

EQUIVARIANT POINT THEOREMS

(Dedicated to Professor A. Komatu on his 70th birthday)

By

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1. Introduction.

This paper is a continuation of my previous paper [13], and is concerned with generalizations of the following two classical theorems on a continuous map f of an n -sphere S^n to itself.

THEOREM 1.1. *If the degree of f is even then there exists $x \in S^n$ such that $f(-x) = f(x)$.*

THEOREM 1.2. *If the degree of f is odd then there exists $x \in S^n$ such that $f(-x) = -f(x)$.*

Throughout this paper, a prime p is fixed, and $G = \{1, T, \dots, T^{p-1}\}$ will denote a cyclic group of order p .

Generalizing the situation in the above theorems, we shall consider the following problems.

PROBLEM 1. *Let $f: N \rightarrow M$ be a continuous map between G -spaces. Under what conditions does f have an equivariant point, i.e., a point $x \in N$ such that*

$$(1.1) \quad f(T^i x) = T^i f(x)$$

for $i = 1, 2, \dots, p-1$?

PROBLEM 2. *Let $f: L \rightarrow M$ and $g: L \rightarrow N$ be continuous maps of a space L to G -spaces M and N . Under what conditions do there exist p points $x_1, \dots, x_p \in L$ such that*

$$(1.2) \quad f(x_{i+1}) = T^i f(x_1), \quad g(x_{i+1}) = T^i g(x_1)$$

for $i = 1, 2, \dots, p-1$?

We shall denote by $A(f)$ the set of points $x \in N$ satisfying (1.1), and by $A(f, g)$ the set of points $(x_1, \dots, x_p) \in L^p$ satisfying (1.2).

If $L = N$ in Problem 2, then $A(f, \text{id})$ may be identified with $A(f)$. Therefore

Problem 2 is more general than Problem 1; still Problem 2 can be reduced to Problem 1. In fact, if we define $h : L^p \rightarrow M \times N$ by

$$(1.3) \quad h(x_1, \dots, x_p) = (f(x_1), g(x_1)) \quad (x_i \in L),$$

and regard L^p and $M \times N$ as G -spaces by cyclic permutations and the diagonal action respectively, then we have $A(h) = A(f, g)$.

Throughout this paper, a manifold will always mean a compact connected topological manifold which is assumed to be oriented if p is odd. The dimension of manifolds M, N, \dots will be denoted by m, n, \dots . By a G -manifold is meant a manifold on which G acts topologically.

In this paper we shall consider Problems 1 and 2 in case M and N are G -manifolds. Some answers have been obtained by Conner-Floyd [3], Munkholm [10], Fenn [5], Lusk [8] and others with respect to generalizations of Theorem 1.1, and by Milnor [9] and the author [13] with respect to generalizations of Theorem 1.2. By pushing the line of [13] we shall prove in this paper more general results.

Throughout this paper the cohomology stands for the Čech cohomology and it takes coefficients from \mathbf{Z}_p , the group of integers mod p .

2. Theorems

In this section we shall state our main theorems answering to Problem 2 and then corollaries answering to Problem 1. The main theorems will be proved in §5 and §6.

Let $\omega_k \in H^k(BG)$ ($k=0, 1, \dots$) denote the usual generators, where BG is the classifying space for G . If X is a paracompact space on which G acts freely, $H^*(X/G)$ can be regarded as an $H^*(BG)$ -module via the homomorphism induced by a classifying map of X ; in particular we have $\omega_k = \omega_k \cdot 1 \in H^k(X/G)$.

The first main theorem is stated as follows, and it generalizes Theorem 1.1 (see Remark 1 below).

THEOREM A. *Let $f : L \rightarrow M$ and $g : L \rightarrow N$ be continuous maps of a compact space L to G -manifolds M and N . Suppose that*

- i) *the action on M is trivial;*
- ii) *the action on N is free and $\omega_n \in H^n(N/G)$ is not zero;*
- iii) *$n \geq (p-1)m$;*
- iv) *$f^* : H^q(M) \rightarrow H^q(L)$ ($q > 0$) is trivial;*
- v) *$g^* : H^n(N) \rightarrow H^n(L)$ is not trivial.*

Then we have $A(f, g) \neq \phi$; if L is moreover a manifold, we have

$$\dim A(f, g) \geq pl - (p-1)(m+n) \geq 0,$$

where $\dim A$ denotes the covering dimension of A .

Putting $L=N$ and $g=\text{id}$, we get

COROLLARY. *Let $f : N \rightarrow M$ be a continuous map of a G -manifold N to a manifold M . Suppose that*

- i) *the action on N is free and $\omega_n \in H^n(N/G)$ is not zero;*
- ii) *$f^* : H^q(M) \rightarrow H^q(N)$ ($q > 0$) is trivial.*

Then we have

$$\dim A(f) \geq n - (p-1)m,$$

where M is regarded as a G -manifold by the trivial action.

REMARK 1. Taking

$$N = \text{a mod } p \text{ homology } n\text{-sphere}$$

in the above corollary, we have the results due to Conner-Floyd [3], Munkholm [10] and the author [12], which are direct generalizations of Theorem 1.1.

REMARK 2. Taking

$$L = N = \text{a mod } p \text{ homology } n\text{-sphere,}$$

$$M = S^m, \text{ deg } f = 0, \text{ deg } g \neq 0 \text{ mod } p$$

in Theorem A, we have the results due to Fenn [5] and Lusk [8].

To state the second main theorem and its corollaries, we shall make some preparations.

For any indexing set I , consider the complement $I_0^p = I^p - dI$ of the diagonal in I^p , and define $(i_1, \dots, i_p), (i'_1, \dots, i'_p) \in I_0^p$ to be equivalent if (i'_1, \dots, i'_p) is a cyclic permutation of (i_1, \dots, i_p) . We denote by $R(I_0^p)$ a set of representatives of the equivalent classes.

Let $f : L \rightarrow M$ and $g : L \rightarrow N$ be continuous maps of a manifold L to G -manifolds M and N . Given homogeneous bases $\{\alpha_i\}_{i \in I}, \{\beta_j\}_{j \in J}$ of $H^*(M), H^*(N)$ and sets $R(I_0^p), R(J_0^p)$, we define $\lambda(f, g), \lambda'(f, g) \in \mathbf{Z}_p$ as follows.

