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A CHARACTERIZATION OF THE R-ELEMENTARY GROUPS AT 2 WITH CHARACTERS OF SCHUR INDEX 2 OVER R

By

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Let H be a group. We call H an $\boldsymbol{R}$-elementary group at 2 if

(i) H is a semi-direct product $P\langle a\rangle$ of a 2-group P and a cyclic $2'$-group $\langle a\rangle$, and

(ii) For each $x\in P$, $xax^{-1}=a$ or a^{-1}.

In order to study the representations of finite groups over $\boldsymbol{R}$, there are some approachs. Our approach is based on the Brauer-Witt theorem ([4], p. 31). By the Brauer-Witt theorem, if χ is an irreducible complex character of Schur index 2 over $\boldsymbol{R}$ for a finite group G, then there exist a subgroup H of G which is $\boldsymbol{R}$-elementary at 2, and an irreducible complex character μ of H of Schur index 2 over $\boldsymbol{R}$ such that $(\chi_H, \mu)\neq 0 \pmod 2$. Therefore it is important to study the representations of $\boldsymbol{R}$-elementary groups at 2 over $\boldsymbol{R}$. The purpose of this paper is to give necessary and sufficient conditions in group theoretical terms for the existence of the characters of Schur index 2 over $\boldsymbol{R}$ in the case where groups are $\boldsymbol{R}$-elementary groups at 2.

First, using the Witt-Roquette theorem for p-groups, we will determine the 2-groups with characters of Schur index 2. Secondly, we will study the representations of an $\boldsymbol{R}$-elementary group H with non-trivial $2'$-group $\langle a\rangle$. In the case where $C_H(a)$ has an abelian Sylow 2-group, the representations of such group over $\boldsymbol{R}$ were studied by Gow in [3]. Let G be a finite group and let a be an element of G. Assume that $C_G(a)$ has an abelian Sylow 2-subgroup P and that there exists an element x in G satisfying $xax^{-1}=a^{-1}$, but $yay^{-1}\neq a^{-1}$ for any involution y in G. Gow showed that in this case G has a complex irreducible character of Schur index 2 over $\boldsymbol{R}$. To prove this theorem Gow used essentially the fact that complex irreducible characters of $P\langle a\rangle$ are linear. So we can not use Gow's method in the case where $C_G(a)$ has a non-abelian Sylow 2-subgroup. However for $\boldsymbol{R}$-elementary groups at 2 we can characterize the groups with real-valued complex irreducible characters of Schur index 2 over $\boldsymbol{R}$ and the groups with non-trivial real-valued

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complex irreducible characters of Schur index 1 over $\boldsymbol{R}$.

In this paper we mean by an irreducible character a complex irreducible character. As usual $\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}$ denote the rational number field, the real number field and the complex number field respectively.

§1. Preliminary lemmas

Let G be a finite group and let χ be an irreducible character of G, and let $m_F(\chi)$ denote the Schur index of χ over a field F. We let $F(\chi)$ denote the field generated over F by the algebraic numbers $\chi(g), g \in G$. Let $\bar{\chi}$ denote the complex conjugate character of χ. Let H be a subgroup of G. We denote by χ_H the restriction of χ to H. For a character μ of H we denote the character of G induced from μ by μ^G. Frobenius and Schur ([2], p. 21) studied the number $\nu(\chi) = |G|^{-1} \sum_{g \in G} \chi(g^2)$ and showed the following remarkable result.

(1.1.) (Frobenius-Schur) *Let G be a finite group and let χ be an irreducible character of G. Then we have*

(i) $\nu(\chi)=1$ *if and only if* $R(\chi)=R$ *and* $m_R(\chi)=1$,

(ii) $\nu(\chi)=-1$ *if and only if* $R(\chi)=R$ *and* $m_R(\chi)=2$,

(iii) $\nu(\chi)=0$ *if and only if* $R(\chi)=C$.

The character χ is called the character of the first kind (respectively, the second kind, the third kind), if $\nu(\chi)=1$ (respectively, $\nu(\chi)=-1$, $\nu(\chi)=0$).

The following well known theorem ([2], p. 73) plays an essential part in the study of representations of p-groups.

(1.2) (Witt-Roquette) *Let P be a p-group. Let F be a field of characteristic* 0. *Suppose that one of the following hypotheses is satisfied*

(i) $p \neq 2$

(ii) $p=2$ *and* $i=\sqrt{-1} \in F$

(iii) $p=2$ *and P does not contain a cyclic subgroup of index* 2.

Then if χ is a nonlinear irreducible faithful character of P there exist $P_0 \lhd P$ and a character ζ of P_0 such that $|P:P_0|=p$, $\chi=\zeta^P$ and $F(\chi)=F(\zeta)$.

REMARK. In the case (iii), if $F(\chi) \not\ni i$, the character ζ is nonlinear.

Let G be a group, let H be a subgroup of G and let ζ be a character of H. Even if ζ is an irreducible character of H, the character ζ^G is not always irreducible. However in the case where ζ is a linear character, using the Mackey decomposition theorem ([2], p. 51), we have following

LEMMA 1.3. *Let G be a finite group, let H be a subgroup of G, and let ζ be a linear character of H. Put $\chi=\zeta^G$. Then χ is not an irreducible character of G if and only if there exists an element x of $G-H$ such that $xhx^{-1}h^{-1}\in \mathrm{Ker}\,\zeta$ for all $h\in H^x\cap H$, where $\mathrm{Ker}\,\zeta=\{h\in H\mid \zeta(h)=\zeta(1)\}$.*

PROOF. Let $Hx_1H(x_1=1)$, $Hx_2H, \cdots, Hx_nH$ be all the (H, H) double cosets in G. By the Mackey decomposition theorem,

$$(\chi, \chi)_G=(\zeta^G, \zeta^G)_G=\sum_{i=1}^{n}((\zeta^{x_i})_{H^{x_i}\cap H}, \zeta_{H^{x_i}\cap H})_{H^{x_i}\cap H}$$

$$=1+\sum_{i=2}^{n}((\zeta^{x_i})_{H^{x_i}\cap H}, \zeta_{H^{x_i}\cap H})_{H^{x_i}\cap H}\,.$$

Therefore χ is not irreducible if and only if for some x_i $(2\leqq i\leqq n)$ $((\zeta^{x_i})_{H^{x_i}\cap H}, \zeta_{H^{x_i}\cap H})_{H^{x_i}\cap H}\neq 0$. Since ζ is a linear character, $((\zeta^{x_i})_{H^{x_i}\cap H}, \zeta_{H^{x_i}\cap H})_{H^{x_i}\cap H}\neq 0$ if and only if $\zeta(x_ihx_i^{-1})=\zeta^{x_i}(h)=\zeta(h)$ for all $h\in H^{x_i}\cap H$. And the condition $\zeta(x_ihx_i^{-1})=\zeta(h)$ means $x_ihx_i^{-1}h^{-1}\in Ker\,\zeta$, because ζ is a linear character. Hence χ is not irreducible if and only if there exists an element x of $G-H$ such that $xhx^{-1}h^{-1}\in \mathrm{Ker}\,\zeta$ for all $h\in H^x\cap H$.

