TSUKUBA J. MATH.
Vol. 2 (1977). 35—68

ON CONJUGATE LOCI AND CUT LOCI OF COMPACT SYMMETRIC SPACES I

By

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Introduction

Let (M, g) be a compact connected Riemannian manifold. Fix a point o of M and denote by $T_o(M)$ the tangent space of M at o. Let Exp: $T_o(M) \longrightarrow M$ be the exponential map of (M, g) at o. A tangent vector $X \in T_o(M)$ is called a *tangential conjugate point* of (M, g), if Exp is degenerate at X. The set $\tilde{Q}$ of all tangential conjugate points of (M, g) in $T_o(M)$ is called the *tangenital conjugate locus* of (M, g) in $T_o(M)$. The image $Q = \operatorname{Exp} \tilde{Q}$ of $\tilde{Q}$ under Exp is called the *conjugate locus* of (M, g) with respect to o.

Let γ: $[0, \infty) \longrightarrow M$ be a geodesic of (M, g) (parametrized by arc-length) emanating from o. Let $X_1 = \dot{\gamma}(0) \in T_o(M)$ denote the initial tangent vector of γ. Assume that the set of $t \in [0, \infty)$ such that $tX_1 \in \tilde{Q}$ is not empty and let t_0 be the infimum of this set. Then the tangent vector $t_0 X_1$ is called the *tangential first conjugate point along* γ. The set $\tilde{F}$ of all $X \in T_o(M)$ which is the tangenital first conjugate point along some geodesic γ emanating from o, is called the *tangenital first conjugate locus* of (M, g) in $T_o(M)$. The image $F = \operatorname{Exp} \tilde{F}$ of $\tilde{F}$ under Exp is called the *first conjugate locus* of (M, g) with respect to o.

Let again γ: $[0, \infty) \longrightarrow M$ be a geodesic emanating from o and $X_1 = \dot{\gamma}(0)$. Let $\bar{t}_0$ be the supremum of the set of $t \in [0, \infty)$ such that $\gamma | [0, t]$ is a minimal geodesic segment from o to $\gamma(t)$. The number $\bar{t}_0$ is always finite since M is compact. Then the tangent vector $\bar{t}_0 X_1$ is called the *tangenital cut point along* γ. The set $\tilde{C}$ af all $X \in T_o(M)$ which is the tangenital cut point along some geodesic γ emanating from o, is called the *tangenital cut locus* of (M, g) in $T_o(M)$. The image $C = \operatorname{Exp} \tilde{C}$ of $\tilde{C}$ under Exp is called the *cut locus* of (M, g) with respect to o.

In the present article, we shall study the structures of the conjugate locus, the first conjugate locus and the cut locus of a compact symmetric space.

Helgason [3] showed by a group theoretical method that the conjugate locus of a compact connected Lie group M, endowed with a bi-invariant Riemannian metric g, is nicely stratified in the sense that it is the disjoint union of smooth submani-

Received December 28, 1977

folds of M. On the other hand, Wong [12], [13], [14] studied conjugate loci and cut loci of Grassmann manifolds by a geometric method and gave stratifications of them. Recently Sakai [7] studied the cut locus of a general compact symmetric space (M, g) and showed that it is determined by the cut locus of a maximal totally geodesic flat submanifold $\hat{A}$ in (M, g). He gave in [6], [7] also stratifications of cut loci of $U(n)/O(n)$, $U(n)$, $SO(n)$, $Sp(2n)/U(n)$ and Grassmann manifolds by his method. These spaces are included in the class of so-called symmetric R-spaces. Naitoh [5] studied the cut locus of $\hat{A}$ and the first conjugate locus in $\hat{A}$ for each irreducible symmetric R-space. Moreover, Sakai [8] gave a stratification of the conjugate locus of a simply connected compact symmetric space, by a refinement of Helgason's approach.

In the present note I, we shall give a stratification of the conjugate locus Q, the first conjugate locus F and the cut locus C of a general (not necessarily simply connected) compact symmetric space (M, g) by a group theoretical method. Our stratification consists of regular submanifolds of M, which are diffeomorphic with fibre bundles over compact manifolds. Our stratification is a generalization of those of Helgason [3] and Sakai [8].

In the forthcoming paper II, we shall study topological structures of Q, F and C. Furthermore we shall give another stratification of the cut locus for a symmetric R-space M. This stratification consists of orbits of a certain group acting on M. Our results include those of Wong and Sakai on cut loci of the previously mentioned symmetric R-spaces.

§ 1. Conjugate loci of compact symmetric spaces

In this section, we shall study the structure of conjugate loci of compact symmetric spaces by a group theoretical approach.

Let G be a compact connected Lie group, K a closed subgroup of G and θ an involutive automorphism of G. Assume that the pair (G, K) is a symmetric pair with respect to θ, i.e., K lies between the subgroup:

$$G_\theta=\{x\in G;\ \theta(x)=x\}$$

and the identity component of G_θ. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. The involutive automorphism of $\mathfrak{g}$ induced by θ will be also denoted by θ. Then the pair $(\mathfrak{g}, \mathfrak{k})$ is a symmetrc pair with respect to θ, i.e., $\mathfrak{k}$ satisfies

$$\mathfrak{k}=\{X\in\mathfrak{g};\ \theta X=X\}.$$

Choose an inner product $(\ ,\ )$ on $\mathfrak{g}$, which is invariant under θ and the adjoint action of G. In what follows, for a subspace $\mathfrak{h}$ of $\mathfrak{g}$, the group of orthogonal transformations of $\mathfrak{h}$ with respect to this inner product (,), will be denoted by $O(\mathfrak{h})$. Consider the homogeneous space:

$$M=G/K,$$

and denote the origin K of M by o. Then the tangent space $T_o(M)$ of M at o is identified with the subspace:

$$\mathfrak{m}=\{X\in\mathfrak{g};\ \theta X=-X\},$$

through the canonical projection $\pi_G: G\longrightarrow M$. This subspace $\mathfrak{m}$ will be called the *canonical complement* for the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. Let g be the unique G-invariant Riemannian metric on M such that it coincides on $T_o(M)$ with the inner product (,) on $\mathfrak{m}$. Then the Riemannian manifold (M, g) is a compact connected symmetric space. Note that any compact connected symmetric space is obtained in this way. It is known that the exponential map Exp of (M, g) at the origin o is given by

$$\operatorname{Exp} X=(\exp X)o \quad \text{for } X\in\mathfrak{m}.$$

Take a Cartan subalgebra $\mathfrak{a}$, i.e., a maximal abelian subalgebra in $\mathfrak{m}$, for the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ and fix it once for all. We denote by A the toral subgroup of G generated by $\mathfrak{a}$. Let $\mathfrak{c}$ and $\mathfrak{g}'=[\mathfrak{g}, \mathfrak{g}]$ be the center and the derived algebra of $\mathfrak{g}$ respectively. Put

$$\mathfrak{k}'=\mathfrak{k}\cap\mathfrak{g}', \quad \mathfrak{m}'=\mathfrak{m}\cap\mathfrak{g}', \quad \mathfrak{a}'=\mathfrak{a}\cap\mathfrak{g}',$$
$$\mathfrak{c}_{\mathfrak{k}}=\mathfrak{c}\cap\mathfrak{k}, \quad \mathfrak{c}_{\mathfrak{m}}=\mathfrak{c}\cap\mathfrak{m}.$$

Then the pair $(\mathfrak{g}', \mathfrak{k}')$ is also a symmetric pair with respect to $\theta'=\theta|\mathfrak{g}'$ with the canonical complement $\mathfrak{m}'$. The subspace $\mathfrak{a}'$ is a Cartan subalgebra for $(\mathfrak{g}', \mathfrak{k}')$. We have

$$\mathfrak{m}=\mathfrak{c}_{\mathfrak{m}}+\mathfrak{m}', \quad \mathfrak{a}=\mathfrak{c}_{\mathfrak{m}}+\mathfrak{a}'.$$

Put

$$r=\dim\mathfrak{a}, \quad r_0=\dim\mathfrak{c}_{\mathfrak{m}}.$$

The integer r is the so-called rank of the symmetric space (M, g). For $\gamma\in\mathfrak{a}$, we define a subspace $\mathfrak{g}_\gamma{}^C$ of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ by

$$\mathfrak{g}_\gamma{}^C=\{X\in\mathfrak{g}^C;\ [H, X]=2\pi\sqrt{-1}(\gamma, H)X \text{ for each } H\in\mathfrak{a}\},$$

and put

$$\Sigma=\{\gamma\in\mathfrak{a}-\{0\};\ \mathfrak{g}_\gamma{}^C\neq\{0\}\}\subset\mathfrak{a}'.$$

An element of Σ is a root (or angular parameter) for $(\mathfrak{g}, \mathfrak{k})$ relative to $\mathfrak{a}$. Take next a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing $\mathfrak{a}$ and put

$$\mathfrak{b}=\mathfrak{t}\cap\mathfrak{k}, \quad \mathfrak{t}'=\mathfrak{t}\cap\mathfrak{g}'.$$

Then we have direct sum decompositions:

$$\mathfrak{t}=\mathfrak{b}+\mathfrak{a}=\mathfrak{c}+\mathfrak{t}'.$$

For $\alpha\in\mathfrak{t}$, we define a subspace $\tilde{\mathfrak{g}}_\alpha$ of $\mathfrak{g}^C$ by

$$\tilde{\mathfrak{g}}_\alpha=\{X\in\mathfrak{g}^C;\ [H, X]=2\pi\sqrt{-1}(\alpha, H)X \text{ for each } H\in\mathfrak{t}\},$$

and put

$$\tilde{\Sigma}=\{\alpha\in\mathfrak{t}-\{0\};\ \tilde{\mathfrak{g}}_\alpha\neq\{0\}\}\subset\mathfrak{t}'.$$