Define $\Delta : M \rightarrow M^p$ by

$$(2.1) \quad \Delta(x) = (x, Tx, \dots, T^{p-1}x) \quad (x \in M),$$

and put

$$(2.2) \quad \Delta_!(1) = \sum_{(i_1, \dots, i_p) \in I_0^p} c_{i_1, \dots, i_p} \alpha_{i_1} \times \dots \times \alpha_{i_p} \quad (c_{i_1, \dots, i_p} \in \mathbf{Z}_p)$$

for the Gysin homomorphism $\Delta_! : H^*(M) \rightarrow H^*(M^p)$.

Similarly, put

$$\Delta_!(1) = \sum_{(j_1, \dots, j_p) \in J^p} d_{j_1 \dots j_p} \beta_{j_1} \times \dots \times \beta_{j_p} \quad (d_{j_1, \dots, j_p} \in \mathbf{Z}_p)$$

for the homomorphism $\Delta_! : H^*(N) \rightarrow H^*(N^p)$.

We define

$$\lambda(f, g) = \langle (f^{*p} \sum_{(i_1, \dots, i_p) \in R(I_0^p)} c_{i_1 \dots i_p} \alpha_{i_1} \times \dots \times \alpha_{i_p}) (g^{*p} \Delta_!(1)), [L]^p \rangle,$$

$$\lambda'(f, g) = \langle (f^{*p} \Delta_!(1)) (g^{*p} \sum_{(j_1, \dots, j_p) \in R(J_0^p)} d_{j_1 \dots j_p} \beta_{j_1} \times \dots \times \beta_{j_p}), [L]^p \rangle.$$

Obviously we have $\lambda(f, g) = \lambda'(g, f)$.

If $L = N$ and $g^* = \text{id}$, we write $\lambda(f) = \lambda(f, g)$. It follows that

$$\lambda(f) = \sum_{(i_1, \dots, i_p) \in R(I_0^p)} c_{i_1 \dots i_p} \langle (f^* \alpha_{i_1}) (T^* f^* \alpha_{i_2}) \dots (T^{*p-1} f^* \alpha_{i_p}), [N] \rangle.$$

REMARK 3. By the definition of $\Delta_!$ we have

$$\langle \Delta^*(\alpha_{k_1} \times \dots \times \alpha_{k_p}), [M] \rangle = \langle (\alpha_{k_1} \times \dots \times \alpha_{k_p}) \Delta_!(1), [M]^p \rangle.$$

From this we get

$$(2.3) \quad y_{k_1 \dots k_p} = \sum_{(i_1, \dots, i_p) \in I^p} (-1)^{\varepsilon(i_1, \dots, i_p, k_1, \dots, k_p)} c_{i_1 \dots i_p} z_{k_1 i_1} \dots z_{k_p i_p},$$

where

$$y_{k_1 \dots k_p} = \langle \alpha_{k_1} (T^* \alpha_{k_2}) \dots (T^{*p-1} \alpha_{k_p}), [M] \rangle,$$

$$z_{ki} = \langle \alpha_k \alpha_i, [M] \rangle,$$

$$\varepsilon(i_1, \dots, i_p, k_1, \dots, k_p) = \sum_{s=1}^{p-1} |\alpha_{i_s}| (|\alpha_{k_{s+1}}| + \dots + |\alpha_{k_p}|),$$

being $|\alpha| = \deg \alpha$. The relations (2.3) for $(k_1, \dots, k_p) \in I^p$ characterize the coefficients $c_{i_1 \dots i_p}$ ([6]). In particular, if $p=2$ we see that the matrix (c_{ij}) is the inverse of the matrix (y_{ij}) .

Now the second main theorem is stated as follows, and it generalizes Theorem 1.2 (see Remark 5 below).

THEOREM B. *Let $f : L \rightarrow M$ and $g : L \rightarrow N$ be continuous maps of a manifold L to G -manifolds M and N . Suppose that*

i) *$i^* : H^q(M) \rightarrow H^q(M^G)$ is trivial for $q \geq m/p$, where M^G is the fixed point set of M , and i is the inclusion;*

ii) *the action on N is free;*

iii) *$pl = (p-1)(m+n)$.*

Then $\lambda(f, g)$ and $\lambda'(f, g)$ are independent of the choices of $\{\alpha_i\}_{i \in I}$, $\{\beta_j\}_{j \in J}$, $R(I_0^p)$,

$R(J_0^p)$, and we have $\lambda(f, g) = \lambda'(f, g)$; If $\lambda(f, g) \neq 0$ we have $A(f, g) \neq \phi$.

Putting $L=N$ and $g=id$ in Theorem B we have

COROLLARY 1. *Let $f: N \rightarrow M$ be a continuous map between G -manifolds, and suppose that*

- i) $i^*: H^q(M) \rightarrow H^q(M^G)$ is trivial for $q \geq m/p$;
- ii) the action on N is free;
- iii) $n = (p-1)m$.

Then $\lambda(f)$ is independent of the choices of $\{\alpha_i\}_{i \in I}$ and $R(I_0^p)$, and if $\lambda(f) \neq 0$ we have $A(f) \neq \phi$.

Put $L=M$ and $f=id$ in Theorem B, and replace the notations M, N, g by N, M, f respectively. Then we get

COROLLARY 2. *Let $f: N \rightarrow M$ be a continuous map between G -manifolds, and suppose that*

- i) $i^*: H^q(N) \rightarrow H^q(N^G)$ is trivial for $q \geq n/p$;
- ii) the action on M is free;
- iii) $n = (p-1)m$.

Then the same conclusions as in Corollary 1 hold.

REMARK 4. The above two corollaries for $p=2$ have been obtained in [13].

The following proposition will be proved in § 4 (see p. 407 of [2] for $p=2$).