LEMMA 1.4. *Let G be a finite group, let H be a subgroup of G and let ζ be a character of H. Put $\chi=\zeta^G$. Then $\mathrm{Ker}\,\chi=\bigcap_{g\in G}(\mathrm{Ker}\,\zeta)^g$.*

PROOF. Let x be an element of $\bigcap_{g\in G}(\mathrm{Ker}\,\zeta)^g$. Since $g^{-1}xg\in\mathrm{Ker}\,\zeta$ for all $g\in G$, we have $\chi(x)=\zeta^G(x)=\zeta^G(1)$. Therefore $x\in\mathrm{Ker}\,\chi$. Conversely, we assume that $x\in\mathrm{Ker}\,\chi$. Let $\{g_1, \cdots, g_n\}$ be a set of (left) representatives of G/H in G. We may assume that $x\in H^{g_i}$ for $i\leqq s$ and $x\notin H^{g_i}$ for $i>s$. Let m be the order of x. As is well known, $\zeta^{g_i}(x)$ is a sum of m-th roots of unity ε_{ij} $(1\leqq j\leqq \zeta(1))$ if $i\leqq s$ and $\zeta^{g_i}(x)=0$ if $i>s$. Thus $\chi(x)=\sum_{i=1}^{s}\sum_{j=1}^{\zeta(1)}\varepsilon_{ij}$. Since $\chi(1)=\sum_{i=1}^{s}\sum_{j=1}^{\zeta(1)}\varepsilon_{ij}$ if and only if $\zeta(1)s=\chi(1)$ and $\varepsilon_{ij}=1$ for all (i, j), we have that $\zeta^{g_i}(x)=\zeta(1)=\zeta^{g_i}(1)$ for all i, and this means that $x\in\bigcap_{g\in G}(\mathrm{Ker}\,\zeta)^g$.

§ 2. 2-groups

Let G be the quaternion group of order 8. It is well known that the ordinary quaternion algebra Λ over $\boldsymbol{Q}$ appears in $\boldsymbol{Q}G$ as a simple component. Let F be a field of characteristic 0. The 2-groups with characters of Schur index 2 over F can be determined in the following

THEOREM 2.1. *Let F be a field of characteristic 0. Let P be a finite 2-group. Then there is a faithful irreducible character of P of Schur index 2 over F if and* *only if P and F satisfy the following conditions:*

(i) *There exist subgroups $Q \supset A \supset K$ of P such that K is a normal subgroup of Q and that A/K is a cyclic subgroup of Q/K of index 2.*

(ii) *Q/K is a generalized quaternion group of order $2^{n+1} \geqq 8$.*

(iii) *If x is an element of $P-A$, there exists an element a in $A^x \cap A$ such that $xax^{-1}a^{-1} \notin K$.*

(iv) $\bigcap_{x \in P} K^x = 1$.

(v) *$F(\varepsilon_{2^n}+\varepsilon_{2^n}^{-1}) \otimes_Q \Lambda$ is a division ring, where ε_{2^n} is a primitive 2^n-th root of unity.*

PROOF. We assume that there exists a faithful irreducible character χ of P of Schur index 2 over F. Since $m_F(\chi)=2$, χ is not linear. If P does not contain a cyclic subgroup of index 2, by (1.2) there exist $P_0 \lhd P$ and a character ζ_0 of P_0 such that $|P:P_0|=2$, $\chi=\zeta_0{}^P$ and $F(\chi)=F(\zeta_0)$. If ζ_0 is realizable in $F(\zeta_0)$, then $\chi=\zeta_0{}^P$ is realizable in $F(\chi)=F(\zeta_0)$, which contradicts the assumption $m_F(\chi)=2$. Therefore $m_F(\zeta_0) \neq 1$ and ζ_0 is not linear. So we can define inductively a subgroup P_i of P and a character ζ_i of P_i $(i=0, 1, 2, \cdots, t)$ such that $|P_i:P_{i+1}|=2$, $\zeta_i=\zeta_{i+1}{}^{P_i}$, $F(\zeta_i)=F(\zeta_{i+1})$, $m_F(\zeta_t) \neq 1$ and $P_t/\mathrm{Ker}\,\zeta_t$ contains a cyclic subgroup of index 2. We denote P_t, $\mathrm{Ker}\,\zeta_t$, ζ_t by Q, K, ζ, respectively. And if we denote by A the inverse image of the cyclic subgroup of Q/K of index 2 in Q, then Q, A, K satisfy the condition (i). As is well known, $\boldsymbol{Q}$ is a splitting field of Q/K if Q/K is not a generalized quaternion group. In the case where Q/K is a generalized quaternion group of order $2^{n+1} \geqq 8$, the faithful irreducible character ζ of Q/K corresponds to the simple component $F(\zeta) \otimes_Q \Lambda$ of $F(\zeta)[Q/K]$ and $m_F(\zeta) \neq 1$ if and only if $F(\zeta) \otimes_Q \Lambda$ is a division algebra. Thus we have (ii) and (v), because $F(\zeta)=F(\varepsilon_{2^n}+\varepsilon_{2^n}^{-1})$. The faithful irreducible character ζ of Q/K is induced by a faithful linear character λ of A/K, and $\chi=\zeta^P=\lambda^P$. Therefore by Lemma 1.3 the condition (iii) is satisfied. Moreover since χ is a faithful irreducible character of G, from Lemma 1.4 the condition (iv) is satisfied.

Conversely we assume that the conditions (i), (ii), (iii), (iv) and (v) are satisfied. Let λ be a faithful linear character of A/K. We denote λ^Q, λ^P by ζ, χ respectively. Then by Lemmas 1.3 and 1.4 χ is a faithful irreducible character of P. The assumptions (ii) and (v) mean $m_F(\zeta)=2$. If χ is realizable in $F(\chi) \subset F(\zeta)$, we have $m_F(\zeta) | (\zeta, \chi)_Q = 1$, which contradicts the fact $m_F(\zeta)=2$. Using the fact that $m_F(\chi)|2$, we have $m_F(\chi)=2$.