An element of $\tilde{\Sigma}$ is a root (or angular parameter) for $\mathfrak{g}$ relative to $\mathfrak{t}$. We put further

$$\tilde{\Sigma}_0=\tilde{\Sigma}\cap\mathfrak{b}.$$

Let $H\longmapsto\bar{H}$ denote the orthogonal projection from $\mathfrak{t}$ onto $\mathfrak{a}$. Then we have

$$\Sigma=\{\bar{\alpha};\ \alpha\in\tilde{\Sigma}-\tilde{\Sigma}_0\}.$$

Choose a compatible order $>$, i.e., a lexicographic order $>$ on $\mathfrak{t}$ such that

$$\alpha>0,\ \alpha\notin\tilde{\Sigma}_0\Longrightarrow -\theta\alpha>0,$$

and fix it one for all. This induces an order on $\mathfrak{a}$, which will be also denoted by $>$. Let $\tilde{\Pi}$ be the fundamental root system for $\tilde{\Sigma}$ with respect to the order $>$ and let

$$\tilde{\Pi}_0=\tilde{\Pi}\cap\tilde{\Sigma}_0.$$

Then the fundamental root system Π for Σ with respect to the order $>$ on $\mathfrak{a}$ is given by

$$\Pi=\{\bar{\alpha};\ \alpha\in\tilde{\Pi}-\tilde{\Pi}_0\}.$$

Let $\tilde{\Sigma}_+$ denote the set of positive roots in $\tilde{\Sigma}$. Then the set Σ_+ of positive roots in Σ is given by

$$\Sigma_+=\{\bar{\alpha};\ \alpha\in\tilde{\Sigma}_+-\tilde{\Sigma}_0\}.$$

Let $\mathfrak{k}_0$ and $\mathfrak{m}_0$ denote the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and $\mathfrak{a}$ respectively. Put

$$\mathfrak{k}_\gamma=\mathfrak{k}\cap(\mathfrak{g}_\gamma{}^C+\mathfrak{g}_{-\gamma}{}^C),\quad \mathfrak{m}_\gamma=\mathfrak{m}\cap(\mathfrak{g}_\gamma{}^C+\mathfrak{g}_{-\gamma}{}^C)$$

for $\gamma\in\Sigma_+$. Then we have the following lemma.

LEMMA 1.1. 1) *We have orthogonal direct sums*:

$$\mathfrak{k}=\mathfrak{k}_0+\sum_{\gamma\in\Sigma_+}\mathfrak{k}_\gamma,$$

$$\mathfrak{m}=\mathfrak{m}_0+\sum_{\gamma\in\Sigma_+}\mathfrak{m}_\gamma.$$

2) *We can choose* $S_\alpha\in\mathfrak{k}$ *and* $T_\alpha\in\mathfrak{m}$ *for each* $\alpha\in\tilde{\Sigma}_+-\tilde{\Sigma}_0$ *in such a way that*:

(1) *For each* $\gamma\in\Sigma_+$, *the sets* $\{S_\alpha;\ \bar{\alpha}=\gamma\}$ *and* $\{T_\alpha;\ \bar{\alpha}=\gamma\}$ *are basis for* $\mathfrak{k}_\gamma$ and $\mathfrak{m}_\gamma$ *respectively*;

(2) $[H, S_\alpha]=2\pi(\alpha, H)T_\alpha$, $[H, T_\alpha]=-2\pi(\alpha, H)S_\alpha$ *for each* $H\in\mathfrak{a}$;

(3) $\mathrm{Ad}(\exp H)S_\alpha=\cos 2\pi(\alpha, H)S_\alpha+\sin 2\pi(\alpha, H)T_\alpha$,
$\mathrm{Ad}(\exp H)T_\alpha=-\sin 2\pi(\alpha, H)S_\alpha+\cos 2\pi(\alpha, H)T_\alpha$

for each $H\in\mathfrak{a}$.

3) *Let* Ψ^K: $K\times\mathfrak{a}\longrightarrow M$ *be the* C^∞ *map defined by*

$$\Psi^K(k, H)=k\,\mathrm{Exp}\,H \quad \text{for } k\in K,\ H\in\mathfrak{a}.$$

Then the differential $d\Psi^K$ *of* Ψ^K *at* (k_0, H_0) *is given by*

$$(d\Psi^K)_{(k_0,H_0)}(d\tau_{k_0}(S_0+\sum_\alpha a_\alpha S_\alpha),\ H)$$

$$=d\tau_{k_0\exp H_0}d\pi_G(H-\sum_\alpha a_\alpha\sin 2\pi(\alpha, H_0)T_\alpha)$$

for $H\in\mathfrak{a}=T_{H_0}(\mathfrak{a})$ *and* $S_0\in\mathfrak{k}_0$, *where* τ_x *denotes the left translation by* x.

PROOF. 1) is an easy consequence of definitions.

2) We define a real reductive subalgebra $\mathfrak{g}^*$ of $\mathfrak{g}^C$ by

$$\mathfrak{g}^*=\mathfrak{k}+\sqrt{-1}\mathfrak{m}$$

and put

$$\mathfrak{g}_\gamma=\mathfrak{g}^*\cap\mathfrak{g}_\gamma{}^C \quad \text{for } \gamma\in\Sigma.$$

Then we have

$$\mathfrak{g}_\gamma+\mathfrak{g}_{-\gamma}=\mathfrak{k}_\gamma+\sqrt{-1}\mathfrak{m}_\gamma \quad \text{for each } \gamma\in\Sigma_+.$$

Choose an $X_\alpha\in\mathfrak{g}^*$ for each $\alpha\in\tilde{\Sigma}_+-\tilde{\Sigma}_0$ in such a way that for each $\gamma\in\Sigma_+$ the set $\{X_\alpha;\ \bar{\alpha}=\gamma\}$ is a basis for $\mathfrak{g}_\gamma$. For $\alpha\in\tilde{\Sigma}_+-\tilde{\Sigma}_0$ with $\bar{\alpha}=\gamma$, let

$$X_\alpha=S_\alpha-\sqrt{-1}T_\alpha \qquad S_\alpha\in\mathfrak{k}_\gamma,\ T_\alpha\in\mathfrak{m}_\gamma.$$

Then these S_α and T_α have the required properties.

3) follows from direct computations. q.e.d.

Let W be the Weyl group for the symmetric pair (G, K), i.e., $W=N_K(A)/Z_K(A)$, where $N_K(A)$ and $Z_K(A)$ are the normalizer and the centralizer of A in K respectively.

It is identified with a finite subgroup of $O(\mathfrak{a})$ through the adjoint action on $\mathfrak{a}$. We define the *diagram* D for the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ by

$$D=\{H\in\mathfrak{a};\ 2(\gamma, H)\in \boldsymbol{Z}\ \text{ for some } \gamma\in\Sigma\}.$$

It is invariant under the Weyl group W. A connected component of $\mathfrak{a}-D$ is called a *fundamental cell* of $\mathfrak{a}$. We define a lattice Γ in $\mathfrak{a}$, a lattice Γ^0 in $\mathfrak{a}'$ and a subgroup Γ^* of $\mathfrak{a}$ by

$$\Gamma=\{H\in\mathfrak{a};\ \exp H\in K\},$$
$$\Gamma^0=\{A_\gamma;\ \gamma\in\Sigma\}_{\boldsymbol{Z}},\qquad \text{where } A_\gamma=(1/(\gamma, \gamma))\gamma,$$
$$\Gamma^*=\{H\in\mathfrak{a};\ 2(\gamma, H)\in\boldsymbol{Z}\ \text{ for each } \gamma\in\Sigma\}.$$

Here $\{*\}_{\boldsymbol{Z}}$ means the subgroup generated by $*$. The following inclusions are known (cf. Takeuchi [11]).

$$\Gamma^0\subset\Gamma\subset\Gamma^*. \tag{1.1}$$

The Weyl group W leaves these groups invariant. Denoting by $t(A)$ the translation: $H\longmapsto H+A$ of $\mathfrak{a}$ by an element $A\in\mathfrak{a}$, we define

$$\bar{W}=t(\Gamma)W,$$
$$\bar{W}^0=t(\Gamma^0)W,$$
$$\bar{W}^*=t(\Gamma^*)W.$$

In virtue of a general relation:

$$st(A)s^{-1}=t(sA)\quad \text{for } s\in O(\mathfrak{a}),\ A\in\mathfrak{a},$$

these are subgroups of the group of Euclidean motions of $\mathfrak{a}$, and the above expressions are semi-direct decompositions. The inclusions (1.1) implies the inclusions:

$$\bar{W}^0\subset\bar{W}\subset\bar{W}^*. \tag{1.2}$$

These groups leave the diagram D invariant, and hence they act on the set of all fundamental cells of $\mathfrak{a}$. The following is classical.

LEMMA 1.2. (E. Cartan [1])

1) *Let*

$$S_\gamma^n=\{H\in\mathfrak{a};\ 2(\gamma, H)=n\}\qquad \gamma\in\Sigma,\ n\in\boldsymbol{Z}$$

be a hyperplane of $\mathfrak{a}$ *contained in the diagram, and denote by* s_γ^n *the symmetry*:

$$H\longmapsto H-(2(H, \gamma)/(\gamma, \gamma))\gamma+(n/(\gamma, \gamma))\gamma\qquad \text{for } H\in\mathfrak{a}$$

of $\mathfrak{a}$ *with respect to* S_γ^n. *Then* $\bar{W}^0$ *is generated by these symmetries* s_γ^n *with* $\gamma\in\Sigma$, $n\in\boldsymbol{Z}$, *and it acts simply transitively on the set of fundamental cells of* $\mathfrak{a}$.