PROPOSITION 2.1. *If M is a G -manifold such that $i^*: H^{m/p}(M) \rightarrow H^{m/p}(M^G)$ is trivial, then*

$$\langle \alpha(T^*\alpha) \cdots (T^{*p-1}\alpha), [M] \rangle = 0 \quad (\alpha \in H^*(M)).$$

Let M be the one in Proposition 2.1 for $p=2$. Then, the proposition and the Poincaré duality show that $H^*(M)$ has a homogeneous basis $\{\mu_1, \dots, \mu_r, \mu'_1, \dots, \mu'_r\}$ such that

$$\langle \mu_i(T^*\mu_k), [M] \rangle = 0, \quad \langle \mu'_i(T^*\mu'_k), [M] \rangle = 0, \quad \langle \mu_i(T^*\mu'_k), [M] \rangle = \delta_{ik}.$$

In terms of this basis we see that

$$\lambda(f) = \sum_{i=1}^r \langle (f^*\mu_i)(T^*f^*\mu'_i), [N] \rangle$$

if $p=2$. In particular, if $M=N$ and $f^*=id$ then $\lambda(f)$ equals the *semi-characteristic*

$$\chi_{1/2}(M) = \dim H^*(M)/2 \pmod{2}.$$

Thus, for $p=2$ we have the following

COROLLARY 3. *Let M be a manifold with a free involution T , and assume $\chi_{1/2}(M) \neq 0$. Let $f, g : M \rightarrow M$ be continuous maps such that $f^* = g^* = \text{id} : H^*(M) \rightarrow H^*(M)$. Then there exist $x, x' \in M$ such that $f(x') = Tf(x)$ and $g(x') = Tg(x)$. In particular, there exists a point $x \in M$ such that $fT(x) = Tf(x)$.*

REMARK 5. Taking

$$M = \text{a mod 2 homology } m\text{-sphere}$$

in Corollary 3, we have the result due to Milnor [9], which is a direct generalization of Theorem 1.2.

3. Method.

In this section we shall explain how to prove Theorems A and B.

Let M be a G -manifold. If we regard M^p as a G -manifold by cyclic permutations, the map $\Delta : M \rightarrow M^p$ in (2.1) is an equivariant embedding. Regard S^{2k+1} as a G -manifold by the standard free action. Then we have a pair $(S^{2k+1} \times_G M^p, S^{2k+1} \times_G \Delta M)$ of manifolds, and hence the Thom isomorphism

$$\theta_k : H^q(S^{2k+1} \times_G \Delta M) \cong H^{q+(p-1)m}(S^{2k+1} \times_G (M^p, M^p - \Delta M))$$

which is the composite of the duality isomorphisms for $S^{2k+1} \times_G \Delta M$ and for $(S^{2k+1} \times_G M^p, S^{2k+1} \times_G \Delta M)$ (see p. 353 of [14]). We denote the Thom class $\theta_k(1)$ by $\hat{U}_M^{(k)}$.

The isomorphisms θ_k for sufficiently large k define the Thom isomorphism

$$\theta : H_G^q(\Delta M) \cong H_G^{q+(p-1)m}(M^p, M^p - \Delta M)$$

of the equivariant cohomology. The element $\theta(1)$ is denoted by \hat{U}_M , and is called the *equivariant fundamental cohomology class of M* .

The image of \hat{U}_M in $H_G^{m(p-1)}(M^p)$ is denoted by \hat{U}'_M , and is called the *equivariant diagonal cohomology class of M* .

If the action of G on M is free, the diagonal set dM is in $M^p - \Delta M$. In this case the image of \hat{U}_M in $H_G^{m(p-1)}(M^p, dM)$ is denoted by \hat{U}''_M , and is called the *modified equivariant diagonal cohomology class of M* .

LEMMA 3.1. *Let M and N be G -manifolds, and regard $M \times N$ as a G -manifold by the diagonal action. If the action on N is free, we have*

$$\hat{U}''_{M \times N} = \pm (q_1^{p*} \hat{U}'_M) (q_2^{p*} \hat{U}''_N),$$

where $q_1^{p*} : H_G^*(M^p) \rightarrow H_G^*((M \times N)^p)$ and $q_2^{p*} : H_G^*(N^p, dN) \rightarrow H_G^*((M \times N)^p, d(M \times N))$ are induced by the projections $q_1 : M \times N \rightarrow M$, $q_2 : M \times N \rightarrow N$.

PROOF. There are the following natural inclusions of manifolds:

$$\begin{aligned} (\mathbb{S}^{2k+1} \times_G \Delta M) \times \mathbb{S}^{2k+1} \times_G \Delta N &\subset (\mathbb{S}^{2k+1} \times_G M^p) \times (\mathbb{S}^{2k+1} \times_G N^p) \\ \cup & \\ \mathbb{S}^{2k+1} \times_G \Delta(M \times N) &\subset \mathbb{S}^{2k+1} \times_G (M \times N)^p \end{aligned}$$

From properties of the Thom class (see 325 of [4]), it follows that the Thom class for the pair in the upper line equals $\pm \hat{U}_M^{(k)} \times \hat{U}_N^{(k)}$, and that it is sent to $\pm \hat{U}_{M \times N}^{(k)}$ by the homomorphism i^* induced by the natural inclusion of the lower line to the upper. Therefore we have

$$\hat{U}_{M \times N}^{(k)} = \pm i^*(\hat{U}_M^{(k)} \times \hat{U}_N^{(k)}) = \pm i^*(p_1^* \hat{U}_M^{(k)} \cdot p_2^* \hat{U}_N^{(k)}) = \pm (q_1^{p^*} \hat{U}_M^{(k)})(q_2^{p^*} \hat{U}_N^{(k)}),$$

where p_1, p_2 are the projections of $(\mathbb{S}^{2k+1} \times_G M^p) \times (\mathbb{S}^{2k+1} \times_G N^p)$ to $\mathbb{S}^{2k+1} \times_G M^p, \mathbb{S}^{2k+1} \times_G N^p$. This fact proves immediately the desired result.

LEMMA 3.2. *Let $f : N \rightarrow M$ be a continuous map of a G -space N to a G -manifold M , and define an equivariant map $\hat{f} : N \rightarrow M^p$ by*

$$\hat{f}(x) = (f(x), fT(x), \dots, fT^{p-1}(x)) \quad (x \in N).$$

If the action on M is free, and if $\hat{f}^(\hat{U}_M^m) \neq 0$ for the homomorphism $\hat{f}^* : H_G^*(M^p, dM) \rightarrow H_G^*(N, N^G)$, then we have $A(f) \neq \emptyset$. If N is moreover a G -manifold, we have*

$$\dim A(f) \geq n - (p-1)m \geq 0.$$

PROOF. In virtue of a commutative diagram

$$\begin{array}{ccc} H_G^*(M^p, M^p - \Delta M) & \xrightarrow{i^*} & H_G^*(M^p, dM) \\ \downarrow \hat{f}^* & & \downarrow \hat{f}^* \\ H_G^*(N, N - A(f)) & \xrightarrow{i^*} & H_G^*(N, N^G), \end{array}$$

$\hat{f}^*(\hat{U}_M^m) \neq 0$ implies $H_G^{m(p-1)}(N, N - A(f)) \neq 0$. Therefore $A(f) \neq \emptyset$. If N is a G -manifold, we have isomorphisms

$$\begin{aligned} H^{n-m(p-1)}(A(f)/G) &\cong H_{m(p-1)}(N'/G, (N' - A(f))/G) \\ &\cong H^{m(p-1)}(N'/G, (N' - A(f))/G) \cong H_G^{m(p-1)}(N', N' - A(f)) \\ &\cong H_G^{m(p-1)}(N, N - A(f)), \end{aligned}$$

where $N' = N - N^G$. Therefore $H^{n-m(p-1)}(A(f)/G) \neq 0$, and so $\dim A(f) \geq n - m(p-1) \geq 0$. This completes the proof.

