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ON THE BAHADUR REPRESENTATION OF SAMPLE QUANTILES FOR MIXING PROCESSES

By

Ryozo YOKOYAMA

1. Introduction.

The asymptotic almost sure (a.s.) representation of sample quantiles for independent and identically distributed random variables was firstly established by Bahadur [1]. Kiefer [5, 6] obtained further developments on this line and also investigated the a.s. representation of quantile process. Here we remark that the representation of sample quantiles in the sense of in probability was obtained first of all by Okamoto [7]. The extensions of Bahadur's by relaxing the assumption of independence of the basic random variables have been studied by a number of authors. Especially, Sen [10] obtained completely analogous results to Bahadur's one for stationary ϕ-mixing processes. The object of the present paper is to show that the Bahadur representation holds, but with a slightly different order of the remainder term, for stationary sequences of strong mixing random variables. We also consider the Bahadur representation for absolutely regular processes and the a.s. representation of quantile processes for ϕ-mixing and strong mixing processes.

Let $\{X_n, n\geqq 1\}$ be a strictly stationary sequence of random variables defined on a probability space $(\Omega, \mathscr{B}, P)$. We shall say that the sequence $\{X_n\}$ is ϕ-mixing if

(1.1) $$\sup|P(AB)-P(A)P(B)|/P(A)=\phi(n)\downarrow 0 \quad (n\to\infty),$$

absolutely regular if

(1.2) $$E\{\sup|P(B|\mathscr{M}_1^k)-P(B)|\}=\beta(n)\downarrow 0 \quad (n\to\infty),$$

and strong mixing if

(1.3) $$\sup|P(AB)-P(A)P(B)|=\alpha(n)\downarrow 0 \quad (n\to\infty).$$

Here the supremum is taken over all $A\in\mathscr{M}_1^k$ and $B\in\mathscr{M}_{k+n}^{\infty}$, and $\mathscr{M}_a^b$ denotes the σ-field generated by $X_n(a\leqq n\leqq b)$. Among these conditions (1.1)-(1.3), the following inequalities hold:

$$\alpha(n)\leqq\beta(n)\leqq\phi(n).$$

For a discussion of mixing conditions, see for example, Ibragimov and Linnik [3]. In late sections, in addition to (1.3), we may need the following condition:

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(1.4) $$\sum_{n=1}^{\infty} n^{2k}\{\alpha(n)\}^{\delta}<\infty \text{ for some } k\geqq 1 \text{ and } 0<\delta<1.$$

We assume that each X_i has a distribution function $(df)F$ on the unit interval $I=[0, 1]$. For $p\in I$, we write $\xi(=\xi_p)$ for the p-quantile of F. Let $0=X_{n,0}\leqq X_{n,1}\leqq \cdots\leqq X_{n,n}\leqq X_{n,n+1}=1$ denote the order statistics of the sample $(X_1,\cdots, X_n)$. Define the empirical df F_n and the sample p-quantile $Z_n(=Z_{p,n})$ as follows:

(1.5) $$F_n(t)=n^{-1}\sum_{i=1}^{n}c(t-X_i),\ t\in I,$$

where $c(u)=1(u\geqq 0);=0(u<0)$, and

(1.6) $$Z_n=X_{n,r},\quad \text{where } r=[np]+1.$$

Further define the empirical process $V_n=\{V_n(t):\ t\in I\}$ and the quantile process $Q_n=\{Q_n(t):\ t\in I\}$ as follows:

(1.7) $$V_n(t)=n^{1/2}[F_n(F^{-1}(t))-t],$$

and

(1.8) $$Q_n(t)=n^{1/2}[Z_{t,n}-F^{-1}(t)].$$

Where no confusion is possible K, K_α, etc., denote generic constants.

The main results are stated in Section 3. Section 2 deals with basic lemmas on strong mixing Bernoullian random variables, which are used in the proofs of the main theorem in Section 4 and the further extension to quantile process in Section 8.

2. Preliminary lemmas.

Let $\{z_i\}$ be stationary sequence satisfying (1.3) with

(2.1) $$Ez_i=0,\ Ez_i^2=\tau;\ 0<\tau<1,$$
$$P(|z_i|>1)=0 \text{ and } E|z_i|<c\tau,\ 0<c<\infty.$$

It is clear that (2.1) holds when z_i are Bernoullian random variables, centered at expectations. Let

$$S_n=z_1+\cdots+z_n,\ n\geqq 1.$$

LEMMA 2.1 (Yokoyama [11]). *Under* (1.4) *and* (2.1), *for every* $n\geqq 1$,

(2.2) $$E(S_n^{2(k+1)})\leqq K_\alpha\{n\tau^{1-\delta}+\cdots+n^{k+1}\tau^{(k+1)(1-\delta)}\},\ K_\alpha<\infty.$$

Consider now a double sequence of Bernoullian random variables

(2.3) $$U_{ni}=U_n(X_i),\ i=1,\cdots, n;$$
$$P(U_{ni}=1)=1-P(U_{ni}=0)=p_n,\ 0\leqq p_n\leqq 1,$$

where the basic random variables $\{X_n\}$ satisfy (1.3). Then we have the following two lemmas.