2) *If G is simply connected, then G_θ is connected.*

3) *If M is simply connected, then $\Gamma=\Gamma^0$.*

Now decompose the symmetric pair $(\mathfrak{g}', \mathfrak{k}')$ into the sum of irreducible symmetric pairs $(\mathfrak{g}_k, \mathfrak{k}_k)$ $(1\leqslant k\leqslant s)$:

$$\mathfrak{g}'=\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_s, \quad \mathfrak{k}'=\mathfrak{g}_0\oplus\mathfrak{k}_1\oplus\cdots\oplus\mathfrak{k}_s,$$

where

$$\mathfrak{g}_0=\{X\in\mathfrak{k}';\ [X, \mathfrak{m}']=\{0\}\}.$$

Then we have also the following decompositions.

$$\begin{aligned}&\mathfrak{a}'=\mathfrak{a}_1\oplus\cdots\oplus\mathfrak{a}_s, \text{ where } \mathfrak{a}_k=\mathfrak{a}'\cap\mathfrak{g}_k \ (1\leqslant k\leqslant s),\\ &\Sigma=\Sigma_1\cup\cdots\cup\Sigma_s, \text{ where } \Sigma_k=\Sigma\cap\mathfrak{a}_k \ (1\leqslant k\leqslant s),\\ &\Pi=\Pi_1\cup\cdots\cup\Pi_s, \text{ where } \Pi_k=\Pi\cap\Sigma_k \ (1\leqslant k\leqslant s).\end{aligned}$$

These imply direct product decompositions:

$$\Gamma^*=\mathfrak{c}_{\mathfrak{m}}+\Gamma_1^*+\cdots+\Gamma_s^*, \tag{1.3}$$

$$\bar{W}^*=t(\mathfrak{c}_{\mathfrak{m}})\times\bar{W}_1^*\times\cdots\times\bar{W}_s^*, \tag{1.4}$$

where Γ_k^* and $\bar{W}_k^*$ are the corresponding groups for the k-th irreducible factor $(\mathfrak{g}_k, \mathfrak{k}_k)$ $(1\leqslant k\leqslant s)$. Let $\delta_k\in\Sigma_k$ denote the highest root in Σ_k $(1\leqslant k\leqslant s)$ and put

$$\Sigma'=\{\delta_k;\ 1\leqslant k\leqslant s\}.$$

Consider disjoint unions:

$$\begin{aligned}&\Pi_k'=\Pi_k\cup\{\delta_k\} \quad (1\leqslant k\leqslant s),\\ &\Pi'=\Pi_1'\cup\cdots\cup\Pi_s'=\Pi\cup\Sigma',\end{aligned}$$

and define

$$\begin{aligned}&S_k=\{H\in\mathfrak{a}_k;\ 0<2(\gamma, H)<1 \quad \text{for each } \gamma\in\Pi_k'\} \quad (1\leqslant k\leqslant s),\\ &S=\{H\in\mathfrak{a};\ 0<2(\gamma, H)<1 \quad \text{for each } \gamma\in\Pi'\},\\ &S'=S\cap\mathfrak{a}'.\end{aligned}$$

Then we have

$$S=\mathfrak{c}_{\mathfrak{m}}\times S_1\times\cdots\times S_s=\mathfrak{c}_{\mathfrak{m}}\times S', \tag{1.5}$$

$$S'=S_1\times\cdots\times S_s. \tag{1.6}$$

Their closures are given by

$$\begin{aligned}&\bar{S}_k=\{H\in\mathfrak{a}_k;\ 0\leqslant2(\gamma, H)\leqslant1 \quad \text{for each } \gamma\in\Pi_k'\} \quad (1\leqslant k\leqslant s),\\ &\bar{S}=\{H\in\mathfrak{a};\ 0\leqslant2(\gamma, H)\leqslant1 \quad \text{for each } \gamma\in\Pi'\},\\ &\bar{S}'=\bar{S}\cap\mathfrak{a}'.\end{aligned}$$

Thus we have also

(1.7) $$\bar{S}=\mathfrak{c}_m\times\bar{S}_1\times\cdots\times\bar{S}_s=\mathfrak{c}_m\times\bar{S}',$$

(1.8) $$\bar{S}'=\bar{S}_1\times\cdots\times\bar{S}_s.$$

It is easy to see that S is an open convex cell in $\mathfrak{a}$ and that it is the unique fundamental cell of $\mathfrak{a}$ such that S is contained in the closed positive Weyl chamber:

$$\mathfrak{a}_+=\{H\in\mathfrak{a};\ (\gamma,\ H)\geqslant 0\quad\text{for each }\gamma\in\Sigma_+\},$$

and such that the closure $\bar{S}$ contains 0. Now we define

$$\bar{W}_S=\{\tau\in\bar{W};\ \tau S=S\},$$
$$\bar{W}_S{}^*=\{\tau\in\bar{W}^*;\ \tau S=S\}.$$

From (1.2), (1.4) and (1.5) we have an inclusion:

(1.9) $$\bar{W}_S\subset\bar{W}_S{}^*$$

and a direct product decomposition:

(1.10) $$\bar{W}_S{}^*=t(\mathfrak{c}_m)\times\bar{W}^*_{S_1}\times\cdots\times\bar{W}^*_{S_s},$$

where $\bar{W}^*_{S_k}$ is the corresponding group for the k-th irreducible factor $(\mathfrak{g}_k,\ \mathfrak{k}_k)$ $(1\leqslant k\leqslant s)$. Note that each $\bar{W}^*_{S_k}$ is a finite group.

LEMMA 1.3. 1) *The group* $\bar{W}^0$ *is a normal subgroup of* $\bar{W}^*$, *and*

$$\bar{W}_S{}^*\cong\bar{W}^*/\bar{W}^0\cong\Gamma^*/\Gamma^0,$$

$$\bar{W}_S\cong\bar{W}/\bar{W}^0\cong\Gamma/\Gamma^0.$$

2) *If M is simply connected, then* $\bar{W}_S=\{1\}$.

PROOF. 1) We show first

(1.11) $$sA-A\in\Gamma^0\quad\text{for each } s\in W,\ A\in\Gamma^*.$$

In fact, if we denote by s_γ the symmetry:

$$H\longmapsto H-(2(H,\gamma)/(\gamma,\gamma))\gamma\quad\text{for } H\in\mathfrak{a}$$

of $\mathfrak{a}$ with respect to $\gamma\in\Sigma$, then

$$s_\gamma A-A=-(2(A,\gamma)/(\gamma,\gamma))\gamma=-2(A,\gamma)A_\gamma\in\Gamma^0.$$

Since W is generated by symmetries s_γ with $\gamma\in\Sigma$, (1.11) holds for any $s\in W$.

Now we define a map $p:\bar{W}^*\longrightarrow\Gamma^*/\Gamma^0$ by

$$p(t(A)s)=A+\Gamma^0\quad\text{for } A\in\Gamma^*,\ s\in W.$$

Then p is a surjective homomorphism in virtue of (1.11). Since kernel $p=\bar{W}^0$, $\bar{W}^0$ is a normal subgroup of $\bar{W}^*$ and $\bar{W}^*/\bar{W}^0 \cong \Gamma^*/\Gamma^0$. Moreover Lemma 1.2, 1) implies $\bar{W}_S{}^* \cong \bar{W}^*/\bar{W}^0$. The same proof shows $\bar{W}_S \cong \bar{W}/\bar{W}^0 \cong \Gamma/\Gamma^0$.

2) follows from 1) and Lemma 1.2, 3). q.e.d.

Now we shall decompose $\bar{S}$ into the union of convex cells. For a subset Δ of Π', let S^Δ be the set of all $H \in \bar{S}$ statisfying the conditions:

$$\begin{aligned} 2(\gamma, H) &> 0 \quad \text{if} \quad \gamma \in \Delta,\ \gamma \in \Pi, \\ 2(\gamma, H) &< 1 \quad \text{if} \quad \gamma \in \Delta,\ \gamma \in \Sigma', \\ 2(\gamma, H) &= 0 \quad \text{if} \quad \gamma \notin \Delta,\ \gamma \in \Pi, \\ 2(\gamma, H) &= 1 \quad \text{if} \quad \gamma \notin \Delta,\ \gamma \in \Sigma'. \end{aligned}$$

It is easily seen that S^Δ is a convex cell in $\bar{S}$. If we denote by S^{Δ_k} the convex cell in $\bar{S}_k$ defined in the same way from the subset Δ_k of Π_k' defined by $\Delta_k = \Delta \cap \Pi_k'$ $(1 \leqslant k \leqslant s)$ and if we put $S'^\Delta = S^\Delta \cap \mathfrak{a}'$, then we have

$$S^\Delta = \mathfrak{c}_m \times S^{\Delta_1} \times \cdots \times S^{\Delta_s} = \mathfrak{c}_m \times S'^\Delta, \tag{1.12}$$

$$S'^\Delta = S^{\Delta_1} \times \cdots \times S^{\Delta_s}. \tag{1.13}$$

Hence, $S^\Delta \neq \phi$ if and only if $\Delta_k \neq \phi$ for each k. A subset Δ of Π' satisfying the latter conditions is said to be *admissible*. For an admissible subset Δ of Π', the dimension k_Δ of S^Δ is given by

$$k_\Delta = |\Delta| + r_0 - s, \tag{1.14}$$

where $|*|$ means the cardinality of the set $*$.

Lemma 1.4. 1) *$\bar{S} = \bigcup_\Delta S^\Delta$ (disjoint union), where Δ ranges over the admissible subsets of Π'.*

2) *The group $\bar{W}_S{}^*$ acts on the set of all S^Δ with Δ admissible.*

3) *For admissible subsets Δ_1, Δ_2 of Π',*

$$\bar{S}^{\Delta_1} \supset S^{\Delta_2} \Longleftrightarrow \Delta_1 \supset \Delta_2.$$

In this case, for $H_1 \in S^{\Delta_1}$ and $H_2 \in S^{\Delta_2}$, we have

$$tH_1 + (1-t)H_2 \in S^{\Delta_1} \quad \text{for each } t \text{ with } 0 < t \leqslant 1.$$

Proof. In virtue of (1.7), (1.10) and (1.13), we may assume that $\mathfrak{g}$ is semisimple and $(\mathfrak{g}, \mathfrak{k})$ is irreducible.

We define a map $\gamma \longmapsto \gamma^\natural$ from Π' into $\mathfrak{a}$ by

(1.15)
$$\gamma^{\natural}=\begin{cases}\gamma & \text{if } \gamma\in\Pi\\ -\gamma & \text{if } \gamma\in\Sigma',\end{cases}$$

and denote its image by $\Pi^{\natural}$. Let $\Pi=\{\gamma_1,\cdots,\gamma_r\}$ and $\{\varepsilon_1,\cdots,\varepsilon_r\}$ the basis of $\mathfrak{a}$ dual to Π:

$$(\varepsilon_i, \gamma_j)=\varepsilon_{ij} \quad (1\leqslant i, j\leqslant r).$$

Let

$$\delta=\sum_{i=1}^{r} n_i\gamma_i \qquad n_i\in\boldsymbol{Z},\ n_i\geqslant 1$$

be the highest root of Σ. We put

$$\gamma_0=-\delta, \quad \varepsilon_0=0, \quad n_0=1,$$

$$P_{\gamma_i}=(1/2n_i)\varepsilon_i \quad (0\leqslant i\leqslant r),$$

so that $\Pi^{\natural}$ is given by

$$\Pi^{\natural}=\{\gamma_0,\gamma_1,\cdots,\gamma_r\}.$$

Then $\bar{S}$ is the ordinary closed Euclidean simplex spanned by the points $\{P_\gamma;\ \gamma\in\Pi^{\natural}\}$:

$$\bar{S}=\{\sum_{\gamma\in\Pi^{\natural}} h_\gamma P_\gamma;\ 0\leqslant h_\gamma\leqslant 1,\ \sum_{\gamma\in\Pi^{\natural}} h_\gamma=1\},$$

and S^Δ is the open Euclidean simplex spanned by the points $\{P_\gamma;\ \gamma\in\Delta^{\natural}\}$:

(1.16)
$$S^\Delta=\{\sum_{\gamma\in\Delta^{\natural}} h_\gamma P_\gamma;\ 0<h_\gamma<1,\ \sum_{\gamma\in\Delta^{\natural}} h_\gamma=1\}.$$

Thus the family $\{S^\Delta\}_\Delta$ gives the ordinary cellular decomposition of the closed simplex $\bar{S}$. This implies the Lemma. q.e.d.