PROPOSITION 3.3. *Let $f : L \rightarrow M$ and $g : L \rightarrow N$ be continuous maps of a space*

L to G -manifolds M and N . Suppose that the action on N is free. Then if

$$(f^{p*}\hat{U}'_M)(g^{p*}\hat{U}''_N) \in H_G^{(m+n)(p-1)}(L^p, dL)$$

is not zero, we have $A(f, g) \neq \phi$. If L is moreover a manifold, we have

$$\dim A(f, g) \geq pl - (p-1)(m+n) \geq 0.$$

PROOF. Consider $h : L^p \rightarrow M \times N$ defined by (1.3). Then, for the map $\hat{h} : L^p \rightarrow (M \times N)^p$ we have $q_1^p \circ \hat{h} = f^p$, $q_2^p \circ \hat{h} = g^p$. Therefore by Lemma 3.1 we have

$$\begin{aligned} \hat{h}^*(\hat{U}''_{M \times N}) &= \pm \hat{h}^*((q_1^{p*}\hat{U}'_M)(q_2^{p*}\hat{U}''_N)) \\ &= \pm (f^{p*}\hat{U}'_M)(g^{p*}\hat{U}''_N). \end{aligned}$$

This proves the desired result by Lemma 3.2.

We shall prove Theorems A and B by making use of Proposition 3.3. For this purpose we are asked to examine the following:

(i) structure of the equivariant cohomologies $H_G^*(X^p)$ and $H_G^*(X^p, dX)$ for a compact space X .

(ii) the equivariant diagonal cohomology class \hat{U}'_M and the modified equivariant diagonal cohomology class \hat{U}''_M for a G -manifold M .

As for (i) we have the results due to Steenrod and Thom, which are stated in §4. Thus Theorems A and B will be proved by examining (ii), as seen in §5 and §6.

4. Preparations

In this section we shall recall some facts needed later.

Let X be a paracompact G -space. Then we have

$$\begin{aligned} H^*(X) &= \varinjlim H^*(K), \\ H_G^*(X, X^G) &= \varinjlim H^*(K/G, K^G/G), \end{aligned}$$

where K ranges over the nerves of G -coverings of X (see Chap III, §6 and Chap VII, §1 of [2]). For each K a cochain map

$$\varphi_K : C^*(K) \longrightarrow C^*(K/G, K^G/G)$$

is defined by

$$\langle \varphi_K(u), \pi(s) \rangle = \sum_{i=0}^{p-1} u(T^i s),$$

where $u \in C^*(K)$, s is a simplex of K , and $\pi : K \rightarrow K/G$ is the projection. Thus

we have a homomorphism

$$(4.1) \quad \pi'_i : H^*(X) \longrightarrow H_G^*(X, X^G)$$

defined by the cochain maps φ_K .

We define

$$(4.2) \quad \pi_i : H^*(X) \longrightarrow H_G^*(X)$$

to be the composite $j^* \circ \pi'_i$, where $j^* : H_G^*(X, X^G) \longrightarrow H_G^*(X)$ is induced by the inclusion. It follows that π_i is the composite of the usual transfer $H^*(X) \longrightarrow H^*(X/G)$ and the canonical homomorphism $H^*(X/G) \longrightarrow H_G^*(X)$.

We call π_i in (4.2) the *transfer*, and π'_i in (4.1) the *modified transfer*.

Put

$$\sigma^* = \sum_{i=0}^{p-1} T^{i*} : H^*(X) \longrightarrow H^*(X).$$

Then it is easily seen that

$$(4.3) \quad \pi^* \circ \pi_i = \sigma^*$$

for the canonical homomorphism $\pi^* : H_G^*(X) \longrightarrow H^*(X)$, and that

$$(4.4) \quad \pi_i(\alpha_1) \cdot \pi'_i(\alpha_2) = \pi'_i(\alpha_1 \cdot \sigma^* \alpha_2) = \pi'_i(\sigma^* \alpha_1 \cdot \alpha_2)$$

($\alpha_1, \alpha_2 \in H^*(X)$). We have also

$$(4.5) \quad \pi_i(\alpha) \cdot \delta^*(\beta) = 0 \quad (\alpha \in H^*(X), \beta \in H_G^*(X^G))$$

for the coboundary homomorphism $\delta^* : H_G^*(X^G) \longrightarrow H_G^*(X, X^G)$.

In fact

$$\begin{aligned} (-1)^{|\alpha|} \pi_i(\alpha) \cdot \delta^*(\beta) &= \delta^*(i^* \pi_i(\alpha) \cdot \beta) \\ &= \delta^*(i^* j^* \pi'_i(\alpha) \cdot \beta) = 0, \end{aligned}$$

where $i^* : H_G^*(X) \longrightarrow H_G^*(X^G)$.

If X is a paracompact G -space, the Smith special cohomology groups $H_{\bar{\rho}}^*(X)$ are defined for $\rho = \sigma = \sum_{i=0}^{p-1} T^i$ and $\rho = \tau = 1 - T$, and we have the exact sequences

$$\begin{array}{ccccccc} \dots & \xrightarrow{\rho^*} & H^q(X) & \xrightarrow{i_{\bar{\rho}}^*} & H_{\bar{\rho}}^q(X) \oplus H^q(X^G) & & \\ & & & & \delta_{\bar{\rho}}^* & \xrightarrow{j_{\bar{\rho}}^*} & H_{\bar{\rho}}^{q+1}(X) \longrightarrow \dots \end{array}$$

for $(\rho, \bar{\rho}) = (\sigma, \tau)$ and (τ, σ) . We have also an isomorphism

$$H_{\sigma}^*(X) \cong H_G^*(X, X^G).$$

(See p. 143 of [2].)

