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POSITIVELY CURVED COMPLEX SUBMANIFOLDS IMMERSED IN A COMPLEX PROJECTIVE SPACE IV

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Introduction

Let $P_m(C)$ be an m-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1, and let M be an n-dimensional compact Kaehler submanifold immersed in $P_m(C)$. It is natural to conjecture the following:

If the holomorphic sectional curvature of $M \geqq 1/2$, *then* M *is a totally geodesic* $P_n(C)$, *a Veronese* $P_n(C)$, *a Segre submanifold* $P_k(C)\times P_{n-k}(C)$ *or a Hermitian symmetric space of rank* 2 *with parallel second fundamental form.*

To prove this conjecture, in view of Theorem 7.4 in [3] it is sufficient to show that the second fundamental form is parallel. Although several partial results are known ([2], [4], [5] etc.), it seems to be difficult to get the complete answer by means of conventional methods.

The purpose of this paper is to prove another partial result that supports the above conjecture.

It is well-known (cf. [1]) that the curvature tensor defines a linear transformation on the space of complex symmetric 2-tensors at each point. More precisely, let $R^a{}_{bc\bar{d}}$ be local components of the curvature tensor of M and let ξ^{ab} be local components of a complex symmetric 2-tensor. The transformation difined by $\xi^{ab} \longrightarrow \sum R^a{}_{bc}{}^d\xi^{bc}$ is called the *holomorphic curvature transformation* or the *holomorphic curvature operator*. It follows from the equation of Gauss that the eigenvalues of the holomorphic curvature operator $\leqq 1$. It is easily seen that if all eigenvalues of the holomorphic curvature operator $\geqq\delta$, then the holomorphic sectional curvature $\geqq\delta$ and the holomorphic bisectional curvature $\geqq\delta/2$. But the converse is not true. Moreover, the holomorphic curvature operator is proportional to the identity if and only if the holomorphic sectional curvature is constant.

We shall prove the following.

THEOREM. *Let* M *be an n-dimensional compact Kaehler submanifold immersed in* $P_m(C)$. *If all eigenvalues of the holomorphic curvature operator* $\geqq 1/2$, *then* M

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is either a totally geodesic $P_n(C)$ or a Veronese $P_n(C)$.

Although our theorem is weaker than the above conjecture, it is best possible in the sense that the lower bound for the eigenvalues cannot be lowered any more.

Proof of Theorem.

We use the same notation as in [4] and [5] unless otherwise stated. It is well-known (cf. [4]) that the second fundamental form of the immersion satisfies a differential equation of Simons type:

(1) $$\frac{1}{2}\Delta\|\sigma\|^2=\|\nabla'\sigma\|^2-8\,\mathrm{tr}(\Sigma A_\alpha^2)^2-\Sigma(\mathrm{tr}A_\lambda A_\mu)^2+\frac{n+2}{2}\|\sigma\|^2 .$$

It is an easy fact that

(2) $$8\,\mathrm{tr}(\Sigma A_\alpha^2)^2\leqq(n+1)\,\Sigma\,(\mathrm{tr}\,A_\lambda A_\mu)^2$$

with equality if and only if M is of constant holomorphic sectional curvature (cf. Lemma 2 in [5]).
Moreover it is easily seen that

(3) $$\Sigma\,(\mathrm{tr}A_\lambda A_\mu)^2=\|\sigma\|^2-\rho+\frac{1}{2}\|R\|^2 .$$

On the other hand, it follows from the equation of Gauss that

(4) $$\rho-\frac{1}{2}\|R\|^2=\Sigma\,R^{a\bar{b}c\bar{d}}k^\alpha_{\bar{a}d}\bar{k}^\alpha_{\bar{b}c} .$$

Since the eigenvalues of the holomorphic curvature operator $\geqq 1/2$, we have

$$\Sigma\,R_{\bar{a}bc\bar{d}}\xi^{bc}\bar{\xi}^{ad}\geqq\frac{1}{2}\,\Sigma\,\xi_{ab}\xi^{ab}$$

for any symmetric (ξ_{ab}).
Thus, in particular, we have

(5) $$\Sigma\,R^{a\bar{b}c\bar{d}}k^\alpha_{\bar{a}d}\bar{k}^\alpha_{\bar{b}c}\geqq\frac{1}{2}\|\sigma\|^2 .$$

From (1), (2), (3), (4) and (5), it follows that

$$\frac{1}{2}\Delta\|\sigma\|^2\geqq\|\nabla'\sigma\|^2\geqq 0 ,$$

and hence, $\|\sigma\|$ is a constant and

(6) $$\nabla'\sigma=0 .$$

Thus all inequalities are reduced to equalities.

In particular, equality holds in (2), and hence M is of constant holomorphic sectional curvature. This, together with (6), implies that M is either a totally geodesic $P_n(C)$ or a Veronese $P_n(C)$ (cf. Theorem 4.4 in [4]).

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