REMARK. Any fundamental cell of $\mathfrak{a}$ can be decomposed to the union of disjoint convex cells in the same way. Thus we get a cellular decomposition of $\mathfrak{a}$, which is invariant under the action of $\bar{W}^*$.

For an element $H\in\bar{S}$, we define a closed subgroup Z^H of K by

$$Z^H=\{k\in K;\ k\operatorname{Exp} H=\operatorname{Exp} H\}.$$

For an admissible subset Δ of $\Pi^{\natural}$, we define a subgroup N^Δ of K and a normal subgroup Z^Δ of N^Δ by

$$N^\Delta=\{k\in K;\ k\operatorname{Exp} S^\Delta=\operatorname{Exp} S^\Delta\},$$

$$Z^\Delta=\{k\in N^\Delta;\ k|\operatorname{Exp} S^\Delta=\mathrm{id}\},$$

where $k|\operatorname{Exp} S^\Delta=\mathrm{id}$ means that $kp=p$ for each $p\in\operatorname{Exp} S^\Delta$. Then Z^Δ is a closed subgroup of K and $Z^{\Delta}\subset Z^{H}$ for each $H\in S^{\Delta}$. Let W^{Δ} be the quotient group:

$$W^{\Delta}=N^{\Delta}/Z^{\Delta}.$$

The class $kZ^{\Delta}\in W^{\Delta}$ containing $k\in N^{\Delta}$ will be denoted by $[k]$. Also we define a subgroup $\bar{N}^{\Delta}$ of $\bar{W}_{S}$, a normal subgroup $\bar{Z}^{\Delta}$ of $\bar{N}^{\Delta}$ and the quotient group $\bar{W}^{\Delta}$ by

$$\bar{N}^{\Delta}=\{\tau\in\bar{W}_{S};\ \tau S^{\Delta}=S^{\Delta}\},$$

$$\bar{Z}^{\Delta}=\{\tau\in\bar{N}^{\Delta};\ \tau|S^{\Delta}=\mathrm{id}\},$$

$$\bar{W}^{\Delta}=\bar{N}^{\Delta}/\bar{Z}^{\Delta}.$$

The class $\tau\bar{Z}^{\Delta}\in\bar{W}^{\Delta}$ containing $\tau\in\bar{N}^{\Delta}$ will be also denoted by $[\tau]$. Let further

$$\Sigma^{\Delta}=\Sigma\cap\{\Pi^{\prime}-\Delta\}_{\boldsymbol{Z}},\ \ \Sigma_{+}{}^{\Delta}=\Sigma^{\Delta}\cap\Sigma_{+},$$

$$\mathfrak{g}^{\Delta}=\mathfrak{k}_{0}+\mathfrak{a}+\sum_{\gamma\in\Sigma_{+}^{\Delta}}(\mathfrak{k}_{\gamma}+\mathfrak{m}_{\gamma}).$$

Then we have a decomposition:

$$\mathfrak{g}^{\Delta}=\mathfrak{k}^{\Delta}+\mathfrak{m}^{\Delta},$$

where

$$\mathfrak{k}^{\Delta}=\mathfrak{k}_{0}+\sum_{\gamma\in\Sigma_{+}^{\Delta}}\mathfrak{k}_{\gamma}=\mathfrak{g}^{\Delta}\cap\mathfrak{k},$$

$$\mathfrak{m}^{\Delta}=\mathfrak{m}_{0}+\sum_{\gamma\in\Sigma_{+}^{\Delta}}\mathfrak{m}_{\gamma}=\mathfrak{g}^{\Delta}\cap\mathfrak{m}.$$

We define moreover a C^{∞} map Ψ^{Δ}: $K/Z^{\Delta}\times S^{\Delta}\longrightarrow M$ by

$$\Psi^{\Delta}(kZ^{\Delta},\ H)=k\,\mathrm{Exp}\,H\quad\text{for }k\in K,\ H\in S^{\Delta}.$$

The image of Ψ^{Δ} will be denoted by M^{Δ}. Our first task is to study the structure of the set M^{Δ}.

LEMMA 1.5. *Let Δ be an admissible subset of $\Pi^{\prime}$. Take an element $H\in S^{\Delta}$. Then*:

1) $\Sigma_{+}{}^{\Delta}=\{\gamma\in\Sigma;\ 2(\gamma,H)=0$ or $1\}$.

2) $\Sigma^{\Delta}=\{\gamma\in\Sigma;\ 2(\gamma,\ H)\in\boldsymbol{Z}\}$.

3) $\mathfrak{g}^{\Delta}=\{X\in\mathfrak{g};\ \mathrm{Ad}(\exp 2H)X=X\}$.

4) $(\mathfrak{g}^{\Delta},\ \mathfrak{k}^{\Delta})$ *is a symmetric pair with the canonical complememt* $\mathfrak{m}^{\Delta}$.

PROOF. 1) We may assume that $\mathfrak{g}$ is semi-simple and $(\mathfrak{g},\ \mathfrak{k})$ is irreducidle. Under the notation in the proof of Lemma 1.4, let $\gamma\in\Sigma_{+}$ be written as

$$\gamma=\sum_{i=1}^{r} m_i\gamma_i \quad m_i\in\boldsymbol{Z},\ m_i\geqslant 0.$$

We shall show that $\gamma\in\{\Pi'-\Delta\}_Z$ if and only if $2(\gamma, H)=0$ or 1.

Case 1: $\delta\notin\Delta$. We have

$$2(\gamma_i, H)>0 \quad \text{if} \quad \gamma_i\in\Delta,$$
$$2(\gamma_j, H)=0 \quad \text{if} \quad \gamma_j\notin\Delta,$$
$$2(\delta, H)=1,$$

and hence

$$0\leqslant 2(\gamma, H)=2\sum_{\gamma_i\in\Delta} m_i(\gamma_i, H)\leqslant 1,$$
$$2(\delta, H)=2\sum_{\gamma_i\in\Delta} n_i(\gamma_i, H)=1.$$

Thus, if $2(\gamma, H)=1$, then

$$\gamma=\sum_{\gamma_j\notin\Delta} m_j\gamma_j,$$

and hence $\gamma\in\{\Pi-\Delta\}_Z\subset\{\Pi'-\Delta\}_Z$. If $2(\gamma, H)=0$, then

$$\gamma=\sum_{\gamma_j\notin\Delta} m_j\gamma_j+\sum_{\gamma_i\in\Delta} n_i\gamma_i=\sum_{\gamma_j\notin\Delta}(m_j-n_j)\gamma_j+\delta,$$

and hence $\gamma\in\{\Pi'-\Delta\}_Z$. Conversely, if $\gamma\in\{\Pi'-\Delta\}_Z$, i.e., γ is written as

$$\gamma=\sum_{\gamma_j\notin\Delta} l_j\gamma_j+l_0\delta \qquad l_j,\ l_0\in\boldsymbol{Z},$$

then $m_i=l_0n_i$ for each i with $\gamma_i\in\Delta$. Thus $l_0=0$ or 1, and hence $2(\gamma, H)=0$ or 1.

Case 2: $\delta\in\Delta$. We have

$$2(\gamma_i, H)>0 \quad \text{if} \quad \gamma_i\in\Delta,$$
$$2(\delta, H)<1,$$
$$2(\gamma_j, H)=0 \quad \text{if} \quad \gamma_j\notin\Delta,$$

and hence

$$0\leqslant 2(\gamma, H)=2\sum_{\gamma_i\in\Delta} m_i(\gamma_i, H),$$
$$2(\delta, H)=2\sum_{\gamma_i\in\Delta} n_i(\gamma_i, H)<1.$$

These imply $2(\gamma, H)<1$. Now

$$2(\gamma, H)=0 \Leftrightarrow \gamma=\sum_{\gamma_j\notin\Delta} m_j\gamma_j$$
$$\Leftrightarrow \gamma\in\{\Pi-\Delta\}_Z=\{\Pi'-\Delta\}_Z.$$

2) follows from 1).

3) The complexification of the right hand side is

$$\{X\in\mathfrak{g}^{C};\ \mathrm{Ad}(\exp 2H)X=X\}=\mathfrak{k}_0{}^{C}+\mathfrak{a}^{C}+\sum_{2(\gamma,H)\in \boldsymbol{Z}}\mathfrak{g}_{\gamma}{}^{C},$$

which is equal to $(\mathfrak{g}^{\Delta})^{C}$ by 2). This implies the assertion 3).

4) is clear, since both $\mathfrak{g}^{\Delta}$ and $\mathfrak{k}^{\Delta}$ are subalgebras of $\mathfrak{g}$ in virtue of the assertion 3). q.e.d.

LEMMA 1.6. 1) *Let Δ_1 and Δ_2 be admissible subsets of Π', $H_1\in S^{\Delta_1}$, $H_2\in S^{\Delta_2}$ and $k\in K$. If $k\,\mathrm{Exp}\,H_1=\mathrm{Exp}\,H_2$, then* $\mathrm{Ad}k\ \mathfrak{m}^{\Delta_1}=\mathfrak{m}^{\Delta_2}$.