It follows that

$$(4.6) \quad i_r^* = (\pi_r^!, i^*) : H^*(X) \longrightarrow H_G^*(X, X^G) \oplus H^*(X^G).$$

LEMMA 4.1. *If M is a G -manifold such that the action is not trivial, then it holds*

$$\pi_r^! : H^m(M) \cong H_G^m(M, M^G).$$

PROOF. In the exact sequence

$$H^m(M) \xrightarrow{i_r^*} H_G^m(M) \oplus H^m(M^G) \xrightarrow{\delta_r^*} H_r^{m+1}(M),$$

we have $H_r^{m+1}(M) = 0$, $H^m(M) \cong \mathbf{Z}_p$, $H^m(M^G) \cong H_0(M, M - M^G) = 0$, and moreover $H_G^m(M) \neq 0$ is proved as follows. Therefore we get the desired result by (4.6).

Suppose $H_G^m(M) = 0$. Then, by the Smith cohomology exact sequence, we see that $i_r^* : H^m(M) \cong H_r^m(M)$ and $\tau^* : H_r^m(M) \longrightarrow H^m(M)$ is onto. This implies that $\tau^* : H^m(M) \longrightarrow H^m(M)$ is onto and so $H^m(M) = 0$, which is a contradiction.

For a paracompact space X , consider the equivariant cohomology $H_G^*(X^p)$, where G acts on X^p by cyclic permutations. Then we have the external Steenrod p -th power operation

$$P : H^q(X) \longrightarrow H_G^{pq}(X^p),$$

which is related to the Steenrod square Sq^i if $p=2$, and to the reduced p -th power \mathcal{S}^i and the Bockstein operation β^* if $p \neq 2$ as follows ([15]):

$$(4.7) \quad d^*P(\alpha) = \begin{cases} \sum_{i=0}^{|\alpha|} \omega_{|\alpha|-i} \times Sq^i \alpha & \text{if } p=2, \\ h_q \sum_{i=0}^{[\frac{|\alpha|}{2}]} (-1)^i (\omega_{(|\alpha|-2i)(p-1)} \times \mathcal{S}^i \alpha - \omega_{(|\alpha|-2i)(p-1)-1} \times \beta^* \mathcal{S}^i \alpha) & \text{if } p \neq 2, \end{cases}$$

where $d^* : H_G^*(X^p) \longrightarrow H_G^*(X) = H^*(BG \times X)$ is induced by the diagonal map, and

$$(4.8) \quad h_q = \begin{cases} (-1)^{q/2} & \text{if } q \text{ is even,} \\ (-1)^{(q-1)/2} ((p-1)/2)! & \text{if } q \text{ is odd.} \end{cases}$$

P is natural, and it satisfies also

$$(4.9) \quad \pi^*P(\alpha) = \alpha^p$$

for the canonical homomorphism $\pi^* : H_G^*(X^p) \longrightarrow H^*(X^p)$.

LEMMA 4.2. *Let M be a G -manifold, and let $\alpha \in H^*(M)$ satisfy $i^*(\alpha) = 0$ for $i^* : H^*(M) \rightarrow H^*(M^G)$. Then $\Delta^*P(\alpha)$ is in the image of $j^* : H_G^*(M, M^G) \rightarrow H_G^*(M)$ induced by the inclusion.*

PROOF. Consider a diagram

$$\begin{array}{ccccc}
 & & H_G^*(M^p) & \xrightarrow{d^*} & H^*(BG \times M) \\
 & & \downarrow \Delta^* & & \downarrow (id \times i)^* \\
 H_G^*(M, M^G) & \xrightarrow{j^*} & H_G^*(M) & \xrightarrow{i^*} & H^*(BG \times M^G),
 \end{array}$$

in which the rectangle is commutative and the lower sequence is exact. Then it follows from (4.7) that $i^*\Delta^*P(\alpha) = (id \times i)^*d^*P(\alpha) = 0$. Therefore $\Delta^*P(\alpha) \in \text{Im } j^*$.

PROOF OF PROPOSITION 2.1. We may assume that the action is non-trivial and $|\alpha| = m/p$. Consider a commutative diagram

$$\begin{array}{ccc}
 H_G^m(M, M^G) & \xrightarrow{j^*} & H_G^m(M) \\
 \uparrow \pi'_1 & \nearrow \pi_1 & \downarrow \pi^* \\
 H^m(M) & \xrightarrow{\sigma^*} & H^m(M)
 \end{array}$$

By Lemmas 4.1 and 4.2, we see

$$\pi^*\Delta^*P(\alpha) \in \text{Im } \sigma^*.$$

Since $\sigma^*H^m(M) = 0$ and

$$\pi^*\Delta^*P(\alpha) = \Delta^*(\alpha^p) = \alpha(T^*\alpha) \cdots (T^{p-1}\alpha)$$

by (4.9), the proof completes.

The following theorem is due to Steenrod [15] (see also [12]).

THEOREM 4.3. *Let X be a compact space, and $\{\alpha_i\}_{i \in I}$ be a homogeneous basis of $H^*(X)$. Then the totality of elements*

$$\begin{aligned}
 \omega_j P(\alpha_i) & \quad (i \in I, j \geq 0), \\
 \pi_1(\alpha_{i_1} \times \cdots \times \alpha_{i_p}) & \quad ((i_1, \dots, i_p) \in R(I_p^p))
 \end{aligned}$$

is a homogeneous basis of $H_G^(X^p)$.*

The following is due to Thom [16] (see also [1], [11], [17]).

THEOREM 4.4. *Let X be a compact space, and $\{\alpha_i\}_{i \in I}$ be a homogeneous basis of $H^*(X)$. Then the totality of elements*

$$\begin{aligned} \delta^*(\omega_j \times \alpha_i) & \quad (i \in I, 0 \leq j < (p-1)|\alpha_i|), \\ \pi_!(\alpha_{i_1} \times \cdots \times \alpha_{i_p}) & \quad ((i_1, \dots, i_p) \in R(I_0^p)) \end{aligned}$$

is a homogeneous basis of $H_G^(X^p, dX)$, where $\delta^* : H^*(BG \times X) = H_G^*(dX) \longrightarrow H_G^*(X^p, dX)$ is the coboundary homomorphism. Furthermore we have*

$$\pi_!(\alpha \times \alpha) = \sum_{i=0}^{|\alpha|-1} \delta^*(\omega_{|\alpha|-i-1} \times Sq^i \alpha)$$

if $p=2$, and

$$\pi_!(\alpha \times \cdots \times \alpha) = \sum_{i=0}^{[\frac{|\alpha|}{2}]} \varepsilon_i \delta^*(\omega_{(p-1)(|\alpha|-2i)-1} \times \mathcal{S}^i \alpha)$$

with some $\varepsilon_i \not\equiv 0 \pmod{p}$ if $p \neq 2$.