LEMMA 2.2. *If* (1.4) *holds, then for every positive* K, *there exist positive constant* K_α *such that, for every* $n \geqq 2$,

$$P(n^{-1}\sum_{i=1}^{n} U_{ni} - p_n > Kn^{-\beta}\log n) \tag{2.4}$$

$$\leqq \begin{cases} K_\alpha n^{-1}(\log n)^{-2} & \text{if } \beta \leqq k/2(k+1), \\ K_\alpha n^{-2}(\log n)^{-2} & \text{if } \beta \leqq (k-1)/2(k+1). \end{cases}$$

PROOF. If we put

$$z_i = U_{ni} - p_n, \tag{2.5}$$

then $z_1, \cdots, z_n$ satisfy (2.1). From Lemma 2.1,

$$E(\sum_{i=1}^{n} z_i)^{2(k+1)} \leqq K_{\alpha,1} n^{k+1}, \quad 0 < K_{\alpha,1} < \infty, \tag{2.6}$$

and hence, by the Markov inequality

$$P(n^{-1}\sum_{i=1}^{n} U_{ni} - p_n > Kn^{-\beta}\log n) \tag{2.7}$$

$$\leqq P(|\sum_{i=1}^{n} z_i| > Kn^{1-\beta}\log n)$$

$$\leqq (Kn^{1-\beta}\log n)^{-2(k+1)} E(\sum_{i=1}^{n} z_i)^{2(k+1)}$$

$$\leqq K_\alpha n^{-(k+1)(1-2\beta)}(\log n)^{-2(k+1)}.$$

Hence, (2.4) follows from (2.7).

LEMMA 2.3. *Suppose that* (1.4) *holds for some* $0 < \delta < (2k+1)/2(k+1)$, *and that* $K_1 n^{-\beta'}\log n \leqq p_n \leqq K_2 n^{-\beta}\log n (0 < K_1, K_2 < \infty)$ *for some* $3/[2\{(k+2)-(k+1)\delta\}] < \beta < \beta' < 1$. *Then, for every positive* K, *there exist positive* K_α $(<\infty)$ *such that*

$$P(n^{-1}\sum_{i=1}^{n} U_{ni} - p_n > Kn^{-\gamma}\log n) \leqq K_\alpha n^{-1-\gamma+\beta}(\log n)^{-2}, \tag{2.8}$$

for all n *sufficiently large, where*

$$\gamma = [k + \{(k+2)-(k+1)\delta\}\beta]/(2k+3). \tag{2.9}$$

Further, if $\beta > 5/[2\{(k+2)-(k+1)\delta\}]$ *and* $0 < \delta < (2k-1)/2(k+1)$, *then for all* n *sufficiently large,*

$$LHS(2.8) \leqq K_\alpha n^{-2-\gamma+\beta}(\log n)^{-2}, \tag{2.10}$$

where

(2.11) $$\gamma=[(k-1)+\{(k+2)-(k+1)\delta\}\beta]/(2k+3).$$

We note that each γ given by (2.9) and (2.11) satisfies

(2.12) $$1/2<\gamma<3/4.$$

PROOF. By assumption on p_n, for all n sufficiently large,

(2.13) $$n\{p_n(1-p_n)\}^{1-\delta}\geqq 1.$$

Hence, it follows from Lemma 2.1 that for all n sufficiently large

(2.14) $$E(\sum_{i=1}^{n} z_i)^{2(k+1)}\leqq K_{\alpha,1}n^{k+1}\{p_n(1-p_n)\}^{(k+1)(1-\delta)}$$
$$\leqq K_{\alpha,2}n^{(k+1)\{1-(1-\delta)\beta\}}(\log n)^{(k+1)(1-\delta)}.$$

Then, (2.8) and (2.10) follow from (2.14) and the Markov inequality.

In the sequel, we shall consider k, β and δ in one of the following three cases.

(2.15) (i) $k\geqq 2,\ 0<\delta<(k-1)/(k+1)$,
$3/[2\{(k+2)-(k+1)\delta\}]<\beta\leqq 1/2$,

(2.16) (ii) $k\geqq 3,\ 0<\delta<(k^2-k-3)/k(k+1)$,
$3/[2\{(k+2)-(k+1)\delta\}]<\beta\leqq k/2(k+1)$,

(2.17) (iii) $k\geqq 6,\ 0<\delta<(k^2-4k-7)/(k^2-1)$,
$5/[2\{(k+2)-(k+1)\delta\}]<\beta\leqq (k-1)/2(k+1)$.