2) *Let Δ be an admissible subset of Π'. Then N^{Δ} is a subgroup of the normalizer $N_K(\mathfrak{m}^{\Delta})$ of $\mathfrak{m}^{\Delta}$ in K. The Lie algebras of Z^{H} are the same $\mathfrak{k}^{\Delta}$ for any $H\in S^{\Delta}$. The Lie algebra of $N_K(\mathfrak{m}^{\Delta})$ is also $\mathfrak{k}^{\Delta}$.*

PROOF. 1) From the assumption, there exists $l\in K$ such that $k\exp H_1=\exp H_2 l$. Applying the automorphism θ of G, we get $k(\exp H_1)^{-1}=(\exp H_2)^{-1}l$ and hence $l=(\exp H_2)k(\exp H_1)^{-1}$. It follows $k\exp H_1=\exp H_2\exp H_2 k(\exp H_1)^{-1}$ and hence $k(\exp 2H_1)k^{-1}=\exp 2H_2$. Now Lemma 1.5, 3) implies $\mathrm{Ad}k\,\mathfrak{g}^{\Delta_1}=\mathfrak{g}^{\Delta_2}$, and thus $\mathrm{Ad}k\ \mathfrak{m}^{\Delta_1}=\mathfrak{m}^{\Delta_2}$.

2) $N^{\Delta}\subset N_K(\mathfrak{m}^{\Delta})$ follows from 1). Let $H\in S^{\Delta}$ and

$$X=S_0+\sum_{\alpha}a_{\alpha}S_{\alpha}\in\mathfrak{k},\quad S_0\in\mathfrak{k}_0.$$

Then, $X\in$ Lie algebra of $Z^{H}\Longleftrightarrow(\exp H)^{-1}(\exp tX)\exp H\in\mathfrak{k}$ for each $t\in\boldsymbol{R}\Longleftrightarrow$ $\mathrm{Ad}(\exp H)^{-1}X\in\mathfrak{k}\Longleftrightarrow 2(\alpha,H)\in\boldsymbol{Z}$ for each $\alpha\in\tilde{\Sigma}_{+}-\tilde{\Sigma}_0$ with $a_{\alpha}\neq 0$ (by Lemma 1.1)$\Longleftrightarrow$ $X\in\mathfrak{k}^{\Delta}$ (by Lemma 1.5). Thus the Lie algebra of Z^{H} coincides with $\mathfrak{k}^{\Delta}$.

To show that the Lie algebra of $N_K(\mathfrak{m}^{\Delta})$ is also $\mathfrak{k}^{\Delta}$, take an element $H\in S^{\Delta}$. Then, $X\in$ Lie algebra of $N_K(\mathfrak{m}^{\Delta})\Rightarrow[H,X]\in\mathfrak{m}^{\Delta}\Rightarrow a_{\alpha}=0$ for each $\alpha\in\tilde{\Sigma}_{+}-\tilde{\Sigma}_0$ with $0<2(\alpha,H)<1$ (by Lemmas 1.1 and 1.5)$\Rightarrow X\in\mathfrak{k}^{\Delta}$(by Lemma 1.5). Conversely, Lemma 1.5, 4) implies $[\mathfrak{k}^{\Delta},\mathfrak{m}^{\Delta}]\subset\mathfrak{m}^{\Delta}$ and hence $\mathfrak{k}^{\Delta}\subset$Lie algebra of $N_K(\mathfrak{m}^{\Delta})$. q.e.d.

The following Corollary 1 is an immediate consequence of the above lemma.

COROLLARY 1. *The group N^{Δ} is a compact subgroup of K. The groups N^{Δ} and Z^{Δ} have the same Lie algebra $\mathfrak{k}^{\Delta}$. Therefore W^{Δ} is a finite group.*

COROLLARY 2. 1) $\dim K/Z^{\Delta}=(1/2)(\dim\mathfrak{g}-\dim\mathfrak{g}^{\Delta})$.

2) *The map Ψ^{Δ} is an immersion.*

PROOF. 1) In virtue of the above lemma, the tangent space of K/Z^{Δ} at the origin Z^{Δ} is linearly isomorphic with

$$\mathfrak{k}/\mathfrak{k}^{\Delta} \cong \sum_{\gamma \in \Sigma_+ - \Sigma_+^{\Delta}} \mathfrak{k}_\gamma,$$

through the canonical projection π_K: $K \longrightarrow K/Z^{\Delta}$. On the other hand, we have

$$\mathfrak{g}/\mathfrak{g}^{\Delta} \cong \sum_{\gamma \in \Sigma_+ - \Sigma_+^{\Delta}} (\mathfrak{k}_\gamma + \mathfrak{m}_\gamma).$$

These imply the assertion 1).

2) It follows from Lemma 1.1 that the differential $d\Psi^{\Delta}$ of Ψ^{Δ} at $(k_0 Z^{\Delta}, H_0) \in K/Z^{\Delta} \times S^{\Delta}$ is given by

$$\begin{cases} d\tau_{k_0} d\pi_K S_\alpha \longmapsto -d\tau_{k_0 \exp H_0} d\pi_G \sin 2\pi(\alpha, H_0) T_\alpha \\ \qquad\qquad\qquad\qquad \text{for } \alpha \in \tilde{\Sigma}_+ - \tilde{\Sigma}_0 \text{ with } 0 < 2(\alpha, H_0) < 1 \\ H \longmapsto d\tau_{k_0 \exp H_0} d\pi_G H \qquad \text{for } H \in T_{H_0}(S^{\Delta}). \end{cases}$$

Therefore $d\Psi^{\Delta}$ is linearly injective at $(k_0 Z^{\Delta}, H_0)$. q.e.d.

LEMMA 1.7. *Let Δ_1 and Δ_2 be admissible subsets of Π', $H_1 \in S^{\Delta_1}$, $H_2 \in S^{\Delta_2}$ and $k \in K$. If $k \operatorname{Exp} H_1 = \operatorname{Exp} H_2$, then there exists $\tau \in \bar{W}_S$ such that*:

i) $\tau S^{\Delta_1} = S^{\Delta_2}$;

ii) $k \operatorname{Exp} H = \operatorname{Exp} \tau H$ *for each* $H \in S^{\Delta}$;

iii) $\tau H_1 = H_2$,

and hence $k \operatorname{Exp} S^{\Delta_1} = \operatorname{Exp} S^{\Delta_2}$.

PROOF. We know $\operatorname{Ad}k\, \mathfrak{m}^{\Delta_1} = \mathfrak{m}^{\Delta_2}$ by Lemma 1.6. Since both $\mathfrak{a}$ and $\operatorname{Ad}k\mathfrak{a}$ are Cartan subalgebras for the symmetric pair $(\mathfrak{g}^{\Delta_2}, \mathfrak{k}^{\Delta_2})$, and since the Lie algebra of Z^{Δ_2} is $\mathfrak{k}^{\Delta_2}$ by the above Corollary 1, we can find $k_1 \in Z^{\Delta_2}$ such that $k_1 k \in N_K(A)$. Therefore, we may assume $k \in N_K(A)$. Put $s = \operatorname{Ad}k|\mathfrak{a} \in W$. Then $(\exp sH_1)o = (\exp H_2)o$ and hence there exists $A \in \Gamma$ such that $sH_1 + A = H_2$. Putting $\tau_1 = t(A)s \in \bar{W}$, we have $\tau_1 H_1 = H_2$. It follows from Remark after Lemma 1.4 that $\tau_1 S^{\Delta_1} = S^{\Delta_2}$. Now Lemma 1.2, 1) implies that there exists $\tau_2 \in \bar{W}^0 \subset \bar{W}$ such that $\tau = \tau_2 \tau_1 \in \bar{W}_S$ and $\tau_2 | S^{\Delta_2} = \mathrm{id}$, and so $\tau S^{\Delta_1} = S^{\Delta_2}$, $\tau H_1 = H_2$. Then, for each $H \in S^{\Delta_1}$ we have

$$\operatorname{Exp} \tau H = \operatorname{Exp} \tau_2 \tau_1 H = \operatorname{Exp} \tau_1 H = \operatorname{Exp} sH = k \operatorname{Exp} H. \qquad \text{q.e.d.}$$

COROLLARY. *We have $Z^H \subset N^{\Delta}$ for each $H \in S^{\Delta}$. Thus $Z^{\Delta} \subset Z^H \subset N^{\Delta} \subset N_K(\mathfrak{m}^{\Delta})$ for each $H \in S^{\Delta}$.*

Put

$$\Gamma_0 = \Gamma \cap \mathfrak{c}_{\mathfrak{m}}$$

and define a homomorphism ι^{Δ}: $\Gamma_0 \longrightarrow \bar{W}^{\Delta}$ by

$$\iota^{\Delta}(A) = [t(A)] \qquad \text{for } A \in \Gamma_0.$$

Then Γ_0 is a lattice in $\mathfrak{c}_{\mathfrak{m}}$ and ι^{Δ} is injective. With these definitions we have the following lemma.

LEMMA 1.8. 1) *There exists a unique homomorphism* π^{Δ}: $\tilde{W}^{\Delta}\longrightarrow W^{\Delta}$ *such that if* $\pi^{\Delta}[\tau]=[k]$ *with* $\tau\in\tilde{N}^{\Delta}$ *and* $k\in N^{\Delta}$, *then*

(1.17) $$k\operatorname{Exp} H=\operatorname{Exp}\tau H \qquad \text{for each } H\in S^{\Delta}.$$

2) *The sequence* $1\longrightarrow\Gamma_0\xrightarrow{\iota^{\Delta}}\tilde{W}^{\Delta}\xrightarrow{\pi^{\Delta}}W^{\Delta}\longrightarrow 1$ *is exact.*

PROOF. 1) Take an arbitrary $\tau\in\tilde{N}^{\Delta}$ and let $\tau=t(A)s$, where $A\in\Gamma$ and $s\in W$. Choose $k\in N_K(A)$ such that $\operatorname{Ad}k|\mathfrak{a}=s$. Then the relation (1.17) holds and hence $k\in N^{\Delta}$. Since $Z_K(A)\subset Z^{\Delta}$, the class $[k]\in\tilde{W}^{\Delta}$ is determined by τ. Moreover, the relation (1.17) implies that $[k]$ depends only on the class $[\tau]$. Now the correspondence $[\tau]\longmapsto[k]$ defines the required homomorphism. The uniqueness is clear from the relation (1.17).