REMARK. In the case where F is an algebraic number field, the necessary and sufficient conditions for $F(\varepsilon_{2^n}+\varepsilon_{2^n}^{-1}) \otimes_Q \Lambda$ to be a division ring were given [1] by Fein, Gordon and Smith.

When $F=\boldsymbol{R}$, we have following

THEOREM 2.2. *Let P be a finite 2-group. Then there is a nonlinear faithful irreducible character of P of the first kind (respectively, the second kind) if and only if P satisfies the following conditions* (i), (ii), (iii) *and* (iv).

(i) *There exist subgroups $Q\supset A\supset K$ of P such that K is a normal subgroup of Q and that A/K is a cyclic subgroup of Q/K of index* 2.

(ii) *Q/K is a dihedral group (respectively, a generalized quaternion group) of order $2^{n+1}\geqq 8$.*

(iii) *If x is an element of $P-A$, there exists an element a in $A^x\cap A$ such that $xax^{-1}a^{-1}\notin K$.*

(iv) $\bigcap_{x\in P} K^x=1$.

PROOF. Let χ be a faithful irreducible character of P. We assume that $\boldsymbol{R}(\chi)=\boldsymbol{R}$. By (1.2) there are a subgroup P_i of P and a character ζ_i of P_i $(i=0,1,2,\cdots,t)$ such that $|P_i:P_{i+1}|=2$, $\zeta_i=\zeta_{i+1}{}^{P_i}$, $\boldsymbol{R}(\zeta_i)=\boldsymbol{R}(\zeta_{i+1})$ and $P_t/\mathrm{Ker}\,\zeta_t$ contains a cyclic subgroup of index 2. In the case where χ is of the second kind, the theorem holds by Theorem 2.1, because $\boldsymbol{R}(\varepsilon_{2^n}+\varepsilon_{2^n}^{-1})\otimes_Q \Lambda=\boldsymbol{R}\otimes_Q\Lambda$ is the Hamilton's quaternion field. Hence we can assume that χ is of the first kind. We denote P_t, $\mathrm{Ker}\,\zeta_t$, ζ_t by Q, K, ζ respectively. Let A be the inverse image of the cyclic subgroup of Q/K of index 2 in Q. Then the conditions (i) and (iv) are satisfied. Since $\boldsymbol{R}(\zeta)=\boldsymbol{R}(\chi)=\boldsymbol{R}$, Q/K is a dihedral group or a generalized quaternion group. If Q/K is a generalized quaternion group, χ is of the second kind. Hence Q/K is a dihedral group. Since the faithful irreducible character ζ of Q/K is induced by a faithful linear character of A/K, by Lemma 1.3 the condition (iii) is satisfied. Conversely we assume that the conditions (i), (ii), (iii) and (iv) are satisfied. Further assume that Q/K is a dihedral group. Let λ be a faithful linear character of A/K. We denote λ^P by χ. By Lemmas 1.3 and 1.4 χ is a faithful irreducible character of P. Since a faithful irreducible character λ^Q of Q/K is realizable as a character of a $\boldsymbol{R}[Q/K]-$module, χ is realizable as a character of a $\boldsymbol{R}P$-module. Hence χ is of the first kind.

Now we will prove some lemmas used in the following section.

LEMMA 2.3. *Let P be a finite 2-group, let Q be a subgroup of P of index 2, let $\{1,g\}$ be a set of representatives of P/Q, and let μ be an irreducible character of Q with $\boldsymbol{R}(\mu^P)=\boldsymbol{R}$. Further we assume that $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)=1$ (respectively, -1).*

If μ^P is irreducible, then

(1) *μ is of the third kind and μ^P is of the first kind (respectively, the second kind).*

If μ^P is not irreducible, the following conditions are satisfied

(2) *There exist characters χ and χ', $\chi\neq\chi'$, of P such that $\mu^P=\chi+\chi'$ and that μ, χ and χ' satisfy one of the following conditions*

(i) *μ, χ and χ' are of the first kind (respectively, the second kind).*

(ii) *μ is of the second kind (respectively, the first kind), and χ and χ' are of the third kind.*

Conversely if one of the conditions (1) *and* (2) *is satisfied, then we have* $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)=1$ *(respectively,* -1).

PROOF. Since $|P:Q|=2$, we obtain $h^2\in Q$ for all $h\in P$, which implies

$$\begin{aligned}|P|^{-1}&\sum_{h\in P}\mu^P(h^2)\\ &=|P|^{-1}\sum_{h\in P}(\mu+\mu^g)(h^2)\\ &=|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)(h^2)+|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2).\end{aligned}$$

Now we assume that μ^P is an irreducible character of P and $\boldsymbol{R}(\mu)=\boldsymbol{R}$. We will show that this assumption induces a contradiction. Remarking that μ is of the first kind (respectively, the second kind) if and only if μ^g is of the first kind (respectively, the second kind), we have $|Q|^{-1}\sum_{h\in Q}\mu(h^2)=|Q|^{-1}\sum_{h\in Q}\mu^g(h^2)$ by (1.1). Thus $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)(h^2)=\pm 1$. By the assumption $\boldsymbol{R}(\mu^P)=\boldsymbol{R}$ we obtain $|P|^{-1}\sum_{h\in P}\mu^P(h^2)=\pm 1$. This contradicts the assumption that $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)=\pm 1$. Therefore we have that μ is of the third kind if μ^P is irreducible. Moreover by (1.1) $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)(h^2)=0$, and so $|P|^{-1}\sum_{h\in P}\mu^P(h^2)=|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)$. Therefore in the case where μ^P is irreducible, μ^P is of the first kind (respectively, the second kind) if and only if $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)=1$ (respectively, -1).

Next we assume that μ^P is not irreducible. We may decompose μ^P into a sum of irreducible characters of P, $\mu^P=\chi+\chi'$. Using the fact $1=(\chi_Q, \mu)_Q=(\chi, \mu^P)_P$ we obtain $(\chi, \chi')=0$. Since $2\mu=\chi_Q+\chi'_Q=(\mu^P)_Q=\mu+\mu^g$, we have $\mu=\mu^g$. Hence

$$\begin{aligned}|P|^{-1}&\sum_{h\in P}\chi(h^2)\\ &=|P|^{-1}\sum_{h\in P}\chi'(h^2)\\ &=|P|^{-1}\sum_{h\in Q}\mu(h^2)+|P|^{-1}\sum_{h\in Q}\mu((gh)^2).\end{aligned}$$

On the other hand by the assumption

$$|P|^{-1}\sum_{h\in Q}\mu((gh)^2)=2^{-1}(|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2))=\pm 2^{-1}.$$

If μ is of the third kind, we have $|P|^{-1}\sum_{h\in P}\chi(h^2)=\pm 2^{-1}$, which contradicts the Frobenius- Schur theorem (1.1). Therefore μ is of the first kind or of the second kind. In the case where μ is of the first kind, $|P|^{-1}\sum_{h\in P}\chi(h^2)=|P|^{-1}\sum_{h\in P}\chi'(h^2)=1$ (respectively, 0) if and only if $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)=1$ (respectively, -1). In the case where μ is of the second kind, $|P|^{-1}\sum_{h\in P}\chi(h^2)=|P|^{-1}\sum_{h\in P}\chi'(h^2)=0$ (respectively, -1) if and only if $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)=1$ (respectively, -1). Hence the lemma was proved.