REMARK. Theorems 4.3 and 4.4 are proved in the literatures for a compact polyhedron. However we can extend them to compact spaces by the device seen in [2].

5. Proof of Theorem A.

The equivariant diagonal cohomology class \hat{U}'_M in case the action on M is trivial has been studied by Haefliger. By Theorem 3.2 in his paper [6] and

$$(5.1) \quad \pi^*(\hat{U}') = \Delta_1(1),$$

we have the following (see the proof of Theorem 9.1 in [13]).

PROPOSITION 5.1. *If the action on M is trivial, then*

$$\hat{U}'_M = \sum_{k=0}^{[m/2]} \omega_{m-2k} P(V_k) + \sum_{i < j} (c_{ij} - c_{ii}c_{jj}) \pi_!(\alpha_i \times \alpha_j)$$

if $p=2$, and

$$\begin{aligned} \hat{U}'_M = h_m \sum_{k=0}^{[m/2p]} (-1)^k \omega_{(p-1)(m-2kp)} P(V_k) \\ + \sum_{(i_1, \dots, i_p) \in R(I_0^p)} (c_{i_1 \dots i_p} - c_{i_1 \dots i_1} \cdots c_{i_p \dots i_p}) \pi_!(\alpha_{i_1} \times \cdots \times \alpha_{i_p}) \end{aligned}$$

if $p \neq 2$, where $\{\alpha_i\}_{i \in I}$ is a homogeneous basis of $H^(M)$, c_{i_1, \dots, i_p} , h_m are those in (2.2), (4.8), and $V_k \in H^*(M)$ are the Wu classes given by*

$$\langle V_k \cdot \alpha, [M] \rangle = \begin{cases} \langle Sq^k \alpha, [M] \rangle & \text{if } p=2, \\ \langle \mathcal{S}^k \alpha, [M] \rangle & \text{if } p \neq 2. \end{cases}$$

We shall next prove

PROPOSITION 5.2. *If the action on M is free and $\omega_m \in H^*(M/G)$ is not zero, it holds*

$$\omega_m \hat{U}_M'' = \delta^*(\omega_{(p-1)m-1} \times \mu),$$

where μ is a generator of $H^m(M)$.

PROOF. Let V be an equivariant open neighbourhood of dM in M^p , and put

$$W = M^p - \Delta M - dM, \quad C = M^p - \Delta M - V.$$

Then C/G is a closed connected and non-compact subset of W/G , and hence we have $H^{mp}(W/G, W/G - C/G) = 0$ (see p. 260 of [4]). Therefore it follows that

$$H_G^{mp}(M^p - \Delta M, V) \cong H_G^{mp}(W, W - C) \cong H^{mp}(W/G, W/G - C/G) = 0.$$

This shows that $i^* : H_G^{mp}(M^p, M^p - \Delta M) \longrightarrow H_G^{mp}(M^p, dM)$ is onto, and so is

$$i^* \circ \theta : H_G^m(\Delta M) \longrightarrow H_G^{mp}(M^p, dM).$$

It follows from Lemma 4.1 and the assumptions that $H_G^m(\Delta M) \cong \mathbf{Z}_p$ is generated by ω_m . By Theorem 4.4, $H_G^{mp}(M^p, dM) \cong \mathbf{Z}_p$ is generated by $\delta^*(\omega_{(p-1)m} \times \mu)$. Since $i^* \circ \theta$ is a homomorphism of $H^*(BG)$ -modules and it sends 1 to \hat{U}_M'' , we have the desired result.

We shall now give

PROOF OF THEOREM A. By the assumption ii) and Proposition 5.2, it holds

$$\omega_n \hat{U}_N'' = \delta^*(\omega_{(p-1)n-1} \times \nu),$$

where ν is a generator of $H^n(N)$. Therefore we have

$$\omega_n(g^{p*} \hat{U}_N'') = \delta^*(\omega_{(p-1)n-1} \times g^* \nu),$$

and this is not zero by the assumption v) and Theorem 4.4. Since $n \geq (p-1)m$ by the assumption iii), it holds

$$\omega_{(p-1)m}(g^{p*} \hat{U}_N'') \neq 0.$$

On the other hand, it follows from the assumptions i), iv) and Proposition 5.1 that

$$f^{p*}(\hat{U}_M') = h_m \omega_{(p-1)m}$$

with $h_m \not\equiv 0 \pmod{p}$. Consequently we have

$$f^{p*}(\hat{U}_M') \cdot g^{p*}(\hat{U}_N'') = h_m \omega_{(p-1)m} g^{p*}(\hat{U}_N'') \neq 0,$$

which completes the proof by Proposition 3.3.

6. Proof of Theorem B and an example.

The following proposition has been proved in [13] if $p=2$. By the similar method we shall prove it for any p .

PROPOSITION 6.1. *If $i^* : H^q(M) \longrightarrow H^q(M^G)$ is trivial for $q \geq m/p$, then we have*

$$\begin{aligned} \hat{U}'_M &= \sum_{(i_1, \dots, i_p) \in R(I^p)} c_{i_1 \dots i_p} \pi_1(\alpha_{i_1} \times \dots \times \alpha_{i_p}), \\ c_{i \dots i} &= 0 \quad (i \in I), \end{aligned}$$

where $\{\alpha_i\}_{i \in I}$ is a homogeneous basis of $H^*(M)$, and $c_{i_1 \dots i_p}$ are those in (2.2).

Before we proceed to proof we make some preparations.

The equivariant homology group $H_*^G(X^p) = H_*(EG \times_G X^p)$ is canonically identified with $H_*(G; H_*(X)^p)$, the homology group of the group G with coefficients in $H_*(X)^p = H_*(X) \otimes \dots \otimes H_*(X)$ on which G acts by cyclic permutations. Taking the standard G -free acyclic complex W , we have an element of $H_*(G; H_*(X)^p)$ represented by $w_k \otimes a \otimes \dots \otimes a$, where $w_k \in W$ is the basis of degree k and $a \in H_*(X)$. The corresponding element in $H_*^G(X^p)$ will be denoted by $P_k(a)$.