2) The surjectivity of π^{Δ} follows from Lemma 1.7. It is clear that $\pi^{\Delta}\circ\iota^{\Delta}$ is trivial. Take $\tau\in\tilde{N}^{\Delta}$ such that $\pi^{\Delta}[\tau]=1$. Let $\tau=t(A''+A')s$, where $A''\in\mathfrak{c}_{\mathfrak{m}}$, $A'\in\mathfrak{a}'$ and $s\in W$. Put $\tau'=t(A)s$ so that $\tau=t(A'')\tau'$. It follows from (1.5), (1.10) and (1.12) that τ' leaves both S' and S'^{Δ} invariant. On the other hand, $\pi^{\Delta}[\tau]=1$ implies

$$\operatorname{Exp} H=\operatorname{Exp}\tau H \qquad \text{for each } H\in S^{\Delta}.$$

Since S^{Δ} is connected and Γ is discrete in $\mathfrak{a}$, we can find $B\in\Gamma$ such that

$$\tau H=H+B \qquad \text{for each } H\in S^{\Delta}.$$

Let $B=B''+B'$, where $B''\in\mathfrak{c}_{\mathfrak{m}}$ and $B'\in\mathfrak{a}'$. Then

$$\tau H'=B''+(H'+B') \qquad \text{for each } H'\in S'^{\Delta}.$$

It follows from the decomposition: $\tau=t(A'')\tau'$ that

$$\tau' H'=H'+B' \qquad \text{for each } H'\in S'^{\Delta}.$$

Since S'^{Δ} is bounded in $\mathfrak{a}'$, we have $B'=0$ and hence $\tau'=1$. Thus we have $\tau=t(A'')$ with $A''\in\mathfrak{c}_{\mathfrak{m}}\cap\Gamma=\Gamma_0$, and hence $[\tau]=\iota^{\Delta}(A'')$. This completes the proof.

q.e.d.

Now we define a C^{∞} right action of the group $\tilde{W}^{\Delta}$ on $K/Z^{\Delta}\times S^{\Delta}$ as follows. Let $kZ^{\Delta}\longmapsto(kZ^{\Delta})\cdot[k']=kk'Z^{\Delta}$ be the natural right action of $[k']\in W^{\Delta}$ on K/Z^{Δ}. We define a right action of $[\tau]\in\tilde{W}^{\Delta}$ on K/Z^{Δ} by $kZ^{\Delta}\longmapsto(kZ^{\Delta})[\tau]=(kZ^{\Delta})\cdot\pi^{\Delta}[\tau]$. Define a right action of $[\tau]\in\tilde{W}^{\Delta}$ on $K/Z^{\Delta}\times S^{\Delta}$ by

$$(kZ^{\Delta},H)\longmapsto((kZ^{\Delta})[\tau],\ \tau^{-1}H) \qquad \text{for } k\in K,\ H\in S^{\Delta}.$$

Then we have the following

LEMMA 1.9. 1) *The group* $\bar{W}^{\Delta}$ *acts on* $K/Z^{\Delta}\times S^{\Delta}$ *freely.*

2) *Let* Ψ^{Δ}: $K/Z^{\Delta}\times S^{\Delta}\longrightarrow M$ *be the previously defined* C^{∞}*map. Then,* $\Psi^{\Delta}(k_1Z^{\Delta}, H_1)=\Psi^{\Delta}(k_2Z^{\Delta}, H_2)$ *if and only if there exists* $[\tau]\in\bar{W}^{\Delta}$ *such that* $(k_1Z^{\Delta}, H_1)[\tau]=(k_2Z^{\Delta}, H_2)$.

PROOF. 1) Assume that $\tau\in\bar{N}^{\Delta}$, $k_0\in K$ and $H_0\in S^{\Delta}$ satisfy $(k_0Z^{\Delta},H_0)[\tau]=(k_0Z^{\Delta}, H_0)$. Since the natural action of W^{Δ} on K/Z^{Δ} is free, we have $\pi^{\Delta}[\tau]=1$ and $\tau^{-1}H_0=H_0$. It follows from Lemma 1.8 that $\tau=t(A)$ with $A\in\Gamma_0$. But $H_0=\tau H_0=H_0+A$ implies $A=0$ and hence $[\tau]=1$.

2) Assume $\Psi^{\Delta}(k_1Z^{\Delta}, H_1)=\Psi^{\Delta}(k_2Z^{\Delta},H_2)$, i.e., $k_1\operatorname{Exp}H_1=k_2\operatorname{Exp}H_2$. Put $k=k_1^{-1}k_2\in K$ so that $k\operatorname{Exp}H_2=\operatorname{Exp}H_1$. It follows from Lemma 1.7 that there exists $\tau\in\bar{N}^{\Delta}$ such that $\pi^{\Delta}[\tau]=[k]$ and $\tau H_2=H_1$. Then $k_2Z^{\Delta}=k_1kZ^{\Delta}=(k_1Z^{\Delta})\cdot[k]=(k_1Z^{\Delta})[\tau]$ and $\tau^{-1}H_1=H_2$. Thus $(k_1Z^{\Delta}, H_1)[\tau]=(k_2Z^{\Delta}, H_2)$. Conversely, assume $(k_1Z^{\Delta}, H_1)[\tau]=(k_2Z^{\Delta}, H_2)$ with $\tau\in\bar{N}^{\Delta}$. Let $\pi^{\Delta}[\tau]=[k]$ where $k\in N^{\Delta}$. Then $k_1kZ^{\Delta}=k_2Z^{\Delta}$ and $\tau^{-1}H_1=H_2$, and hence $k_2\operatorname{Exp}H_2=k_1k\operatorname{Exp}H_2=k_1\operatorname{Exp}\tau H_2=k_1\operatorname{Exp}H_1$, i.e., $\Psi^{\Delta}(k_1Z^{\Delta}, H_1)=\Psi^{\Delta}(k_2Z^{\Delta}, H_2)$. q.e.d.

For an admissible subset Δ of Π', let

$$E^{\Delta}=K/Z^{\Delta}\times_{\bar{W}^{\Delta}}S^{\Delta}$$

be the quotient manifold of $K/Z^{\Delta}\times S^{\Delta}$ relative to the above free right action of $\bar{W}^{\Delta}$. The class in E^{Δ} of a point $(kZ^{\Delta}, H)\in K/Z^{\Delta}\times S^{\Delta}$ will be denoted by $[kZ^{\Delta}, H]$. Note that K/Z^{Δ} is connected since K is generated by $K\cap A$ and the identity component of K (cf. Takeuchi [11]). Thus E^{Δ} is also connected. We will show in Part II that E^{Δ} is diffeomorphic with a fibre bundle over a compact manifold. With these definitions, we have

THEOREM 1.1. 1) *A compact connected symmetric space M is the (not necessarily disjoint) union:*

$$M=\bigcup_{\Delta}M^{\Delta}$$

of connected regular submanifolds M^{Δ}, *where* Δ *ranges over the admissible subsets of* Π'. *Each* M^{Δ} *is diffeomorphic with* E^{Δ} *by the diffeomorphism* ψ^{Δ}: $E^{\Delta}\longrightarrow M^{\Delta}$ *induced by the* C^{∞} *map* Ψ^{Δ}: $K/Z^{\Delta}\times S^{\Delta}\longrightarrow M$.

2) *The dimension of* M^{Δ} *is given by*

$$\dim M^{\Delta}=(1/2)(\dim\mathfrak{g}-\dim\mathfrak{g}^{\Delta})+|\Delta|+r_0-s.$$

In particular, $\dim M^{\Delta}\leqq\dim M-2$ *for any proper admissible subset* Δ *of* Π'.

3) $M^{\Delta_1}\cap M^{\Delta_2}\neq\phi \Longleftrightarrow M^{\Delta_1}=M^{\Delta_2} \Longleftrightarrow$ *There exists* $\tau\in\bar{W}_S$ *such that* $\tau S^{\Delta_1}=S^{\Delta_2}$.

4) $\bar{M}^{\Delta_1}\supset M^{\Delta_2} \Longleftrightarrow$ *There exists* $\tau\in\bar{W}_S$ *such that* $\tau\bar{S}^{\Delta_1}\supset S^{\Delta_2}$.

PROOF. 1) Let p be an arbitrary point of M. Take $X\in\mathfrak{m}$ such that $\mathrm{Exp}X=p$ and then take $k_1\in K$ such that $H_1=Adk_1X\in\mathfrak{a}$. It follows from Lemma 1.2, 1) that there exists $\tau\in\bar{W}$ such that $H=\tau H_1\in S$. By Lemma 1.4, we heve an admissible subset Δ of Π' with $H\in S^{\Delta}$. Let $\tau=t(A)s$, where $A\in\Gamma$ and $s\in W$, and take $k_2\in N_K(A)$ such that $\mathrm{Ad}k_2|\mathfrak{a}=s$. Put $k=(k_2k_1)^{-1}\in K$. Then $k^{-1}p=k_2k_1\mathrm{Exp}X=k_2\mathrm{Exp}\mathrm{Ad}k_1X=k_2\mathrm{Exp}\,H_1=\mathrm{Exp}\,sH_1=\mathrm{Exp}\tau H_1=\mathrm{Exp}\,H$, and hence $p=k\mathrm{Exp}H=\Psi^{\Delta}(kZ^{\Delta},H)\in M^{\Delta}$. Thus $M=\cup M^{\Delta}$.

For each admissible subset Δ of Π', Ψ^{Δ} is a C^{∞} immersion by Corollary 2 of Lemma 1.6, and it induces a C^{∞} imbedding ψ^{Δ}: $E^{\Delta}\longrightarrow M$ by Lemma 1.9. Thus it suffices to show that ψ^{Δ}: $E^{\Delta}\longrightarrow M^{\Delta}$ is an open map with respect to the topology of M^{Δ} induced by that of M. We prove this in the same way as in Sakai [8]. Suppose that this would not hold. Then, there would exist sequences $k_n\in K$, $H_n\in S^{\Delta}$ such that $k_n\mathrm{Exp}H_n$ would converge in M to a point $k_0\mathrm{Exp}H_0$ with $k_0\in K$, $H_0\in S^{\Delta}$, but $[k_nZ^{\Delta}, H_n]$ would not converge to $[k_0Z^{\Delta}, H_0]$ in E^{Δ}. We shall show that this assumption leads to a contradiction. From the assumption, there exist a neighborhood $\mathscr{U}$ of $[k_0Z^{\Delta}, H_0]$ in E^{Δ} and subsequences k_{n_i}, H_{n_i} such that $[k_{n_i}Z^{\Delta}, H_{n_i}]\notin\mathscr{U}$. Since both K and $\mathfrak{c}_{\mathfrak{m}}/\Gamma_0$ are compact, we may assume that subsequences k_{n_i} and H_{n_i} converge to $k'\in K$ and to $H'\in\bar{S}^{\Delta}$ respectively, so that $k'\,\mathrm{Exp}\,H' = k_0\,\mathrm{Exp}\,H_0$. Putting $k=k_0^{-1}k'\in K$, we get

$$k\,\mathrm{Exp}\,H'=\mathrm{Exp}H_0 \qquad \text{where} \quad H'\in\bar{S}^{\Delta},\ H_0\in S^{\Delta}.$$

It follows from Lemma 1.7 that there exists $\tau\in\bar{W}_S$ such that $\tau H'=H_0$. Thus $H'\in S^{\Delta}$, and hence

$$k\,\mathrm{Exp}\,H'=k_0\,\mathrm{Exp}\,H_0 \qquad \text{where} \quad H',\ H_0\in S^{\Delta}.$$

Now Lemma 1.9, 2) implies $[k'Z^{\Delta}, H']=[k_0Z^{\Delta}, H_0]$. But the sequence $[k_{n_i}Z^{\Delta}, H_{n_i}]$ converges to $[k'Z^{\Delta}, H']$ in E^{Δ}. This contradicts to the assumption: $[k_{n_i}Z^{\Delta}, H_{n_i}]\notin\mathscr{U}$.

