_1+2\gamma_2\rangle$ is of type B_2 with highest root $-\alpha_1$. Hence we have $X_{-\alpha_1}\subseteqq G[\Psi]$. q.e.d.

3. Main result

In this section we shall prove the following theorem:

THEOREM. *Let Φ be an irreducible root system of rank $\geqq 2$, R be a commutative ring with an identity such that the additive group R^+ of R is finitely generated and ρ be any faithful representation of the Lie algebra $\mathfrak{g}(\Phi)$. Then the elementary subgroup $E_\rho(\Phi, R)$ has the property (FA).*

To prove this theorem we need the following result due to Jean-Pierre Serre (cf. [5], Proposition 2 and its corollaries).

PROPOSITION 5. *Let G be a finitely generated nilpotent group. Assume that G acts without inversions on a tree X.*

(i) *Let $\{g_i\}$ be a finite set of generators of G. If X^{g_i} is non-empty for all i, then X^G is non-empty.*

(ii) *Let g be an element of G. If g^n is in $[G, G]$ for some positive integer n, then X^g is non-empty.*

PROPOSITION 6. *Assume that the elementary subgroup $E_\rho(\Phi, R)$ acts without inversions on a tree X, where ρ, Φ and R are as in the theorem. Let $U = U_\rho(\Phi, R, \Delta)$ be as in Section 2. Then X^U is non-empty.*

PROOF. Let Δ be any fixed base of Φ. Since $U = U_\rho(\Phi, R, \Delta)$ is finitely generated and nilpotent, we can apply (i) of Proposition 5 to the group U. It is enough to prove that for each generator $g = x_\alpha(t)$, $\alpha \in \Phi^+$, $t \in R$, of U, X^g is non-empty. On the other hand, by Proposition 2, for any root $\alpha \in \Phi$ and any element $t \in R$ there exist a base Δ' of Φ and a positive integer n such that $x_\alpha(t) \in U'$ and $x_\alpha(t)^n = x_\alpha(nt) \in [U', U']$, where $U' = U_\rho(\Phi, R, \Delta')$. Applying (ii) of Proposition 5 to the group U' and an element $g = x_\alpha(t) \in U'$, we have $X^g \neq \phi$. q.e.d.

PROOF OF THE THEOREM. Given an action of $E_\rho(\Phi, R)$ on a tree X, let $\{r_i \in R \mid i = 1, \cdots, n\}$ be a finite set of generators of R^+. For each $\alpha \in \Phi$ and $r_i \in R$, put

$$g_{i,\alpha} = x_\alpha(r_i), \quad X_{i,\alpha} = X^{g_{i,\alpha}}.$$

First we claim that $X_{i,\alpha} \cap X_{j,\beta}$ is non-empty for any $\alpha, \beta \in \Psi$ and integers i, j ($1 \leqq i, j \leqq n$), where Ψ is as in Proposition 4. We may assume $\alpha \neq \beta$. Since $\alpha + \beta$ is non-zero, there is a base Δ' of Φ such that α and β are positive roots with respect to Δ'. Take $U' = U_\rho(\Phi, R, \Delta')$, then $X^{U'}$ is non-empty by Proposition 6. On the other hand, since $g_{i,\alpha}, g_{j,\beta} \in U'$, we have $X_{i,\alpha} \cap X_{j,\beta} \supseteqq X^{U'}$. Thus $X_{i,\alpha} \cap X_{j,\beta}$ is non-empty. Hence we have, by Proposition 4,

$$X^{E_\rho(\Phi, R)} = \bigcap_{\substack{1 \leqq i \leqq n \\ \alpha \in \Psi}} X_{i,\alpha}$$

and this is non-empty by (Proposition 1). q.e.d.

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