3. The Bahadur representation of sample quantiles for strong mixing sequences.

Let the stationary sequence $\{X_n\}$ satisfy (1.3). We assume that $F(t)$ is absolutely continuous in some neighborhood of its p-quantile ξ, and has a continuous density function $f(t)$, such that

(3.1) $$0<f(\xi)<\infty.$$

Let

(3.2) $$\upsilon^2=\upsilon_0+2\sum_{h=1}^{\infty}\upsilon_h;$$
$$\upsilon_h=E\{c(\xi-X_1)c(\xi-X_{1+h})\}-p^2,\ h\geqq 0.$$

We note that $\sum\alpha(n)<\infty$ implies $\upsilon<\infty$. Further, let

(3.3) $$\sigma=\upsilon/f(\xi),$$

and assume that

(3.4) $$\sigma>0$$

For β $(0<\beta\leqq 1/2)$, we define

(3.5) $$\mathrm{I}_n(\beta)=\{t:\xi-n^{-\beta}\log n\leqq t\leqq\xi+n^{-\beta}\log n\}$$

Then, we have the following.

THEOREM 3.1. *If* (1.4) *holds for some k and* δ *in* (2.15), *and if* (3.1) *holds, then as* $n\to\infty$,

(3.6) $$\sup\{|[F_n(t)-F(t)]-[F_n(\xi)-F(\xi))]|:t\in I_n(\beta)\}$$
$$=O(n^{-\gamma}\log n)\ \text{a.s.},$$

where γ *and* β *are given by* (2.9) *and* (2.15), *respectively. If, in addition to* (3.1),

(3.7) $f'(t)$ *is bounded in some neighborhood of* ξ,

and if

(3.8) (1.4) *holds for some k and* δ *in* (2.16),

then as $n\to\infty$

(3.9) $$|n^{1/2}\{[Z_n-\xi]f(\xi)+[F_n(\xi)-p]\}|$$
$$=O(n^{-\gamma_1}\log n)\ \text{a.s.},$$

where $\gamma_1=\gamma-1/2$ *and* γ *is given by* (2.9) *corresponding to* β *in* (2.16).

Under (3.1), (3.4) *and* (3.8),

(3.10) $$\mathscr{L}(n^{1/2}[Z_n-\xi]/\sigma)\to\mathscr{N}(0,1).$$

Finally, under (3.1), (3.4), (3.7) *and* (3.8),

(3.11) $$\limsup_{n\to\infty} n^{1/2}(Z_n-\xi)/[\sigma(2\log\log n)^{1/2}]=1\quad\text{a.s.},$$

(3.12) $$\liminf_{n\to\infty} n^{1/2}(Z_n-\xi)/[\sigma(2\log\log n)^{1/2}]=-1\quad\text{a.s.}$$

4. Proof of Theorem 3.1.

The proof follows on the same line as in Theorem 3.1 of Sen [10]. We consider a set of real numbers

(4.1) $$\eta_{r,n}=\xi+rn^{-\gamma}\log n\quad\text{for}\quad r=0,\pm1,\cdots,\pm b_n;$$
$$b_n=[n^{\gamma-\beta}]+1.$$

Then we have

$$\sup\{|F_n(t)-F(t)-F_n(\xi)+p|:t\in I_n(\beta)\}$$
$$\leqq \max_{-b_n\leqq r\leqq b_n}|F_n(\eta_{r,n})-F(\eta_{r,n})-F_n(\xi)+p|$$
$$+\max_{-b_n\leqq r\leqq b_n-1}|F(\eta_{r+1,n})-F(\eta_{r,n})|. \tag{4.2}$$

By (3.1) and the continuity of $f(t)$ in some neighborhood of ξ, if we choose n sufficiently large, the second term on the *RHS* of (4.2) is $O(n^{-\gamma}\log n)$. To complete the proof of (3.6), it suffices to show that the first term on the *RHS* of (4.2) is also $O(n^{-\gamma}\log n)$, with probability one, as $n\to\infty$. Let

$$U_{ni}^{(r)}=c(\eta_{r,n}-X_i)-c(\xi-X_i), \tag{4.3}$$
$$i=1,\cdots,n;\ r=1,\cdots,b_n.$$