LEMMA 6.2. *Suppose that $i^* : H^q(M) \longrightarrow H^q(M^G)$ is trivial for $q \geq m/p$. Then, for any $k \geq 0$ and for any $\alpha \in H^*(M)$, we have*

$$\langle \omega_1 \hat{U}'_M, P_{k+1}(\alpha \frown [M]) \rangle = 0$$

if $p=2$, and

$$\langle \hat{U}'_M, P_{2k+1}(\alpha \frown [M]) \rangle = 0,$$

$$\langle \omega_1 \hat{U}'_M, P_{2k+1}(\alpha \frown [M]) \rangle = 0$$

if $p \neq 2$.

PROOF. Similarly to Lemma 4.4 in [13], the result for $p \neq 2$ is proved as follows.

It follows that $P_{2k+1}([M])$ is in the image of

$$i_{k*} : H_{2k+1+pm}(S^{2k+1} \times_G M^p) \longrightarrow H_{2k+1+pm}^G(M^p)$$

induced by the inclusion, and that $i_k^*(\hat{U}'_M)$ is the image of 1 under the homomorphism

$$(\text{id} \times \mathcal{A})_! : H^*(S^{2k+1} \times_G M) \longrightarrow H^*(S^{2k+1} \times_G M^p).$$

From these facts we see that $\hat{U}'_M \frown P_{2k+1}([M])$ is in the image of

$$H_{2k+1+m}^G(S^{2k+1} \times_G M) \xrightarrow{i_{k*}} H_{2k+1+m}^G(M) \xrightarrow{\Delta_*} H_{2k+1+m}^G(M^p).$$

Therefore it follows that

$$\begin{aligned} \langle \hat{U}'_M, P_{2k+1}(\alpha \frown [M]) \rangle &= \langle \hat{U}'_M, P(\alpha) \frown P_{2k+1}([M]) \rangle \\ &= \langle P(\alpha), \hat{U}'_M \frown P_{2k+1}([M]) \rangle = \varepsilon_k \langle P(\alpha), \Delta_* i_{k*} [S^{2k+1} \times_G M] \rangle \\ &= \varepsilon_k \langle \Delta_* P(\alpha), i_{k*} [S^{2k+1} \times_G M] \rangle \quad (\varepsilon_k \in \mathbf{Z}_p), \end{aligned}$$

and similarly

$$\langle \omega_1 \hat{U}'_M, P_{2k+1}(\alpha \frown [M]) \rangle = \varepsilon_k \langle \omega_1 \Delta_* P(\alpha), i_{k*} [S^{2k+1} \times_G M] \rangle.$$

To prove the desired two equalities, we may suppose $p|\alpha| \geq m+1$ in the first, and $p|\alpha| \geq m$ in the second. Consequently it suffices to prove that

$$\begin{aligned} \Delta_* P(\alpha) &= 0 \quad \text{if } p|\alpha| \geq m+1, \\ \omega_1 \Delta_* P(\alpha) &= 0 \quad \text{if } p|\alpha| \geq m. \end{aligned}$$

By Lemma 4.2 $\Delta_* P(\alpha)$ and $\omega_1 \Delta_* P(\alpha)$ are in the image of $j^* : H_G^*(M, M^G) \rightarrow H_G^*(M)$, and the Smith cohomology exact sequence implies $H_G^q(M, M^G) = 0$ ($q > m$). Therefore we have the desired results, and the proof completes.

PROOF OF PROPOSITION 6.1. In virtue of Theorem 4.3 it can be written uniquely that

$$\hat{U}'_M = \sum_{i,j} \xi_{ij} \omega_j P(\alpha_i) + \sum_{(i_1, \dots, i_p) \in R(I_0^p)} \eta_{i_1 \dots i_p} \pi_1(\alpha_{i_1} \times \dots \times \alpha_{i_p})$$

with some $\xi_{ij}, \eta_{i_1 \dots i_p} \in \mathbf{Z}_p$. Since it is easily seen that

$$\begin{aligned} \text{Im } \pi_1 \frown P_k(a) &= 0, \\ \langle \omega_j P(\alpha), P_k(a) \rangle &= \delta_{jk} \langle \alpha, a \rangle \end{aligned}$$

($\alpha \in H^*(M), a \in H_*(M)$), it follows from Lemma 6.2 that $\xi_{ij} = 0$. We see from (5.1) that $\eta_{i_1 \dots i_p} = c_{i_1 \dots i_p}$ if $(i_1, \dots, i_p) \in R(I_0^p)$ and $c_{i \dots i} = 0$ for any $i \in I$. This completes the proof.

REMARK 1. Working in the smooth category, Hattori [7] has given formulae for \hat{U}'_M with no assumption on M^G .

The following is immediate from Proposition 6.1 and Theorem 4.5.

PROPOSITION 6.3. *If the action on M is free, then it can be written uniquely that*

$$\begin{aligned}\hat{U}_M'' &= \sum_{(i_1, \dots, i_p) \in R(I_0^p)} c_{i_1 \dots i_p} \pi_!(\alpha_{i_1} \times \dots \times \alpha_{i_p}) \\ &\quad + \sum_{|\alpha_i| \geq m-m/p} \varepsilon_i \delta^*(\omega_{(p-1)m-|\alpha_i|-1} \times \alpha_i)\end{aligned}$$

with some $\varepsilon_i \in \mathbf{Z}_p$.

REMARK 2. The author does not know how to determine ε_i in the above. If M is a mod p homology sphere, it follows from Propositions 5.2 and 6.3 that

$$\hat{U}_M'' = \begin{cases} \pi_!(1 \times \mu) & \text{if } p=2, \\ \pi_!(1 \times \mu \times \dots \times \mu) + \varepsilon \delta^*(\omega_{(p-2)m-1} \times \mu) & \text{if } p \neq 2, \end{cases}$$

where $\varepsilon \not\equiv 0 \pmod{p}$, and $\mu \in H^m(M)$ is a generator such that $\langle \mu, [M] \rangle = 1$.