2) follows from Corollary 2 of Lemma 1.6 and (1.14).

3) Let $M^{\Delta_1}\cap M^{\Delta_2}\neq\phi$. Then there exist k_1, $k_2\in K$, $H_1\in S^{\Delta_1}$ and $H_2\in S^{\Delta_2}$ such that $k_1\,\mathrm{Exp}\,H_1=k_2\,\mathrm{Exp}\,H_2$. Putting $k=k_2^{-1}k_1\in K$, we get $k\,\mathrm{Exp}\,H_1=\mathrm{Exp}\,H_2$. By Lemma 1.7, there exists $\tau\in\bar{W}_S$ such that $\tau S^{\Delta_1}=S^{\Delta_2}$. Assume conversely that there exists $\tau\in\bar{W}_S$ such that $\tau S^{\Delta_1}=S^{\Delta_2}$. Let $\tau=t(A)s$, where $A\in\Gamma$ and $s\in W$, and take $k\in N_K(A)$ such that $\mathrm{Ad}\,k|\mathfrak{a}=s$. Then $k\,\mathrm{Exp}\,H=\mathrm{Exp}\,\tau H$ for each $H\in S^{\Delta_1}$, and hence $M^{\Delta_1}=K\,\mathrm{Exp}\,S^{\Delta_1}=K\,\mathrm{Exp}\,\tau S^{\Delta_1}=K\,\mathrm{Exp}\,S^{\Delta_2}=M^{\Delta_2}$. These prove the assertion 3).

4) Assume $\bar{M}^{\Delta_1}\supset M^{\Delta_2}$. Then there exist sequences $k_n\in K$, $H_n\in S^{\Delta_1}$ such that

$k_n \mathrm{Exp} H_n$ converges to a point $k_0 \mathrm{Exp} H_0 \in M^{\Delta_2}$ with $k_0 \in K$, $H_0 \in S^{\Delta_2}$. In the same way as in the proof of 1), we may assume that sequences k_n and H_n converge to $k' \in K$ and to $H' \in \bar{S}^{\Delta_1}$ respectively. The same argument as there shows the existence of of $\tau \in \bar{W}_S$ such that $\tau H' = H_0$. Thus $\tau \bar{S}^{\Delta_1} \supset S^{\Delta_2}$.

Conversely, assume the existence of $\tau \in \bar{W}_S$ with $\tau \bar{S}^{\Delta_1} \supset S^{\Delta_2}$. Let $\tau = t(A)s$, where $A \in \Gamma$ and $s \in W$. Take an arbitrary point $k_0 \mathrm{Exp}\, H_0 \in M^{\Delta_2}$, where $k_0 \in K$ and $H_0 \in S^{\Delta_2}$. Choose $k_1 \in N_K(A)$ with $\mathrm{Ad}\, k_1 | \mathfrak{a} = s$ and a sequence $H_n \in S^{\Delta_1}$ such that τH_n converges to H_0. Then the sequence $k_0 k_1 \mathrm{Exp}\, H_n = k_0 \mathrm{Exp}\, \tau H_n$ in M^{Δ_1} converges to $k_0 \mathrm{Exp} H_0$. This shows $\bar{M}^{\Delta_1} \supset M^{\Delta_2}$. q.e.d.

COROLLARY 1. (Sakai [8])

Let (M, g) be a simply connected compact symmetric space. Then:

1) *M is the disjoint union*:

$$M = \bigcup_{\Delta} M^{\Delta}$$

of connected regular submanifolds M^{Δ}, which are diffeomorphic with $K/Z^{\Delta} \times S^{\Delta}$;

2) $\bar{M}^{\Delta_1} \supset M^{\Delta_2} \Longleftrightarrow \Delta_1 \supset \Delta_2$;

3) $Z^{H_0} = Z^{\Delta}$ *for each* $H_0 \in S^{\Delta}$.

PROOF. 1) and 2) follow from Lemma 1.3, 2): $\bar{W}_S = \{1\}$.

3) Let $k \in Z^{H_0}$, so that $k \,\mathrm{Exp}\, H_0 = \mathrm{Exp}\, H_0$. We have to show $k \in Z^{\Delta}$. Lemma 1.7 implies the existence of $\tau \in \bar{N}^{\Delta}$ such that

$$k \,\mathrm{Exp}\, H = \mathrm{Exp}\, \tau H \qquad \text{for each } H \in S^{\Delta}.$$

Since $\bar{N}^{\Delta} = \{1\}$ in our case, we have $\tau = 1$, and hence $k \in Z^{\Delta}$. q.e.d.

Consider the map Ψ^{Δ} in the case where $\Delta = \Pi'$. Our $\Psi^{\Pi'}$ will be abbreviated to Ψ and $M^{\Pi'}$ will be denoted by R. Note that R is connected. An element of R is called a ***regular point*** of (M, g) with respect to the origin o. In this case, we have $Z^{\Pi'} = Z_K(A)$, $S^{\Pi'} = S$ and $\bar{W}^{\Pi'} = \bar{W}_S$. Thus we have the following

COROLLARY 2. *The C^{∞}map Ψ: $K/Z_K(A) \times S \longrightarrow R$ defined by*

$$\Psi(kZ_K(A), H) = k \,\mathrm{Exp}\, H \qquad \text{for } k \in K,\ H \in S$$

is a covering map, and it induces a diffeomorphism ψ: $K/Z_K(A) \times_{\bar{W}_S} S \longrightarrow R$. In particular, the C^{∞}map Ψ is a diffeomorphism if M is simply connected.

It is known (cf. Helgason [3]) that the conjugate locus Q of (M, g) with respect to o is given by

$$Q=M-R.$$

The tangential first conjugate locus $\tilde{F}$ of (M, g) in $T_o(M)$ is given as follows. Recall first that $\mathfrak{m}=\mathrm{Ad}K\mathfrak{a}_+$ and thus $\tilde{F}=\mathrm{Ad}K(\tilde{F}\cap\mathfrak{a}_+)$. It is known (cf. Helgason [3]) that $\tilde{F}\cap\mathfrak{a}_+$ is given by

$$\tilde{F}\cap\mathfrak{a}_+=\{H\in\mathfrak{a}_+;\ 2(\gamma, H)=1 \quad \text{for some } \gamma\in\Sigma'\}.$$

Thus we get

$$\tilde{F}=\mathrm{Ad}K(\tilde{F}\cap\bar{S}).$$

where $\tilde{F}\cap\bar{S}$ is given by

$$\tilde{F}\cap\bar{S}=\bigcup_{\Delta\not\supset\Sigma'}S^\Delta.$$

Recall that the first conjugate locus F of (M, g) with respect to o is defined by $F=\mathrm{Exp}\,\tilde{F}$. Now we get stratifications of Q and F.

COROLLARY 3. *We have*

$$Q=\bigcup_{\Delta\subsetneqq\Pi'}M^\Delta,$$

$$F=\bigcup_{\Delta\not\supset\Sigma'}M^\Delta,$$

where Δ ranges in admissible subsets of Π'.

§ 2. Fundamental groups of compact symmetric spaces

In this section, we shall prove that the group $\bar{W}_S$ is isomorphic with the fundamental group $\pi_1(M)$ of M. Furthermore we shall investigate the relations between submanifolds M^Δ making use of the group $\bar{W}_S$.

LEMMA 2.1. *Let R be the set of regular points of (M, g) with respect to o, and let ι: $R\longrightarrow M$ be the inclusion map. Then the induced homomorphism ι_*: $\pi_1(R)\longrightarrow\pi_1(M)$ is surjective.*

PROOF. By Theorem 1.1, $Q=M-R$ is the union of submanifolds M^Δ with $\dim M^\Delta\leqslant\dim M-2$. Thus a theorem of the dimension theory (cf. Helgason [3]) yields the Lemma. q.e.d.

LEMMA 2.2. *Let G_0' be the simply connected compact Lie group with the Lie algebra $\mathfrak{g}'$ and let θ_0' be the involutive automorphism of G_0' whose differential is $\theta'=\theta|\mathfrak{g}'$, Put*

$$K_0'=\{x\in G_0';\ \theta_0'(x)=x\}.$$

Let A_0' denote the toral subgroup of G_0' generated by $\mathfrak{a}'$. Then $K/Z_K(A)$ is diffeomorphic with $K_0'/Z_{K_0'}(A_0')$ in the natural way.

PROOF. (i) Let K^0 denote the identity component of K. Then the inclusion

$K_0 \longrightarrow K$ induces a diffeomorphism $K^0/Z_{K^0}(A) \longrightarrow K/Z_K(A)$, since K is generated by K^0 and $K \cap A$.

(ii) Let G' and A' be connected Lie subgroups of G generated by $\mathfrak{g}'$ and $\mathfrak{a}'$ respectively, and put $K' = G' \cap K$. We have $K^0 = C_{\mathfrak{t}} K'^0$, where $C_{\mathfrak{t}}$ is the toral subgroup of G generated by $\mathfrak{c}_{\mathfrak{t}}$ and K'^0 is the identity component of K'. Thus the inclusion $K'^0 \longrightarrow K^0$ induces a diffeomorphism $K'^0/Z_{K'^0}(A') \longrightarrow K^0/Z_{K^0}(A)$.