Then $U_{ni}^{(r)}$ are zero-one-valued random variables, for which

$$P(U_{ni}^{(r)}=1)=F(\eta_{r,n})-F(\xi)=p_{r,n}; \tag{4.4}$$

$$K_1n^{-\gamma}\log n\leqq p_{r,n}\leqq K_2n^{-\beta}\log n,\quad 0<K_1,\ K_2<\infty, \tag{4.5}$$

where K_1 and K_2 depend on $f(\xi)$, and

$$F_n(\eta_{r,n})-F_n(\xi)=n^{-1}\sum_{i=1}^{n}U_{ni}^{(r)}. \tag{4.6}$$

From Lemma 2.3, it follows that

$$P(|F_n(\eta_{r,n})-F(\eta_{r,n})-F_n(\xi)+p|>Kn^{-\gamma}\log n)$$
$$\leqq K_\alpha n^{-1-\gamma+\beta}(\log n)^{-2} \tag{4.7}$$

for all n sufficiently large and all $r=1,\cdots,b_n$. The same inequality holds for $r=-b_n,\cdots,-1$. Hence,

$$P(\max_{-b_n\leqq r\leqq b_n}|F_n(\eta_{r,n})-F(\eta_{r,n})-F_n(\xi)+p|>Kn^{-\gamma}\log n)$$
$$\leqq 2K_\alpha b_nn^{-1-\gamma+\beta}(\log n)^{-2}=O(n^{-1}(\log n)^{-2}), \tag{4.8}$$

and the proof of (3.6) follows from (4.2), (4.8) and the Borel-Cantelli lemma.

To prove (3.9), we note that

$$P(Z_n<\xi-n^{-\beta}\log n) \tag{4.9}$$
$$=P([n^{-1}\sum_{i=1}^{n}c(\xi-n^{-\beta}\log n-X_i)-F(\xi-n^{-\beta}\log n)]$$
$$\geqq r/n-\mathrm{F}(\xi-n^{-\beta}\log n)),$$

where by (3.1), as $n\to\infty$

(4.10) $$r/n-F(\xi-n^{-\beta}\log n)=f(\xi)n^{-\beta}\log n(1+o(1)).$$

Hence, by (2.4), (4.9) and (4.10), we have

(4.11) $$Z_n>\xi-n^{-\beta}\log n \quad \text{a.s.,} \quad \text{as } n\to\infty.$$

In a similar way, we have

(4.12) $$Z_n<\xi+n^{-\beta}\log n \quad \text{a.s.,} \quad \text{as } n\to\infty.$$

From (3.6), (3.7), (4.11) and (4.12), (3.9) follows.

The proofs of (3.10)−(3.12) are the same as those of Sen [10] and so are omitted.

REMARK. We remark that

(4.13) $$\mathscr{L}(n^{1/2}[F_n(\xi)-p]/\upsilon)\to\mathscr{N}(0, 1)$$

holds under $\sum\alpha(n)<\infty$(cf. [3]), and that

(4.14) $$\limsup_{n\to\infty} n^{1/2}[F_n(\xi)-p]/[\upsilon(2\log\log n)^{1/2}]=1 \quad \text{a.s.,}$$

(4.15) $$\liminf_{n\to\infty} n^{1/2}[F_n(\xi)-p]/[\upsilon(2\log\log n)^{1/2}]=-1 \quad \text{a.s.}$$

hold under $\alpha(n)=O(n^{-1-\epsilon})$ for some $\epsilon>0$ (cf. [8]).

5. The Bahadur representation for absolutety regular processes.

In this section, we consider Bahadur's representation of sample quantiles for stationary sequence satisfying (1.2).

Let $\{Y_n, n\geqq 1\}$ be a sequence of zero-one-valued random variables satisfying (1.2), and

(5.1) $$P(Y_i=1)=1-P(Y_i=0)=p;\ 0<p<1.$$

Let

$$S_n=Y_1+\cdots+Y_n, \quad n\geqq 1.$$

Then, we have the following.

LEMMA 5.1. *For δ: $0<\delta<1$, and all $s>0$,*

(5.2) $$P(|S_n-np|>s)\leqq 2(n^\delta+1)\{exp(-h)+n^{1-\delta}\beta(n^\delta)\},$$

where

(5.3) $$\begin{aligned} h&=h(n, p, s)\\ &=t^2/[2(n^{1-\delta}p(1-p)+(t/3)\max\{p, 1-p\})] \end{aligned}$$

and

$$t = t_n = s/(n^{\delta}+1). \tag{5.4}$$

RROOF. Choose an integer $k = k_n = [n^{\delta}]+1$, and write

$$S_n = U_1 + \cdots + U_k, \tag{5.5}$$

where

$$U_j = Y_j + Y_{j+k} + \cdots + Y_{j+m_j k}, \quad 1 \leqq j \leqq k \tag{5.6}$$

and $m_j = m_{n,j}$ is the largest positive integer for which $j + m_j k \leqq n$. We note that

$$m_j \leqq m_1 \leqq n^{1-\delta}-1, \text{ for } j=1,\cdots, k \text{ and } k<n. \tag{5.7}$$

From (5.5), it follows that

$$\begin{aligned} &P(|S_n - np| > s) \\ &\leqq P(\sum_{j=1}^{k} |U_j - (m_j+1)p| > s) \\ &\leqq \sum_{j=1}^{k} P(|U_j - (m_j+1)p| > k^{-1}s). \end{aligned} \tag{5.8}$$

