TSUKUBA J. MATH.
Vol. 3, No. 1 (1979), 123–128

REMARK ON LOCALIZATIONS OF NOETHERIAN RINGS WITH KRULL DIMENSION ONE

By

Hideo SATO

Let R be a left noetherian ring with left Krull dimension α. For a left R-module M which has Krull dimension, we denote its Krull dimension by $K\text{-}dim\, M$ in this note. In the previous paper [6], we have shown that the family $\boldsymbol{F}_{\beta}(R)=\{{}_RI\subseteq R \mid K\text{-}dim\, R/I\leq\beta\}$ is a left (Gabriel) topology on R for any ordinal $\beta<\alpha$. We are most interested in the case when R is (left and right) noetherian, $\alpha=1$ and $\beta=0$. Let R be such a ring and we denote $\boldsymbol{F}_0(R)$ by $\boldsymbol{F}$. Let A be the artinian radical of R. Then Lenagan [3] showed that R/A has a two-sided artinian, two-sided classical quotient ring $Q(R/A)$. In this note, we shall show that $R_{\boldsymbol{F}}$, the quotient ring of R with respect to $\boldsymbol{F}$, is isomorphic to $Q(R/A)$ as ring and we shall investigate a more precise structure of $R_{\boldsymbol{F}}$ under some additional assumptions.

In this note, a family of left ideals of R is said to be a topology if it is a Gabriel topology in the sense of Stenström's book [7]. So a perfect topology in this note is corresponding to a perfect Gabriel topology in [7]. Let $\boldsymbol{G}$ be a left topology on R, and M a left R-module. A chain of submodules of M;

$$M_0\supseteq M_1\supseteq\cdots\cdots\supseteq M_r$$

is called a $\boldsymbol{G}$-chain if each M_{i-1}/M_i is not a $\boldsymbol{G}$-torsion module. A $\boldsymbol{G}$-chain of M is said to be maximal if it has no proper refinement of $\boldsymbol{G}$-chain.

The following lemma can be proved easily.

LEMMA 1. *If ${}_RM$ has a finite maximal $\boldsymbol{G}$-chain of length r, then any $\boldsymbol{G}$-chain of M has a finite length s and $s\leq r$.*

Hence we can give a definition of $\boldsymbol{G}$-dimension of M, denoted by $\boldsymbol{G}\text{-}dim\, M$, as follows; if M has a finite maximal $\boldsymbol{G}$-chain of length r, define $\boldsymbol{G}\text{-}dim\, M=r$, and $\boldsymbol{G}\text{-}dim\, M=\infty$ otherwise.

COROLLARY 2. *For any short exact sequence of R-modules;*

$$0\rightarrow M'\rightarrow M\rightarrow M''\rightarrow 0$$

we have $\boldsymbol{G}\text{-}dim\, M=\boldsymbol{G}\text{-}dim\, M'+\boldsymbol{G}\text{-}dim\, M''$.

Received October 27, 1978. Revised February 6, 1979

COROLLARY 3. *Let* $\boldsymbol{G}\subseteq\boldsymbol{G}'$ *be left topologies on* R, *and* M *a left* R-*module. Then* $\boldsymbol{G}$-*dim* $M\geq\boldsymbol{G}'$-*dim* M.

We apply Lenagan's results ([3, Theorem 3.6] and [2, Theorem 3.1]) in the following form.

THEOREM (Lenagan) *Let* R *be a* (*left and right*) *noetherian ring with left Krull dimension one, and* A *its artinian radical. Denote* R/A *by* $\bar{R}$, *and* $x+A$ *by* $\bar{x}$ *for* $x\in R$. *Let* $S=\{s\in R|\bar{s}$ *is a regular element in* $\bar{R}$.$\}$ *Then the following statements hold.*

(1) $\Sigma(S)=\{Rs|s\in S\}$ *is a cofinal family of* $\boldsymbol{F}$.

(2) $\bar{R}$ *has a two-sided classical quotient ring* $Q(\bar{R})$.

We should remark that Lenagan showed that $Q(\bar{R})$ is a (left and right) artinian ring. But in the assertion (2) we need only the existence of $Q(\bar{R})$ for our purpose.