LEMMA 2.4. *Let ε_k be a primitive 2^{n_k}-th root of unity* ($k=1, \cdots, m$). *Assume that* $n_k\geqq 2$ and $\sum_{k=1}^{m}\varepsilon_k\in\boldsymbol{Q}$. *Then we have* $\sum_{k=1}^{m}\varepsilon_k=0$.

PROOF. This is easy, therefore we omit it.

LEMMA 2.5. *Let P be a finite 2-group, let Q be a subgroup of P of index 2, let $\{1, g\}$ be a set of representatives of P/Q and let μ be a character of Q. Let T be a subgroup of Q, let λ be a linear character of T and let $X=\{h_1, \cdots, h_m\}$ be a set of (left) representatives of Q/T. We put*

$$A=\{(h_i, h)\in X\times Q \mid h_i(gh)^2h_i^{-1}\in T-Ker\,\lambda,\ h_i(gh)^4h_i^{-1}\in \mathrm{Ker}\,\lambda\},$$

$$B=\{(h_i, h)\in X\times Q \mid h_i(gh)^2h_i^{-1}\in \mathrm{Ker}\,\lambda\},$$

$$\alpha=|A| \quad and \quad \beta=|B|.$$

Assume that $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)\in\boldsymbol{Q}$.

(1) *If $\mu=\mu^g$ and $\lambda^Q=\mu$, then we have*

$$|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)=|Q|^{-1}(\beta-\alpha).$$

(2) *If $\lambda^Q=(\mu^P)_Q$, then we have*

$$|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)=|P|^{-1}(\beta-\alpha).$$

PROOF. First we assume that $\mu=\mu^g$ and $\lambda^Q=\mu$. Then $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)=|Q|^{-1}\sum_{h\in Q}\mu((gh)^2)=|Q|^{-1}\sum_{h\in Q}\lambda^Q((gh)^2)=|Q|^{-1}\sum_{h\in Q}\sum_{i=1}^{m}\lambda(h_i(gh)^2h_i^{-1})$, where $\lambda(h_i(gh)^2h_i^{-1})=0$ if $h_i(gh)^2h_i^{-1}\notin T$. If $h_i(gh)^2h_i^{-1}\in T$ and the order of $h_i(gh)^2h_i^{-1}\,\mathrm{Ker}\,\lambda$ in $T/\mathrm{Ker}\,\lambda$ is 2^n, then $\lambda(h_i(gh)^2h_i^{-1})$ is a primitive 2^n-th root of unity. Thus by the assumption that $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)\in\boldsymbol{Q}$ and by Lemma 2.4 it holds that

$$\begin{aligned}&|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)\\&=|Q|^{-1}\sum_{(h,h_i)\in A}\lambda(h_i(gh)^2h_i^{-1})+|Q|^{-1}\sum_{(h,h_i)\in B}\lambda(h_i(gh)^2h_i^{-1})\\&=|Q|^{-1}(\beta-\alpha).\end{aligned}$$

Secondly we assume $\lambda^Q=(\mu^P)_Q$. Then along the same line as above we obtain

$$\begin{aligned}
&|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)\\
&\quad=|P|^{-1}\sum_{h\in Q}\lambda^Q((gh)^2)\\
&\quad=|P|^{-1}\sum_{h\in Q}\sum_{i=1}^{m}\lambda(h_i(gh)^2h_i^{-1})\\
&\quad=|P|^{-1}(\beta-\alpha).
\end{aligned}$$

§ 3. *R*-elementary groups at 2

Let H be an $\boldsymbol{R}$-elementary group at 2, i. e.

(i) H is a semi-direct product $P\langle a\rangle$ of a 2-group P and a cyclic $2'$-group $\langle a\rangle$, and

(ii) For each $x\in P$, $xax^{-1}=a$ or a^{-1}.

We denote $\langle a\rangle$, $C_P(a)$ by A, Q respectively. If H is a direct product $P\times A$ of P and A, an irreducible character χ of H is decomposed into a product of an irreducible character μ of P and an irreducible character τ of A, $\chi=\mu\tau$. In this case χ is of the first kind (respectively, the second kind) if and only if μ is of the first kind (respectively, the second kind) and $\tau=1_A$. But in § 2 we already studied when P has irreducible characters of the first kind (respectively, the second kind). So in this section we assume that $A\neq 1$ and $|P:Q|=2$, and we only study the irreducible characters of H which are faithful on A. We denote by $\{1, g\}$ a set of representatives of P/Q.

Let χ be an irreducible character of H. We assume that $\boldsymbol{R}(\chi)=\boldsymbol{R}$ and $\operatorname{Ker}\chi\cap A=1$. Let χ_1 be an irreducible character of QA satisfying $(\chi_{QA}, \chi_1)\neq 0$. Then we can decompose χ_1 into a product of an irreducible character μ of Q and an irreducible character τ of A, $\chi_1=\mu\tau$. We assume that $\operatorname{Ker}\tau\neq 1$. If x is an element of $\operatorname{Ker}\tau$, then we have

$$\begin{aligned}
\chi_1{}^H(x)&=\mu(1)\tau(x)+\mu(1)\tau(gxg^{-1})=\mu(1)\tau(x)+\mu(1)\tau(x^{-1})\\
&=\mu(1)\tau(1)+\mu(1)\tau(1)=\chi_1{}^H(1).
\end{aligned}$$

Therefore $(\chi_1{}^H)_A$ is not faithful, which contradicts the fact that χ_A is faithful. Hence τ is faithful. Since A is not trivial, $\boldsymbol{R}(\tau)=\boldsymbol{C}$, which means $\boldsymbol{R}(\mu\tau)=\boldsymbol{C}$. Since $\tau\neq\bar{\tau}=\tau^g$, $(\mu\tau, (\mu\tau)^g)=1$. By the Frobenius reciprocity theorem $((\mu\tau)^H, (\mu\tau)^H)_H=(\mu\tau+(\mu\tau)^g, \mu\tau)_{QA}=1$, and so $\chi=(\mu\tau)^H$. Moreover we have $\mu^g=\bar{\mu}$, because $\boldsymbol{R}(\chi)=\boldsymbol{R}$.