For fixed j $(1 \leqq j \leqq k)$, let A_j be the Borel subset of the (m_j+1)-dimensional Euclidean space R^{m_j+1} defined by

$$A_j = \{(y_0, \cdots, y_{m_j}) : |\sum_{i=0}^{m_j} (y_i - p)| > k^{-1}s\} \tag{5.9}$$

and define the Borel function g by

$$g(y_0, \cdots, y_{m_j}) = \begin{cases} 1 & \text{if } (y_0, \cdots, y_{m_j}) \in A_j \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

Using Lemma 1 of Yoshihara [14] repeatedly, we have

$$\begin{aligned} &P(|U_j - (m_j+1)p| > k^{-1}s) \\ &= Eg(Y_j, \cdots, Y_{j+m_j}) \\ &\leqq \int_{R^{m_j+1}} \cdots \int g(y_0, \cdots, y_{m_j}) dG(y_0) \cdots dG(y_{m_j}) + 2m_j \beta(k) \\ &= P(|\sum_{i=0}^{m_j} (Y_i' - p)| > k^{-1}s) + 2m_j \beta(k), \end{aligned} \tag{5.11}$$

where $\{Y_n'\}$ are independent and identically distributed random variables with same df G as that of Y_1. By the Bernstein inequality

(5.12) $$P(|\sum_{i=0}^{m_j}(Y_i'-p)|>k^{-1}s)\leqq 2\exp(-h),$$

where h is given by (5.3). Since (5.11) holds for each j ($1\leqq j\leqq k$), the proof of (5.2) follows from (5.8), (5.11) and (5.12).

THEOREM 5.1. *Under* (3.1) *and*

(5.13) $$\sum n^{5/4}\beta(n^{3/4-\gamma})<\infty \text{ for some } 1/2<\gamma<3/4,$$

as $n\to\infty$

(5.14) $$\sup\{|[F_n(t)-F(t)]-[F_n(\xi)-F(\xi)]| : t\in I_n(\gamma-1/4)\}$$
$$=O(n^{-\gamma}\log n) \quad \text{a.s.}$$

If, in addition to (3.1), (3.7) *holds, as* $n\to\infty$

(5.15) $$|n^{1/2}\{(Z_n-\xi)f(\xi)+[F_n(\xi)-p]\}|$$
$$=O(n^{-\gamma+1/2}\log n) \quad \text{a.s.}$$

Further, under (3.1), (3.4) *and* (5.13), (3.10) *holds, and under* (3.1), (3.4), (3.7) *and* (5.13), (3.11) *and* (3.12) *both hold.*

PROOF. We have only to prove that the first term on the *RHS* of (4.2) is $O(n^{-\gamma}\log n)$ and that (4.11) and (4.12) both hold for $\beta=\gamma-1/4$. Define b_n in (4.1) by

(5.16) $$b_n=[n^{1/4}]+1.$$

Then, $p_{r,n}$ in (4.4) satisfies

(5.17) $$p_{r,n}\leqq K_1 n^{-\gamma+1/4}\log n \quad (0<K_1<\infty).$$

From Lemma 5.1, we have

$$P(\max_{-b_n\leqq r\leqq b_n}|F_n(\eta_{r,n})-F(\eta_{r,n})-F_n(\xi)+p|>Kn^{-\gamma}\log n)$$

(5.18) $$\leqq 4b_n(n^\delta+1)\{\exp(-h_n)+n^{1-\delta}\beta(n^\delta)\},$$

where $h_n=h(n, p_{r,n}, Kn^{1-\gamma}\log n)$. Since $h(n,p,s)\geqq s^2/4(n^{1+\delta}p+n^\delta s)$, if we put $\delta=3/4-\gamma$, for all n sufficiently large,

(5.19) $$h_n\geqq K^2\log n/4(K_1+Kn^{-1/4})$$
$$=\lambda_n \text{ say,}$$

and then,

(5.20) $$\log\{b_n n^\delta \exp(-\lambda_n)\}/\log n\to(1-\gamma)-(K^2/4K_1).$$

If, given K_1, K is chosen suficiently large, the limit in (5.20) is less than -1, then by (5.19),

(5.21) $$\sum b_n n^{\delta} \exp(-h_n) < \infty,$$

and hence (5.14) follows. By (4.9), (4.10) and Lemma 5.1, as $n \to \infty$

(5.22) $$P(Z_n < \xi - n^{-\gamma+1/4} \log n)$$
$$\leqq 2(n^{\delta}+1)\{\exp(-h_n') + n^{1-\delta}\beta(n^{\delta})\},$$

where $h_n' = h(n, p, f(\xi)n^{5/4-\gamma} \log n)$. If we put $\delta = 3/2 - 2\gamma$, for sufficiently large n,