We shall now give

PROOF OF THEOREM B. By the assumption i) and Proposition 6.1 we have

$$f^{p*} \hat{U}_M' = \pi_! f^{*p} \left(\sum_{(i_1, \dots, i_p) \in R(I_0^p)} c_{i_1 \dots i_p} \alpha_{i_1} \times \dots \times \alpha_{i_p} \right),$$

and by the assumption ii) and Proposition 6.3 we have

$$\begin{aligned}g^{p*} \hat{U}_N'' &= \pi_! g^{*p} \left(\sum_{(j_1, \dots, j_p) \in R(J_0^p)} d_{j_1 \dots j_p} \beta_{j_1} \times \dots \times \beta_{j_p} \right) \\ &\quad + \sum_{|\beta_j| \geq n-n/p} \varepsilon_j \delta^*(\omega_{(p-1)n-|\beta_j|-1} \times g^* \beta_j).\end{aligned}$$

It follows from (5.1) and Proposition 6.1 that

$$\begin{aligned}\sigma^* \sum_{(i_1, \dots, i_p) \in R(I_0^p)} c_{i_1 \dots i_p} \alpha_{i_1} \times \dots \times \alpha_{i_p} &= \Delta_!(1), \\ \sigma^* \sum_{(j_1, \dots, j_p) \in R(J_0^p)} d_{j_1, \dots, j_p} \beta_{j_1} \times \dots \times \beta_{j_p} &= \Delta_!(1).\end{aligned}$$

Thus, by (4.4), (4.5) and the assumption ii), we have

$$\begin{aligned}& (f^{p*} \hat{U}_M') \cdot (g^{p*} \hat{U}_N'') \\ &= \pi_! \left(f^{*p} \sum_{(i_1, \dots, i_p) \in R(I_0^p)} c_{i_1 \dots i_p} \alpha_{i_1} \times \dots \times \alpha_{i_p} \right) (g^{*p} \Delta_!(1)) \\ &= \pi_! \left(f^{*p} \Delta_!(1) \right) \left(g^{*p} \sum_{(j_1, \dots, j_p) \in R(J_0^p)} d_{j_1, \dots, j_p} \beta_{j_1} \times \dots \times \beta_{j_p} \right)\end{aligned}$$

in $H_G^{pl}(L^p, dL)$.

It follows from Theorem 4.4 that $H_G^{pl}(L^p, dL) \cong \mathbf{Z}_p$ is generated by $\delta^*(\omega_{(p-1)l-1} \times \rho)$ or $\pi_!(\rho \times \dots \times \rho)$, where $\rho \in H^l(L)$ is a generator such that $\langle \rho, [L] \rangle = 1$.

Consequently we have

$$\begin{aligned}(f^{p*} \hat{U}_M') (g^{p*} \hat{U}_N'') &= \lambda(f, g) \pi_!(\rho \times \dots \times \rho) \\ &= \lambda'(f, g) \pi_!(\rho \times \dots \times \rho),\end{aligned}$$

which completes the proof by Proposition 3.3.

Theorem B for $p=2$, particularly corollary 3 in § 2, has interesting applications as is seen in [13]. The author does not know so interesting applications of Theorems B for $p \neq 2$. However there is the following example for which Theorem B for $p=3$ is applicable.

Let $n=1, 3$ or 7 , and take in Theorem B

$$L = S^n \times S^n, \quad M = S^n \times S^n, \quad N = S^n,$$

where the action on N is any free G -action, and action on M is given as follows:

$$T(x, y) = (y, y^{-1}x^{-1}),$$

x, y being complex numbers, quaternions or Cayley numbers according as $n=1, 3$ or 7 . It follows that the fixed point set of M is homeomorphic to $S^{n-1} + \text{point}$. Thus the assumptions i), ii), iii) in Theorem B are satisfied.

Let $\nu \in H^n(S^n)$ denote a generator, and put $\nu_1 = \nu \times 1, \nu_2 = 1 \times \nu \in H^n(S^n \times S^n)$. Then, by Remark 3 in § 2, it can be seen that

$$\Delta_1(1) = \sigma^*(1 \times \nu_1 \nu_2 \times \nu_1 \nu_2 - \nu_1 \times \nu_1 \times \nu_1 \nu_2 - \nu_2 \times \nu_2 \times \nu_1 \nu_2 - \nu_2 \times \nu_1 \times \nu_1 \nu_2)$$

for the homomorphism $\Delta_1 : H^*(M) \rightarrow H^*(M^3)$, and

$$\Delta_1(1) = \sigma^*(1 \times \nu \times \nu)$$

for the homomorphism $\Delta_1 : H^*(N) \rightarrow H^*(N^3)$. Therefore, if continuous maps $f : L \rightarrow M, g : L \rightarrow N$ satisfy

$$f^*(\nu_i) = a_{i1}\nu_1 + a_{i2}\nu_2, \quad g^*(\nu) = b_1\nu_1 + b_2\nu_2$$

($a_{ij}, b_i \in \mathbf{Z}_3$), simple calculation shows

$$\lambda(f, g) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \left(\begin{vmatrix} a_{11} & a_{12} \\ b_1 & b_2 \end{vmatrix} - \begin{vmatrix} a_{21} & a_{22} \\ b_1 & b_2 \end{vmatrix} \right).$$

This yields by Theorem B the following

THEOREM 6.4. *Let $n=1, 3$ or 7 , and let $f_1, f_2, g : S^n \times S^n \rightarrow S^n$ be continuous maps of type $(a_{11}, a_{12}), (a_{21}, a_{22}), (b_1, b_2)$ respectively. Let $T : S^n \rightarrow S^n$ be a homomorphism of period 3 without fixed points. Then, if*

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \left(\begin{vmatrix} a_{11} & a_{12} \\ b_1 & b_2 \end{vmatrix} - \begin{vmatrix} a_{21} & a_{22} \\ b_1 & b_2 \end{vmatrix} \right) \not\equiv 0 \pmod{3},$$

there exist $x, y, z \in S^n \times S^n$ such that

$$(f_2(x), f_2(y), f_2(z)) = (f_1(y), f_1(z), f_1(x)),$$

$$(Tg(x), Tg(y), Tg(z)) = (g(y), g(z), g(x)),$$

$$f_1(x)f_1(y)f_1(z) = 1.$$

In particular, taking f_i =projection to the i -th factor, we have

COROLLARY. *If $b_1 + b_2 \neq 0$ then there exist $x, y, z \in S^n$ such that*

$$Tg(x, y) = g(y, z), \quad Tg(y, z) = g(z, x), \quad xyz = 1,$$

where n, g and T are those in Theorem 6.4.

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