In this section, for an irreducible character χ of H satisfying $\operatorname{Ker}\chi\cap A=1$ and $\boldsymbol{R}(\chi)=\boldsymbol{R}$, we mean by μ, τ above μ, τ respectively.

LEMMA 3.1. (1) χ *is of the first kind if and only if*

$$|P|^{-1}\sum_{h\in Q}(\mu+\mu^{g})((gh)^{2})=1.$$

(2) χ *is of the second kind if and only if*

$$|P|^{-1}\sum_{h\in Q}(\mu+\mu^{g})((gh)^{2})=-1.$$

PROOF. Since $\boldsymbol{R}(\mu\tau)=\boldsymbol{R}((\mu\tau)^{g})=\boldsymbol{C}$, by (1.1) we have

$$|QA|^{-1}\sum_{h\in QA}\mu\tau(h^{2})=|QA|^{-1}\sum_{h\in QA}(\mu\tau)^{g}(h^{2})=0.$$

Thus

$$\begin{aligned}
&|H|^{-1}\sum_{h\in H}\chi(h^{2})\\
&\quad=|H|^{-1}\sum_{h\in H}(\mu\tau+(\mu\tau)^{g})(h^{2})\\
&\quad=|H|^{-1}\sum_{h\in QA}(\mu\tau+(\mu\tau)^{g})(h^{2})+|H|^{-1}\sum_{h\in QA}(\mu\tau+(\mu\tau)^{g})((gh)^{2})\\
&\quad=|H|^{-1}\sum_{h\in QA}(\mu\tau+(\mu\tau)^{g})((gh)^{2})\\
&\quad=|H|^{-1}\sum_{h\in Q}\sum_{k\in A}(\mu\tau+(\mu\tau)^{g})((ghk)^{2})\\
&\quad=|H|^{-1}\sum_{h\in Q}\sum_{k\in A}(\mu\tau+(\mu\tau)^{g})((gh)^{2})\\
&\quad=|P|^{-1}\sum_{h\in Q}(\mu\tau+(\mu\tau)^{g})((gh)^{2})\\
&\quad=|P|^{-1}\sum_{h\in Q}(\mu+\mu^{g})((gh)^{2}).
\end{aligned}$$

Therefore by (1.1) Lemma 3.1 holds.

Now we will characterize $\boldsymbol{R}$-elementary groups at 2 with characters of the first kind and $\boldsymbol{R}$-elementary groups at 2 with characters of the second kind.

First we study in the case where $\boldsymbol{R}(\mu)=\boldsymbol{R}$ and μ is a linear character of Q.

PROPOSITION 3.2. *H has an irreducible character χ of the first kind (respectively, the second kind) such that* $\operatorname{Ker}\chi\cap A=1$, $\boldsymbol{R}(\mu)=\boldsymbol{R}$ *and μ is linear, if and only if H satisfies the following condition* (A).

(A) *There exists a subgroup K of Q such that $K\triangleleft P$ and P/K is an elementary abelian group of order* $\leqq 4$ *(respectively, a cyclic group of order* 4).

PROOF. If $\boldsymbol{R}(\mu)=\boldsymbol{R}$, then we have $\mu^{g}=\bar{\mu}=\mu$. We denote $\operatorname{Ker}\mu$ by K. For $x\in K$ $\mu(gxg^{-1})=\mu^{g}(x)=\mu(x)=\mu(1)$, which implies $g\in N_{P}(K)$. Since $N_{P}(K)\supset Q$, we obtain that $P\triangleright K$. Because μ is linear and $\boldsymbol{R}(\mu)=\boldsymbol{R}$, it holds that $|Q/K|\leqq 2$. Thus $|P/K|\leqq 4$. From the assumption $\boldsymbol{R}(\chi)=\boldsymbol{R}$ we have $|P|^{-1}\sum_{h\in Q}(\mu+\mu^{g})((gh)^{2})=\pm 1$ by Lemma 3.1. Since $\boldsymbol{R}(\mu)=\boldsymbol{R}$, by Lemma 2.3 there exist characters ξ and ξ', $\xi\neq\xi'$, of P such that $\mu^P=\xi+\xi'$. Moreover by Lemma 2.3 ξ and ξ' are of the first kind (respectively, the third kind) if χ is of the first kind (respectively, the second kind), because μ is of the first kind. First we will show that $\mathrm{Ker}\,\xi\cap Ker\,\xi'=K$. In fact $(\xi+\xi')_Q=(\mu^P)_Q=2\mu$, and so $\mathrm{Ker}\,\xi\cap\mathrm{Ker}\,\xi'\supset K$. Let $x\in\mathrm{Ker}\,\xi\cap\mathrm{Ker}\,\xi'$. If $x\in P-Q$, we have $0=\mu^P(x)=\xi(x)+\xi'(x)=\xi(1)+\xi'(1)$, which is impossible. Therefore $x\in Q$, and $2\mu(1)=\xi(1)+\xi'(1)=\xi(x)+\xi'(x)=2\mu(x)$, which implies $\mu(x)=\mu(1)$. Hence we have $x\in K$, and it holds that $\mathrm{Ker}\,\xi\cap\mathrm{Ker}\,\xi'=K$. Therefore we may assume that P/K is a subgroup of $P/\mathrm{Ker}\,\xi\times P/\mathrm{Ker}\,\xi'$. In the case where χ is of the first kind, we have $\boldsymbol{R}(\xi)=\boldsymbol{R}(\xi')=\boldsymbol{R}$. Thus $|P/\mathrm{Ker}\,\xi|$, $|P/\mathrm{Ker}\,\xi'|\leqq 2$, and P/K is an elementary abelian group of order $\leqq 4$. In the case where χ is of the second kind, we have $\boldsymbol{R}(\xi)=\boldsymbol{R}(\xi')=\boldsymbol{C}$. Since $K=\mathrm{Ker}\,\xi\cap\mathrm{Ker}\,\xi'\subset\mathrm{Ker}\,\xi$, we obtain $4\geqq|P/K|\geqq|P/\mathrm{Ker}\,\xi|\geqq 4$. Hence $|P/K|=4$ and $K=\mathrm{Ker}\,\xi$. Further $\boldsymbol{R}(\xi)=\boldsymbol{C}$ means that P/K is a cyclic group of order 4.