(5.23) $$h_n' \geqq f^2(\xi)(\log n)^2/p(1-p).$$

Since $f^2(\xi)/p(1-p) > 0$, it follows from (5.23) that $\sum n^{\delta} \exp(-h_n') < \infty$. It is clear that $\sum n\beta(n^{\delta}) < \infty$, and hence

(5.24) $$Z_n > \xi - n^{-\gamma+1/4} \log n \quad \text{a.s.} \quad \text{as } n \to \infty.$$

It also follows that

(5.25) $$Z_n < \xi + n^{-\gamma+1/4} \log n \quad \text{a.s.} \quad \text{as } n \to \infty.$$

REMARK. Recently, Yoshihara [15] proved that under the condition $\beta(n) = O(e^{-tn})$ for some $t > 0$,

$$\sup\{|[F_n(t) - F(t)] - [F_n(\xi) - F(\xi)]| : t \in I_n(1/2)\}$$
$$= O(n^{-3/4}(\log n)^2) \quad \text{a.s.}$$

6. Functional central limit theorems for sample quantiles.

Let $D = D[0, 1]$ be the space of functions on I that are right-continuous and have left-hand limit, with uniform topology

(6.1) $$d(f, g) = \sup_{t \in I} |f(t) - g(t)| \quad \text{for } f, g \in D.$$

Let $\{X_n\}$ satisfy (1.3), and assume that F is twice differentiable on I, and

(6.2) $$\inf_{t \in I} f(t) > 0 \quad \text{and} \quad \sup_{t \in I} f'(t) < \infty \quad (f = F').$$

We define for every $s, t \in I$,

(6.3) $$v(s, t) = \lim_{n \to \infty} \{n \operatorname{cov}[F_n(F^{-1}(s)), F_n(F^{-1}(t))]\}$$

where we note that $\sum \alpha(n) < \infty$ implies $|v(s,t)| < \infty$ for all $s, t \in I$. Let, under (6.2),

(6.4) $$\sigma(s, t) = v(s,t)/f(F^{-1}(s))f(F^{-1}(t)), \quad s, t \in I.$$

Finally, let Q_n be the quantile process defined by (1.8), and let $Z = \{Z(t) : t \in I\}$ be a Gaussian random function on I such that $EZ(t) = 0$ and $E\{Z(s)Z(t)\} = \sigma(s, t)$ for all s, $t\in I$. Then we have the following.

THEOREM 6.1. *Under* (3.8) *and* (6.2),

$$Q_n \xrightarrow{\mathscr{D}} Z \quad (in\ D). \tag{6.5}$$

Next, we consider the space $C=C[0, 1]$ of all continuous functions on I with uniform topology. Define a process $Y_n=\{Y_n(t):t\in I\}$ by

(6.6) $Y_n(0)=0$, $Y_n(i/n)=i[Z_i-\xi]/(\sigma n^{1/2})$, $i=1,\cdots, n$, and linear interpolation for $t\in[(i-1)/n, i/n]$, $i=1,\cdots, n$.

THEOREM 6.2. *Under* (3.4), (3.7) *and* (3.8),

$$Y_n \xrightarrow{\mathscr{D}} W \quad (in\ C) \tag{6.7}$$

where $W=\{W(t):t\in I\}$ *is a Wiener process on* I.

Let $\{N_r\}$ be a sequence of positive integer-valued random variables, such that as $r\to\infty$

$$r^{-1}N_r\to\lambda, \quad \text{in probability,} \tag{6.8}$$

where λ is a positive random variable defined on the same probability space $(\Omega, \mathscr{B}, P)$. Then we have

THEOREM 6.3. *Under* (6.8) *and the assumptions of Theorem* 6.2,

$$Y_{N_r} \xrightarrow{\mathscr{D}} W \quad (in\ C) \tag{6.9}$$

7. Proofs of Theorems 6.1—6.3.

The proofs of Theorems 6.2 and 6.3 are the same as those of Sen [10, Theorem 6.2], and so are omitted.

RROOF OF THEOREM 6.1. On D, define another empirical process $V_n^*=\{V_n^*(t): t\in I\}$ by

$$V_n^*(t)=V_n(t)/f(F^{-1}(t)). \tag{7.1}$$

where V_n is the empirical process defined by (1.7).
Since Z lies in C with probability one, V_n^* converges weakly in the uniform topology on D to Z under the condition $\alpha(n)=O(n^{-5/2-\varepsilon})$ for some $\epsilon>0$ (cf. [13]). We complete the proof of theorem by showing that

$$d(Q_n, V_n^*) \xrightarrow{P} 0 \quad \text{as } n\to\infty. \tag{7.2}$$

We consider the case $F(t)=t$. We shall prove (7.2) in three steps. Define another quantile process $\tilde{Q}_n=\{\tilde{Q}_n(t):t\in I\}$ by