In the following Lemmas 4, 5 and 6, R is assumed to be a left noetherian ring with left Krull dimension α.

LEMMA 4. (See [6, Theorem 3.1].) For any $\beta<\alpha$, $\boldsymbol{F}_{\beta}=\{{}_{R}I\subseteq R|K\text{-}dim\ R/I\leq\beta\}$ *is a left topology on* R.

LEMMA 5. *Let* $t_{F_{\beta}}$ *be the torsion radical corresponding to the topology* $\boldsymbol{F}_{\beta}$. *Then* $rad^{\beta}({}_{R}R)=t_{F_{\beta}}(R)$ *where* $rad^{\beta}({}_{R}R)$ *is the largest left ideal of* R *whose Krull dimension is at most* β. (Cf. [6])

PROOF. Clear by definitions.

LEMMA 6. *For every left ideal* I *of* R, $I\in\boldsymbol{F}_{\beta}(R)$ *if and only if* $I+A/A\in\boldsymbol{F}_{\beta}(R/A)$ *where* $A=t_{F_{\beta}}(R)$.

PROOF. Since $(R/A)/(I+A/A)\cong R/I+A$ as R/A-module and as R-module, $I\in\boldsymbol{F}_{\beta}(R)$ implies that $K\text{-}dim\ R/I+A\leq K\text{-}dim\ R/I\leq\beta$. Thus $I+A/A\in\boldsymbol{F}_{\beta}(R/A)$. Conversely assume that $I+A/A\in\boldsymbol{F}_{\beta}(R/A)$. Then $K\text{-}dim_{R/A}(R/I+A)\leq\beta$. Since $I+A/I\cong A/A\cap I$, $K\text{-}dim\ I+A/I\leq K\text{-}dim\ A\leq\beta$. Thus $K\text{-}dim\ R/I\leq\beta$. Hence we have $I\in\boldsymbol{F}_{\beta}(R)$.

In the sequel, R is assumed to be a left and right noetherian ring with left Krull dimension one. Denote $F_{0}(R)$ by $\boldsymbol{F}$ and $\boldsymbol{F}_{0}(\bar{R})$ by $\boldsymbol{F}'$ respectively. Here $\bar{R}=R/A$ and $A=t_{F}(R)$.

LEMMA 7. *$R_F \cong \bar{R}_{F'} \cong Q(\bar{R})$ as ring.*

PROOF. For any left ideal I of R, consider the following exact sequence: $0 \to I \cap A \to I \to I/I \cap A \to 0$. Since $I \cap A$ is an $\boldsymbol{F}$-torsion module and $\bar{R} = R/t_F(R)$ is $\boldsymbol{F}$-torsion-free, $\mathrm{Hom}_R(I, \bar{R}) \cong \mathrm{Hom}_R(I/I \cap A, \bar{R}) \cong \mathrm{Hom}_{\bar{R}}(\bar{I}, \bar{R})$ where $\bar{I} = I + A/A$. Clearly the above isomorphisms are natural in I. Thus $R_F = \varinjlim_{I \in F} \mathrm{Hom}_R(I, \bar{R}) \cong \varinjlim_{\bar{I} \in F} \mathrm{Hom}_{\bar{R}}(\bar{I}, \bar{R}) = \bar{R}_{F'}$ by Lemma 6. It follows from Lenagan's theorem that $\bar{R}_{F'} \cong Q(\bar{R})$ as ring. This completes the proof.

LEMMA 8. *$\boldsymbol{F}$ is a perfect topology.*

PROOF. Let $S = \{s \in R \mid \bar{s}$ is a regular element in $\bar{R}\}$. Then by Lenagan's theorem, it is sufficient to prove that $bs = 0$ for $b \in R$ and $s \in S$ implies $ub = 0$ for some $u \in S$ (see [7, XI, Proposition 6.3]). Now we have then $\bar{b}\bar{s} = 0$ and hence $\bar{b} = 0$, that is, $b \in A$. Thus $Rb \cong R/l(b)$ is artinian and hence $l(b) \in \boldsymbol{F}$. Here $l(b)$ is the left annihilator ideal of b. It follows from Lenagan's theorem that $l(b) \cap S \neq \emptyset$. This shows that $\Sigma(S)$ is a cofinal family of $\boldsymbol{F}$.