Conversely suppose that H satisfies the condition (A). Let μ be a faithful linear character of Q/K and let τ be a faithful linear character of A. We will prove that $(\mu\tau)^H$ is an irreducible character of H of the first kind (respectively, the second kind). First we assume that $Q=K$. Then $\mu=1_Q$, and $((1_Q\tau)^{PA})_{QA}=1_Q\tau+1_Q\bar{\tau}$, which implies that $(1_Q\tau)^{PA}$ is an irreducible character of H. We define the characters ξ and ξ' of P by $\xi(x)=\xi'(x)=1$ for $x\in Q$ and $\xi(x)=1$, $\xi'(x)=-1$ for $x\in P-Q$. Then $1_Q{}^P=\xi+\xi'$. Since ξ and ξ' are of the first kind, by Lemmas 2.3 and 3.1 $(1_Q\tau)^H$ is of the first kind. Secondly we assume that $|Q/K|=2$. In this case $((\mu\tau)^H)_{QA}=\mu\tau+\mu\bar{\tau}$, and $(\mu\tau)^H$ is an irreducible character of H. Since $\boldsymbol{R}((\mu\tau)^H)=\boldsymbol{R}$ and $\boldsymbol{R}(\mu)=\boldsymbol{R}$, by Lemma 2.3 there exist characters ξ and ξ', $\xi\neq\xi'$, of P such that $\mu^P=\xi+\xi'$. In the case where P/K is an elementary abelian group of order 4 both ξ and ξ' are of the first kind. So in this case, using Lemmas 2.3 and 3.1, we obtain that $(\mu\tau)^H$ is of the first kind. Next we assume that P/K is a cyclic group of order 4. We put $P/K=\langle y\rangle$. We define the linear characters ξ and ξ' of P/K by $\xi(y)=i$ and $\xi'(y)=-i$. Then we have $\mu^P=\xi+\xi'$. Since $\boldsymbol{R}(\xi)=\boldsymbol{R}(\xi')=\boldsymbol{C}$, by Lemmas 2.3 and 3.1 we conclude that $(\mu\tau)^H$ is of the second kind.

Secondly we study in the case where $\boldsymbol{R}(\mu)=\boldsymbol{R}$ and μ is a nonlinear character of Q.

PROPOSITION 3.3. *H has an irreducible character χ of the first kind (respectively, the second kind) such that* $\mathrm{Ker}\,\chi\cap A=1$, *$\boldsymbol{R}(\mu)=\boldsymbol{R}$ and μ is nonlinear, if and only if H satisfies the following condition* (B).

(B) *There exist subgroups T, K of Q which satisfy the following conditions*

(i) *$T\triangleright K$ and T/K is a cyclic group.*

(ii) *For $x\in Q-T$ there exists an element a of $T^x\cap T$ such that $xax^{-1}a^{-1}\notin K$.*

There exists an element y *of* $P-Q$ *which satisfies* $yby^{-1}b^{-1}\in K$ *for all* $b\in T^y\cap T$.

(iii) *Let* $X=\{h_1, \cdots, h_m\}$ *be a set of representatives of* Q/T. *We put*

$$C=\{(h_i, h)\in X\times Q \mid h_i(gh)^2h_i^{-1}\in T-K,\ h_i(gh)^4h_i^{-1}\in K\},$$

$$D=\{(h_i, h)\in X\times Q \mid h_i(gh)^2h_i^{-1}\in K\},$$

$$\gamma=|C| \quad and \quad \delta=|D|.$$

Then we have $|Q|^{-1}(\delta-\gamma)=1$ (*respectively,* -1).

PROOF. Since μ is nonlinear, by Theorem 2.2 there exist a subgroup S of Q and a character ζ of S such that $\boldsymbol{R}(\zeta)=\boldsymbol{R}$ and $\zeta^Q=\mu$. Moreover in the case where μ is of the first kind (respectively, the second kind) $S/\mathrm{Ker}\,\zeta$ is a dihedral group (respectively, a generalized quaternion group). Therefore $S/\mathrm{Ker}\,\zeta$ contains a cyclic group $T/\mathrm{Ker}\,\zeta$ of index 2 and ζ is induced by a faithful linear character λ of $T/\mathrm{Ker}\,\zeta$. We denote $\mathrm{Ker}\,\zeta$ by K. Since $\lambda^Q=\mu$ is irreducible but $\lambda^P=\mu^P$ is not irreducible, by Lemma 1.3 the condition (ii) is satisfied. From Lemma 2.5 we obtain (iii). So the condition (B) is satisfied.

Conversely we assume that the condition (B) is satisfied. Let λ be a faithful linear character of T/K and let τ be a faithful linear character of A. By Lemma 1.3 the condition (ii) means that λ^Q is irreducible but λ^P is not irreducible. We denote λ^Q by μ. Then $(\mu^P)_Q=\mu+\mu^g$ and $\mu=\mu^g$, because μ^P is not irreducible. Since $(\mu\tau)^g=\mu\bar{\tau}\neq\mu\tau$, $(\mu\tau)^H$ is irreducible. On the other hand by (1.1) $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)$ $=|P|^{-1}\sum_{h\in P}\mu^P(h^2)-|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)(h^2)\in\boldsymbol{Q}$. Therefore the assumption of Lemma 2.5 is satisfied, and the condition (iii) means $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)=1$ (respectively, -1). Using Lemma 3.1 we conclude that $(\mu\tau)^H$ is of the first kind (respectively, the second kind).

Thirdly we study in the case where $\boldsymbol{R}(\mu)=\boldsymbol{C}$.

PROPOSITION 3.4. *H has an irreducible character χ of the first kind (respectively, the second kind) such that* $\mathrm{Ker}\,\chi\cap A=1$ *and* $\boldsymbol{R}(\mu)=\boldsymbol{C}$, *if and only if H satisfies the following condition (C).*

(C) *There exist subgroups $S\supset T\supset K$ of P which satisfy the following conditions*

(i) *$S\triangleright K$, S/K is a dihedral group (respectively, a generalized quaternion group) and T/K is a cyclic subgroup of S/K of index 2.*

(ii) *For $x\in P-T$ there exists an element a of $T^x\cap T$ such that $xax^{-1}a^{-1}\notin K$.*

(iii) *One of the following conditions is satisfied*

(a) *$K\subset T\cap Q$ and $(S\cap Q)/K$ is a cyclic subgroup of S/K of index 2.*

(b) *$K\subset T\cap Q$ and $(S\cap Q)/K$ is not abelian, or $K\not\subset T\cap Q$. Let $X=\{h_1, \cdots, h_m\}$ be* *a set of representatives of* $Q/(T\cap Q)$. *We put*

$$C=\{(h_i, h)\in X\times Q \mid h_i(gh)^2h_i^{-1}\in T-K,\ h_i(gh)^4h_i^{-1}\in K\},$$

$$D=\{(h_i, h)\in X\times Q \mid h_i(gh)^2h_i^{-1}\in K\},$$

$$\gamma=|C| \quad and \quad \delta=|D|.$$

Then we have $|P|^{-1}(\delta-\gamma)=1$ (*respectively*, -1) *and* $|T:(T\cap Q)|=2$.