(7.3) $$\tilde{Q}_n(i/n)=Q_n(i/n), \quad i=0,\cdots, n,$$

and by linear interpolation for $t\in[(i-1)/n, i/n]$, $i=1,\cdots, n$. Also define another empirical process $\tilde{V}_n=\{\tilde{V}_n(t): t\in I\}$ in the same manner, i.e., $\tilde{V}_n(t)$ is the linear interpolation of $V_n(t)$ between the points $t=i/n$ for $i=0,\cdots, n$.
Firstly, we show that

(7.4) $$\mathrm{d}(V_n, \tilde{V}_n)\xrightarrow{P}0 \quad \text{as } n\to\infty.$$

Let $I_{ni}=[(i-1)/n, i/n]$ for $i=1,\cdots, n$. Then we have

(7.5) $$\begin{aligned}&\sup|V_n(t)-\tilde{V}_n(t)|\\ &\leqq\max_{1\leqq i\leqq n}\sup_{t\in I_{ni}}|V_n(t)-\tilde{V}_n(t)|\\ &\leqq 2\max_{1\leq i\leq n}\sup_{t\in I_{ni}}|V_n(t)-V_n(i/n)|.\end{aligned}$$

Hence, from (2.8), (7.5) and the arguments of the proof of (3.6) as in (4.1)–(4.8), for $K>1$ and all n sufficiently large, it follows that

(7.6) $$\begin{aligned}&P(\sup_{t\in I}|V_n(t)-\tilde{V}_n(t)|>2Kn^{-\gamma+1/2}\log n)\\ &\leqq\sum_{i=1}^{n}P(\sup_{t\in I_{ni}}|F_n(t)-F_n(i/n)-t+i/n|>Kn^{-\gamma}\log n)\\ &=O((\log n)^{-2})\end{aligned}$$

which implies (7.4).

By repeating the method of the proof of (3.9) as in (4.9)–(4.12), we have, as $n\to\infty$

(7.7) $$\begin{aligned}&P(\max_{0\leq i\leq n}|Q_n(i/n)-V_n(i/n)|>Kn^{-\gamma+1/2}\log n)\\ &=O((\log n)^{-2}).\end{aligned}$$

From (7.3), (7.7) and the definition of $\tilde{V}_n$, it follows that

(7.8) $$d(\tilde{V}_n, \tilde{Q}_n)\xrightarrow{P}0 \quad \text{as } n\to\infty.$$

Finally, we show that

(7.9) $$d(Q_n, \tilde{Q}_n)\xrightarrow{P}0 \quad \text{as } n\to\infty,$$

then (7.2) follows from (7.4), (7.8) and (7.9). By (7.3)

(7.10) $$\sup_{t\in I}|Q_n(t)-\tilde{Q}_n(t)|\leqq\max_{1\leqq i\leqq n}n^{1/2}|X_{n,i+1}-X_{n,i}|,$$

and, for $i=1,\cdots, n$,

(7.11)
$$\begin{aligned} & n^{1/2}|X_{n,i+1}-X_{n,i}| \\ & \leqq |Q_n(i/n)-V_n(i/n)|+|Q_n((i-1)/n)-V_n((i-1)/n)| \\ & \quad +|V_n(i/n)-V_n((i-1)/n)|+n^{-1/2}. \end{aligned}$$

From (7.5), (7.6), (7.7), (7.10) and (7.11), (7.9) follows. Thus, the proof is completed in the case $F(t)=t$.

For an arbitrary twice differentiable F, the proof of (7.2) follows from the essentially same line as the above proof and so is omitted.

8. Almost sure representation of quantile processes.

Let $\{X_n\}$ satisfy (1.3), and let Q_n be quantile process and V_n be empirical process defined by (1.8) and (1.7), respectively.

THEOREM 8.1. *If* (1.4) *holds for somd k and δ in* (2.17), *and if* (6.2) *holds, then as $n\to\infty$*

(8.1)
$$\sup_{t\in I}|Q_n(t)f(F^{-1}(t))-V_n(t)|=O(n^{-\gamma_1}\log n) \quad \text{a.s.},$$

where $\gamma_1=\gamma-1/2$ and γ is given by (2.11) *corresponding to β in* (2.17).