Recall that R is said to satisfy the restricted minimum condition for left ideals if R/I is an artinian module for every dense left ideal I. (Cf. [6])

THEOREM 9. *Let R be a noetherian QF-3 ring satisfying the restricted minimum condition for left ideals. Then R_F is a QF ring where $\boldsymbol{F} = \boldsymbol{F}_0$.*

PROOF. By assumption, it follows from [6, Theorem 5.1] and [8, Proposition 1] that R has left Krull dimension at most one. Denote R_F by Q. Then by Lemma 8, Q_R is flat, and $Q \otimes_R N = 0$ if and only if N is an $\boldsymbol{F}$-torsion module for any left R-module N. Let M be any finitely generated left Q-module. Then it follows from the above facts that there exists a finitely generated, $\boldsymbol{F}$-torsion-free left R-module N such that $M \cong Q \otimes_R N$ as left Q-module. Since R satisfies the restricted minimum condition for for left ideals, ${}_RN$ is $\boldsymbol{D}$-torsion-free where $\boldsymbol{D}$ is the topology of dense left ideals. Since R is QF-3, ${}_RN$ is a finitely generated torsionless module. Thus ${}_RN$ can be embedded into a finitely generated free R-module because R is noetherian. Thus ${}_QM$ can be embedded a finitely generated free Q-module. Since Q is a noetherian ring, it follows from the above facts that any proper descending chain of left ideals of Q is an $\boldsymbol{F}$-chain of ${}_RQ$. Since an R-module $Q/\bar{R}$ is an $\boldsymbol{F}$-torsion module, we have $\boldsymbol{F}\text{-}dim_R Q = \boldsymbol{F}\text{-}dim_R \bar{R} = \boldsymbol{F}\text{-}dim_R R \leq \boldsymbol{D}\text{-}dim_R R < \infty$ by Corollary 3 and [8, Proposition 1]. This shows that ${}_QQ$ has finite length. Therefore it follows from [4, Corollary 6] that Q is a QF ring.

THEOREM 10. *For a noetherian ring R, the following statements are equivalent.*

(1) *R is a two-sided order in a QF ring and $K\text{-}dim_R R \leq 1$.*

(2) *R can be decomposed into a ring direct sum, say $R=A\oplus B$, where A is a QF ring and B is a QF-3 ring satisfying the restricted minimum condition for left ideals and $Soc(B)=0$.*

PROOF. Assume the statement (1). By [1, Theorem 10] we have a decomposition $R=A\oplus B$ where A is the artinian radical of R and $Soc(B)=0$. By assumption, it is clear that A is a QF ring and B has a QF classical two-sided quotient ring $Q(B)$. Thus B is QF-3 (see [5, Theorem 1.5]). Let S be the set of all regular elements in B. Let $\Sigma(S)=\{Bs|s\in S\}$. Then $\Sigma(S)\subseteq \boldsymbol{F}_0(B)$. Conversely assume $I\in\boldsymbol{D}(B)$ where $\boldsymbol{D}(B)$ is the topology of dense left ideal of B. Since ${}_{Q(B)}Q(B)$ is a cogenerator, we have $Q(B)=Q(B)I$ and hence $I\cap S\neq\emptyset$. Hence $\boldsymbol{F}_0(B)=\boldsymbol{D}(B)$. This shows that B satisfies the restricted minimum condition for left ideals.

Conversely assume the statement (2). Then it is immediate from Lenagan's theorem, Theorem 9 and Lemma 7.

In the remainder of this note, we assume that R is a noetherian QF-3 ring satisfying the restricted minimum condition for left ideals. So R has left Krull dimension one. We denote the topology of dense left ideals by $\boldsymbol{D}$ and $\boldsymbol{F}_0$ by $\boldsymbol{F}$. Let $S=R_F$ and $Q=R_D$. Then we shall give a remark on the connection between two rings, S and Q. Now, we have a commutative diagram of canonical ring homomorphisms;

$$\begin{tikzcd} R \arrow[rr] \arrow[dr, "\phi"'] & & Q \arrow[dl, "\psi"] \\ & S & \end{tikzcd}$$

because $\boldsymbol{D}\subseteq\boldsymbol{F}$ by assumption.