PROOF. Since $\boldsymbol{R}(\mu)=\boldsymbol{C}$ and $\mu^g=\bar{\mu}$, μ^P is a nonlinear irreducible character of P. Thus by Theorem 2.2 there exist a subgroup S and an irreducible character ζ of S such that $\zeta^P=\mu^P$, $\boldsymbol{R}(\zeta)=\boldsymbol{R}$ and $S/\mathrm{Ker}\,\zeta$ is a dihedral group (respectively, a generalized quaternion group) if μ^P is of the first kind (respectively, the second kind). Further applying Lemmas 2.3 and 3.1 we have that $S/\mathrm{Ker}\,\zeta$ is a dihedral group (respectively, a generalized quaternion group). Let T be the inverse image of the cyclic subgroup of $S/\mathrm{Ker}\,\zeta$ of index 2 in S. Then there exists a faithful linear character λ of $T/\mathrm{Ker}\,\zeta$ such that $\lambda^S=\zeta$. Since $\lambda^P=\zeta^P=\mu^P$ is irreducible, by Lemma 1.3 the condition (ii) is satisfied for $K=\mathrm{Ker}\,\zeta$. Thus if we put $K=\mathrm{Ker}\,\zeta$, then the conditions (i) and (ii) are satisfied.

To prove the condition (iii) first we assume that $K\subset T\cap Q$ and $(S\cap Q)/K$ is an abelian group. Under this assumption we will show that $(S\cap Q)/K$ is a cyclic subgroup of index 2. In the case where χ is of the first kind S/K is a dihedral group. We put $S/K=\langle x, y|x^{2^n}=1, y^2=1, y^{-1}xy=x^{-1}\rangle$, and we assume that $(S\cap Q)/K\cap\langle x\rangle=\langle x^2\rangle$. Since $(yx^i)^2=1$ and $(yx^i)^{-1}x(yx^i)=x^{-1}$, we may assume that $y\in(S\cap Q)/K$. Since $(S\cap Q)/K=\langle y, x^2\rangle$ is abelian, we have $n=2$. We define characters ξ and ξ' of $\langle y, x^2\rangle$ by $\xi(1)=1$, $\xi(x^2)=-1$, $\xi(y)=1$, $\xi(x^2y)=-1$, $\xi'(1)=1$, $\xi'(x^2)=-1$, $\xi'(y)=-1$, $\xi'(x^2y)=1$. Then $\zeta_{S\cap Q}=\xi+\xi'$ and $\xi^S=\zeta$. This means $\xi^P=(\xi^S)^P=\zeta^P=\mu^P$. Therefore $\xi^Q+\xi^{Qg}=(\xi^P)_Q=(\mu^P)_Q=\mu+\bar{\mu}$. Since $\xi^P=(\xi^Q)^P$ is irreducible, ξ^Q is also irreducible and $\xi^Q=\mu$ or $\bar{\mu}$. However $\boldsymbol{R}(\xi^Q)=\boldsymbol{R}$, which contradicts the assumption $\boldsymbol{R}(\mu)=\boldsymbol{C}$. Hence $(S\cap Q)/K\cap\langle x\rangle\neq\langle x^2\rangle$. Since $|\langle x\rangle:(S\cap Q)/K\cap\langle x\rangle|\leqq|S:(S\cap Q)|\leqq|P:Q|=2$, we have $(S\cap Q)/K\cap\langle x\rangle=\langle x\rangle$. We recall that $(S\cap Q)/K$ is abelian, and we conclude that $(S\cap Q)/K=\langle x\rangle$. Thus $(S\cap Q)/K$ is a cyclic subgroup of S/K of index 2. In the case where χ is of the second kind S/K is a generalized quaternion group. We put $S/K=\langle x, y|x^{2^n}=1, y^2=x^{2^{n-1}}, y^{-1}xy=x^{-1}\rangle$. Since $|\langle x\rangle:(S\cap Q)/K\cap\langle x\rangle|\leqq2$, $(S\cap Q)/K\cap\langle x\rangle=\langle x^2\rangle$ or $\langle x\rangle$. When $(S\cap Q)/K\cap\langle x\rangle=\langle x^2\rangle$, we mayass ume $y\in(S\cap Q)/K$, and $(S\cap Q)/K=\langle x^2, y\rangle$. Since $\langle x^2, y\rangle$ is abelian, we have $n=2$ and $(S\cap Q)/K=\langle y\rangle$, which implies that $(S\cap Q)/K$ is a cyclic subgroup of S/K of index 2. Thus, in any case, $(S\cap Q)/K$ is a cyclic subgroup of S/K of index 2.

Next we assume that $K\subset T\cap Q$ and $(S\cap Q)/K$ is not abelian or that $K\not\subset T\cap Q$. First we show that $|T:T\cap Q|=2$. Suppose that $T=T\cap Q$. Since $K\subset T=T\cap Q$,

$(S\cap Q)/K$ is not abelian. Since $S\supset S\cap Q\supset T$ and T/K is cyclic, we obtain that $S=S\cap Q$. Thus $\zeta^P=(\zeta^Q)^P=\mu^P$, which implies $\zeta^Q=\mu$ or $\bar{\mu}$, because ζ^Q is irreducible. But $\boldsymbol{R}(\zeta^Q)=\boldsymbol{R}$, which contradicts the assumption $\boldsymbol{R}(\mu)=\boldsymbol{C}$. Therefore it holds that $|T:T\cap Q|=2$. We put $\eta=\lambda_{T\cap Q}$. Since $|T:T\cap Q|=2$, we may assume $g\in T$. Let $\{x_1=1, \cdots, x_n\}$ be a set of representatives of P/T. Since $|P:T|=|Q:T\cap Q|$, we may assume $x_i\in Q$. For $h\in Q$ we can easily see that $\lambda^P(h)=\sum_{k=1}^{n}\lambda^{x_k}(h)=\sum_{k=1}^{n}\eta^{x_k}(h)=\eta^Q(h)$. Therefore $(\mu^P)_Q=(\lambda^P)_Q=\eta^Q$, and so $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)=|P|^{-1}\sum_{h\in Q}\eta^Q((gh)^2)$. By Lemma 2.5 $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)=|P|^{-1}(\delta-\gamma)$. Further using Lemma 3.1 we have $|P|^{-1}(\delta-\gamma)=1$ (respectively, -1).