PROOF. We use the notations and methods of previous section, and prove the theorem for the case of uniform distribution as in Theorem 6.1. If we use the second inequality in (2.4), then (7.7) is replaced by

(8.2)
$$\begin{aligned} & P(\max_{0\leqq i\leqq n}|Q_n(i/n)-V_n(i/n)|>Kn^{-\gamma_1}\log n) \\ & =O(n^{-1}(\log n)^{-2}). \end{aligned}$$

Further, applying the inequality (2.10) to (7.6), we have

(8.3)
$$\begin{aligned} & P(\sup_{t\in I}|(V_n(t)-\tilde{V}_n(t)|>Kn^{-\gamma_1}\log n) \\ & =O(n^{-1}(\log n)^{-2}). \end{aligned}$$

By (8.2), (8.3) and the Borel-Cantelli lemma, as $n\to\infty$

(8.4)
$$\sup_{t\in I}|\tilde{Q}_n(t)-\tilde{V}_n(t)|=O(n^{-\gamma_1}\log n) \quad \text{a.s.},$$

and

(8.5)
$$\sup_{t\in I}|V_n(t)-\tilde{V}_n(t)|=O(n^{-\gamma_1}\log n) \quad \text{a.s.}$$

Finally, from (7.5), (7.10), (7.11), (8.4), and (8.5), it follows that, as $n\to\infty$

(8.6) $$\sup_{t\in I}|Q_n(t)-\tilde{Q}_n(t)|=O(n^{-\gamma_1}\log n) \quad \text{a.s.}$$

and the proof of (8.1) follows from (8.4), (8.5) and (8.6).

9. Concluding remarks.

(i) Quantile processes for ϕ-mixing random variables.

For ϕ-mixing random variables, Sen [10] investigated the Bahadur representation of sample p-quantile and the weak convergence of quantile processes which are different from ours. But his results are immediately applicable to our Section 6 and Section 8. Let $\{X_n\}$ satisfy (1.1) with

(9.1) $$\phi(n)=O(n^{-2}),$$

and let each X_i have a *df F on I*. We note that (9.1) is a sufficient condition for weak convergence of empirical *df*'s to a Gaussian random function (cf. [12]). It is easy to check that Lemmas 4.1, 4.3 and 4.4 of Sen [10] hold under (9.1), and hence, his Theorem 3.1 holds under (9.1), instead of his (2.2), by using the central limit theorem for bounded random variables (cf. [3]). From these remarks and our methods, we have the following.

THEOREM 9.1. *Under* (6.2) *and* (9.1), *as* $n\to\infty$

(9.2) $$\sup_{t\in I}|Q_n(t)f(F^{-1}(t))-V_n(t)|=O(n^{-1/8}\log n) \quad \text{a.s.}$$

Further, under (6.2) *and*

(9.3) $$\phi(n)=O(e^{-tn}) \text{ for some } t>0,$$

as $n\to\infty$

(9.4) $$LHS(9.2)=O(n^{-1/4}\log n) \quad \text{a.s.}$$

(ii) Quantiles for multivariate distributions.

As in [10], we can consider quantiles for a strictly stationary strong mixing sequence $\{\bar{X}_i\}$ of stochastic vectors. We assume that $\bar{X}_i$ has a q-variate distribution $F(\bar{x})$, $\bar{x}\in R^q$, and $\{\bar{X}_i\}$ satisfies (1.3). Let $\bar{\xi}=(\xi_1,\cdots,\xi_q)$, be a point in R^q and

(9.5) $$P(X_{ij}\leqq\xi_j)=p_j:\ 0<p_j<1,\ j=1,\cdots,q,\ i=1,2,\cdots$$

where X_{ij} is the j-th variate of $\bar{X}_i$. Further, we assume that is some neighborhood of $\bar{\xi}$, $F(\bar{x})$ is strictly monotonic is each of its q coordinates and admits of a continuous density function $f(\bar{x})$, such that

(9.6) $$0<f(\bar{\xi})<\infty.$$

Define

(9.7) $$\bar{I}_n(\beta)=\{\bar{x}=(x_1,\cdots, x_q): \max_{1\leq i\leq q}|x_i-\xi_i|\leqq n^{-\beta}\log n\}$$

and put $F(\bar{\xi})=p$ where we assume that $0<p<1$. Finally, let

(9.8) $$F_n(\bar{x})=n^{-1}[\sharp \text{ of } \bar{X}_i=(X_{i1},\cdots, X_{iq}): X_{ij}\leqq x_j,\ j=1,\cdots, q \text{ for } i=1,\cdots, n].$$

THEOREM 9.2. *If* (1.4) *holds for some k and δ in* (2.15) *and if* (9.6) *holds, then as $n\to\infty$*

[9.9] $$\sup\{|[F_n(\bar{x})-F(\bar{x})]-[F_n(\bar{\xi})-F(\bar{\xi})]| : \bar{x}\in\bar{I}_n(\beta)\} = O(n^{-\gamma}\log n) \quad \text{a.s.}$$

where γ and β are given by (2.9) *and* (2.15), *respectively.*

The proof follows along the same line as in Theorem 3.1 and so is omitted (cf. Theorem 6.4 in [10]).

Analogous results to Theorems 6.5 and 6.6 in [10] may also be proved, but we shall not enter into details.

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Institute of Mathematics
The University of Tsukuba

Present address:
Department of Mathematics
Osaka Kyoiku University
Ikeda, Osaka, Japan