Then we have

PROPOSITION 11. *Both ϕ and ψ are left flat epimorphisms. Moreover S is injective both as left R-module and as left Q-module.*

PROOF. By Lemma 8, ϕ is a left flat epimorphism and hence ψ is an epimorphism. By Theorem 9, S is an injective left S-module. So we see by adjointness that S is also injective as left R-module. Denote the artinian radical by A, and R/A by $\bar{R}$. Consider a canonical exact sequense;

$$0\rightarrow R\rightarrow Q\rightarrow Q/R\rightarrow 0.$$

Since Q/R is a $\boldsymbol{D}$-torsion R-module, it is an $\boldsymbol{F}$-torsion module. On the other hand, ${}_R\bar{R}$ is $\boldsymbol{F}$-torsion-free and ${}_RS$ is an essential extension of ${}_RR$. Thus ${}_RS$ is $\boldsymbol{F}$-torsion-free. Hence we have $\mathrm{Hom}_R(Q/R, S)=0$. For any left ideal I of Q and Q-homomorphism g of ${}_QI$ into ${}_QS$, there exists an R-homomorphism $\bar{g}$ which makes the below diagram commutative;

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "j"] \arrow[d, "g"'] & Q \arrow[dl, "\bar{g}"] \\ & S & \end{tikzcd}$$

where j is an inclusion. Fix any element q_0 in Q. Define an R-homomorphism h of Q into S as follows;

$$qh=q(q_0\bar{g})-(qq_0)\bar{g} \qquad \text{for any } q\in Q.$$

It is clear that $Rh=0$. We have the induced R-homomorphism $\bar{h}$ such that the following diagram is commutative;

$$\begin{tikzcd} Q \arrow[rr, "h"] \arrow[dr, "\pi"'] & & S \\ & Q/R \arrow[ur, "\bar{h}"'] & \end{tikzcd}$$

where π is the canonical map. By the above remark, $\bar{h}=0$ and hence $h=0$. This shows that $\bar{g}$ is a Q-homomorphism and hence ${}_QS$ is injective. It remains to show that S_Q is flat. Consider an exact sequence of left Q-modules:

$$0 \to X \to Y.$$

Since ${}_QS$ is injective, we have the following exact sequence;

$$\mathrm{Hom}_Q(Y, S) \to \mathrm{Hom}_Q(X, S) \to 0.$$

Since ${}_SS$ is a cogenerator, it is immediate by adjointness that the following sequence is exact.

$$0 \to S\otimes_Q X \to S\otimes_Q Y.$$

Thus S_Q is flat. This completes the proof.

References

[1] Ginn, S.M. and Moss, P.B.: A decomposition theorem for noetherian orders in artinian rings, Bull. London Math. Soc. 9 (1977) 177–181.

[2] Lenagan, T.H.: Noetherian rings with Krull dimension one, J. London Math. Soc. (2) 15 (1977) 43–47.

[3] Lenagan, T. H.: Artinian quotient rings of Macaulay rings, Lecture Notes in Math. 545, Springer, Berlin (1976).

[4] Rutter, E. A.: A characterization of QF-3 rings, Pacific J. Math. 51 (1974) 533–536.

[5] Sato, H.: On localizations of a 1-Gorenstein ring, Sci. Rep. Tokyo Kyoiku Daigaku A. 13 (1977) 188–193.

[6] Sato, H.: On maximal quotient rings of QF-3 1-Gorenstein rings with zero socle, to appear in Osaka J. Math.

[7] Stenström, B.: Rings of quotients, Grundlehren Wissenschaften 217, Springer, Berlin (1975).

[8] Sumioka, T.: On non-singular QF-3′ rings with injective dimension ≤ 1, Osaka J. Math. 15 (1978) 1–11.

Wakayama University