Conversely we assume that the conditions (i), (ii), (iii)-(a) are satisfied. Let ξ be a faithful linear character of $(S\cap Q)/K$ and let τ be a faithful linear character of A. Since ξ^S is a faithful irreducible character of S/K, there exists a faithful irreducible character λ of T/K such that $\lambda^S=\xi^S$. By Lemma 1.3 the condition (ii) means that λ^P is an irreducible character. We denote ξ^Q by μ. Then μ is irreducible, because $(\xi^Q)^P=\xi^P=\lambda^P$ is irreducible. Therefore $\mu\tau$ is irreducible, and $\mu\tau\neq\mu^g\bar{\tau}=(\mu\tau)^g$ means $(\mu\tau)^H$ is irreducible. We will prove that $(\xi\tau)^{SA}$ is an irreducible character of SA of the first kind (respectively, the second kind). If it is true, $(\xi\tau)^H=((\xi\tau)^{QA})^H=(\mu\tau)^H$ means that $(\mu\tau)^H$ is of the first kind (respectively, the second kind). In fact, under the condition $\boldsymbol{R}((\xi\tau)^{SA})=\boldsymbol{R}(((\xi\tau)^{SA})^H)=\boldsymbol{R}$ it is easily seen that $m_{\boldsymbol{R}}((\xi\tau)^{SA})=m_{\boldsymbol{R}}(((\xi\tau)^{SA})^H)$. Since $|S:S\cap Q|=2$, we may assume that $g\in S$. Since $(S\cap Q)/K$ is a cyclic subgroup of S/K of index 2 and S/K is a dihedral group (respectively, a generalized quaternion group), $\boldsymbol{R}(\xi)=\boldsymbol{C}$ and $\xi^g=\bar{\xi}$. Therefore $(\xi\tau)^g=\bar{\xi}\bar{\tau}$, which implies that $\boldsymbol{R}((\xi\tau)^{SA})=\boldsymbol{R}$ and $(\xi\tau)^{SA}$ is irreducible. Since S/K is a dihedral group (respectively, a generalized quaternion group), ξ^S is an irreducible character of S of the first kind (respectively, the second kind). Hence by Lemma 2.3 we have

$$|S|^{-1}\sum_{h\in S\cap Q}(\xi+\xi^g)((gh)^2)=1 \text{ (respectively, } -1).$$

Further by Lemma 3.1 we conclude that $(\xi\tau)^{SA}$ is of the first kind (respectively, the second kind). So under the assumption that $K\subset T\cap Q$ and $(S\cap Q)/K$ is abelian the proof of the proposition is completed.

Finally we assume that the conditions (i), (ii), (iii)-(b) are satisfied, and we will prove that H has an irreducible character of the first kind (respectively, the second kind). Let λ be a faithful linear character of T/K. We denote $\lambda_{T\cap Q}$ by η. Then η is a faithful linear character of $(T\cap Q)/(T\cap Q\cap K)$. By Lemma 1.3 the condition (ii) means that λ^P is an irreducible character of P. Since $|P:T|=|Q:T\cap Q|$, we have $(\lambda^P)_Q=\eta^Q$, and by Lemma 2.5 we obtain that $|P|^{-1}(\delta-\gamma)=|P|^{-1}\sum_{h\in Q}\eta^Q((gh)^2)$.

By Theorem 2.2 λ^P is of the first kind (respectively, the second kind). Therefore $\boldsymbol{R}(\eta^Q)=\boldsymbol{R}$. On the other hand

$$|P|^{-1}\sum_{h\in P}\lambda^P(h^2)=|P|^{-1}\sum_{h\in P}\eta^Q(h^2)$$
$$=|P|^{-1}\sum_{h\in Q}\eta^Q(h^2)+|P|^{-1}\sum_{h\in Q}\eta^Q((gh)^2).$$

Using (1.1), we have $|P|^{-1}\sum_{h\in Q}\eta^Q(h^2)=0$, which implies that η^Q is of the third kind if η^Q is irreducible. But it contradicts $\boldsymbol{R}(\eta^Q)=\boldsymbol{R}$. Now we may decompose η^Q into a sum of irreducible characters μ_i of $Q, \eta^Q=\mu_1+\mu_2+\cdots+\mu_t$. Since $(\lambda^P)_Q=\eta^Q$, we have $((\lambda^P)_Q, \mu_i)\neq 0$, which implies $(\lambda^P, \mu_i{}^P)\neq 0$. Thus $\lambda^P(1)\leqq\mu_i{}^P(1)$, because λ^P is irreducible. Hence $2\lambda^P(1)=2\eta^Q(1)=\eta^P(1)=\mu_1{}^P(1)+\mu_2{}^P(1)+\cdots+\mu_t{}^P(1)\geqq t\lambda^P(1)$. Therefore $t=2$ and $\mu_i{}^P=\lambda^P$ $(i=1,2)$. We put $\mu_1=\mu$. Then $\mu^g=\mu_2$ and $\eta^Q=\mu+\mu^g$. Now we have

$$|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2)=1 \text{ (respectively, } -1)$$

and by Lemma 2.3 we have μ is of the third kind and $\mu^g=\bar{\mu}$. Since $((\mu\tau)^H)_{QA}=\mu\tau+\bar{\mu}\bar{\tau}$, $(\mu\tau)^H$ is irreducible and $\boldsymbol{R}((\mu\tau)^H)=\boldsymbol{R}$. Finally by Lemma 3.1 we have that $(\mu\tau)^H$ is of the first kind (respectively, the second kind).

From Propositions 3.2, 3.3 and 3.4 we get

THEOREM 3.5. *Let H be an $\boldsymbol{R}$-elementary group at* 2. *Let P be a Sylow* 2-*subgroup of H and $A=\langle a\rangle$ a cyclic normal $2'$-group such that $PA=H$. We denote $C_P(a)$ by Q. We assume that $A\neq 1$ and $P\neq Q$. Then there exists an irreducible character χ of H of the first kind* (*respectively, the second kind*) *satisfying* $\mathrm{Ker}\,\chi\cap A=1$ *if and only if one of the conditions* (A), (B) *and* (C) *in Proposition* 3.2, 3.3 *or* 3.4 *is satisfied.*

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