(iii) Let π: $G_0' \longrightarrow G'$ be the covering homomorphism. Since K_0' is connected by Lemma 1.2, 2), π induces a covering homomorphism π: $K_0' \longrightarrow K'^0$. This induces a diffeomorphism $K_0'/Z_{K_0'}(A_0') \longrightarrow K'^0/Z_{K'^0}(A')$.

The composition of the above three diffeomorphisms is the required one. q.e.d.

THEOREM 2.1. *The group $\bar{W}_S$ is isomorphic with the fundamental group $\pi_1(M)$ of M.*

PROOF. This theorem, in a restricted case, was proved by Takeuchi [9]. We prove the present theorem in the same way as [9].

Let $M_0' = G_0'/K_0'$. Since K_0' is connected, M_0' is a compact simply connected symmetric space. Let R_0' denote the set of regular points of M_0'. Then, by Corollary 2 of Theorem 1.1, the C^∞map Ψ_0': $K_0'/Z_{K_0'}(A_0') \times S' \longrightarrow R_0'$ defined by

$$\Psi_0'(kZ_{K_0'}(A_0'), H) = k \operatorname{Exp}' H \qquad \text{for } k \in K_0',\ H \in S'$$

is a diffeomorphism. Here Exp' denotes the exponential map of M_0' at the origin.

Identifying $K/Z_K(A) \times S$ with $\mathfrak{c}_{\mathfrak{m}} \times K_0'/Z_{K_0'}(A_0') \times S'$ by Lemma 2.2 and (1.5), we define a C^∞ map $\tilde{\iota}: K/Z_K(A) \times S \longrightarrow \mathfrak{c}_{\mathfrak{m}} \times M_0'$ by

$$\tilde{\iota}(H'', kZ_{K_0'}(A_0'), H') = (H'', \Psi_0'(kZ_{K_0'}(A_0'), H'))$$

$$\text{for } H' \in \mathfrak{c}_{\mathfrak{m}},\ k \in K_0',\ H' \in S'.$$

From the above argument we see that $\tilde{\iota}$ is an imbedding with the image $\mathfrak{c}_{\mathfrak{m}} \times R_0'$. We define further a covering map Π: $\mathfrak{c}_{\mathfrak{m}} \times M_0' \longrightarrow M$ by

$$\Pi(H'', xK_0') = (\exp H'' \pi(x))o \qquad \text{for } H'' \in \mathfrak{c}_{\mathfrak{m}},\ x \in G_0'.$$

Then it is verified that the following diagram is commutative.

$$\begin{array}{ccc} K/Z_K(A) \times S & \xrightarrow{\ \tilde{\iota}\ } & \mathfrak{c}_{\mathfrak{m}} \times M_0' \\ \downarrow \Psi & & \downarrow \Pi \\ R & \xrightarrow[\ \iota\ ]{} & M \end{array}$$

Fix points $p \in R$ and $\tilde{p} \in K/Z_K(A) \times S$ with $\Psi(\tilde{p}) = p$. For a continuous closed curve $c: [0, 1] \longrightarrow R$ in R with $c(0) = c(1) = p$, let $\tilde{c}: [0, 1] \longrightarrow K/Z_K(A) \times S$ denote the lift of c relative to Ψ with $\tilde{c}(0)=\tilde{p}$. The terminal point $\tilde{c}(1)$ of $\tilde{c}$ depends only on the homotomy class $\{c\}\in\pi_1(R)$ of c. From Corollary 2 of Theorem 1.1, there exists uniquely $\tau\in\bar{W}_S$ such that $\tilde{p}\tau=\tilde{c}(1)$. Then the correspondence $\{c\}\longmapsto\tau$ defines a homomorphism ϕ: $\pi_1(R)\longrightarrow\bar{W}_S$. It is surjective since $K/Z_K(A)\times S$ is connected. For $\{c\}\in\pi_1(R)$, we have $\phi(\{c\})=1$ if and only if the lift $\tilde{c}$ of c relative to Ψ is a closed curve, which is equivalent to that the lift $\widetilde{\iota\circ c}=\tilde{\iota}\circ\tilde{c}$ of $\iota\circ c$ relative to Π is a closed curve. Since $\mathfrak{c}_{\mathfrak{m}}\times M_0'$ is simply connected, the above is equivalent to that the closed curve $\iota\circ c$ is homotopic to zero in M. Thus we get

$$\pi_1(R)/\operatorname{kernel}\iota_*\cong\bar{W}_S.$$

On the other, ι_* is surjective by Lemma 2.1, and hence

$$\pi_1(R)/\operatorname{kernel}\iota_*\cong\pi_1(M).$$

Thus $\bar{W}_S\cong\pi_1(M)$. q.e.d.

Now Lemma 1.3 implies the following

COROLLARY. *The fundamental group $\pi_1(M)$ of a compact connected symmetric space (M, g) is isomorphic with Γ/Γ^0. Therefore $\pi_1(M)$ is an abelian group.*

Now we shall study the detailed structure of $\bar{W}_S$.

We define a surjective map π_{Γ^*}: $\bar{W}^*\longrightarrow\Gamma^*$ by

$$\pi_{\Gamma^*}(\tau)=\tau(0)\quad\text{for }\tau\in\bar{W}^*,$$

or equivalently, by

$$\pi_{\Gamma^*}(t(A)s)=A\quad\text{for }A\in\Gamma^*,\ s\in W.$$

It induces also a surjective map π_{Γ^*}: $\bar{W}\longrightarrow\Gamma$. Let π_W: $\bar{W}^*\longrightarrow W$ be a homomorphism defined by

$$\pi_W(t(A)s)=s\quad\text{for }A\in\Gamma^*,\ s\in W.$$

It induces also a homomorphism π_W: $\bar{W}\longrightarrow W$. Recall the decomposition:

$$\mathfrak{a}=\mathfrak{c}_{\mathfrak{m}}+\mathfrak{a}'.$$

Let $p_{\mathfrak{c}}$: $\mathfrak{a}\longrightarrow\mathfrak{c}_{\mathfrak{m}}$ and $p_{\mathfrak{a}'}$: $\mathfrak{a}\longrightarrow\mathfrak{a}'$ denote orthogonal projections onto $\mathfrak{c}_{\mathfrak{m}}$ and $\mathfrak{a}'$ respectively. We define a map $\pi_{\mathfrak{c}}$: $\bar{W}^*\longrightarrow\mathfrak{c}_{\mathfrak{m}}$ by

$$\pi_{\mathfrak{c}}=p_{\mathfrak{c}}\circ\pi_{\Gamma^*}.$$

In general, for $\tau_i=t(A_i''+A_i')s_i$, where $A_i''\in\mathfrak{c}_{\mathfrak{m}}$, $A_i'\in\mathfrak{a}'$, $s_i\in W$ $(i=1,2)$, we have

$$\tau_1\tau_2=t(A_1''+A_2''+(A_1'+s_1A_2'))s_1s_2,\tag{2.1}$$

where $A_1'+A_2''\in\mathfrak{c}_{\mathfrak{m}}$, $A_1'+s_1A_2'\in\mathfrak{a}'$, $s_1s_2\in W$. Therefore $\pi_{\mathfrak{c}}$ is a homomorphism. It induces also a homomorphism $\pi_{\mathfrak{c}}$: $\bar{W} \longrightarrow \mathfrak{c}_{\mathrm{m}}$. We define subgroups $\bar{W}_{S'}{}^*$ and $(\bar{W}_S)_*$ of $\bar{W}_S{}^*$ by

$$\bar{W}^*_{S'} = \{\tau \in \bar{W}^*_S;\ \pi_{\mathfrak{c}}(\tau)=0\},$$

$$(\bar{W}_S)_* = \{\tau \in \bar{W}_S;\ \pi_{\mathfrak{c}}(\tau)=0\} \subset \bar{W}^*_{S'}.$$

The group $\bar{W}^*_{S'}$ acts on $\mathfrak{c}_{\mathrm{m}}$ trivially, and hence it is identified with a subgroup of the group of Euclidean motions of $\mathfrak{a}'$. Actually we have an isomorphism:

(2.2) $$\bar{W}^*_{S'} \cong \bar{W}^*_{S_1} \times \cdots \times \bar{W}^*_{S_t}.$$

Thus $\bar{W}^*_{S'}$ is a finite group, and hence $(\bar{W}_S)_*$ is also a finite group. Next we define a subgroup Z of $\mathfrak{c}_{\mathrm{m}}$ by

$$Z = \pi_{\mathfrak{c}}(\bar{W}_S).$$

Since Z contains the lattice Γ_0 of $\mathfrak{c}_{\mathrm{m}}$, Z is also a lattice of $\mathfrak{c}_{\mathrm{m}}$. Thus Z is isomorphic with $\boldsymbol{Z}^{r_0}$. From definitions, we have an exact sequence:

(2.3) $$0 \longrightarrow (\bar{W}_S)_* \longrightarrow \bar{W}_S \xrightarrow{\pi_{\mathfrak{c}}} Z \longrightarrow 0.$$

This exact sequence splits since Z is free, and hence

(2.4) $$\bar{W}_S \cong (\bar{W}_S)_* \times Z, \quad Z \cong \boldsymbol{Z}^{r_0}.$$

We define a map π': $\bar{W}_S \longrightarrow \bar{W}^*_{S'}$ by

$$\pi'(t(A''+A')s) = t(A')s \quad \text{for } A'' \in \mathfrak{c}_{\mathrm{m}},\ A' \in \mathfrak{a}',\ s \in W.$$

The map π' is a homomorphism in virtue of (2.1), and satisfies

(2.5) $$t(\pi_{\mathfrak{c}}(\tau))\pi'(\tau) = \tau \quad \text{for each } \tau \in \bar{W}_{S'},$$

(2.6) $$\pi_W(\pi'(\tau)) = \pi_W(\tau) \quad \text{for each } \tau \in \bar{W}_S.$$

We define subgroups $\boldsymbol{F}^*$, $\boldsymbol{F}$ and $\boldsymbol{F}_*$ of W by

$$\boldsymbol{F}^* = \pi_W(\bar{W}^*{}_{S'}),$$
$$\boldsymbol{F} = \pi_W(\bar{W}_S),$$
$$\boldsymbol{F}_* = \pi_W((\bar{W}_S)_*).$$

Since π_W is injective on $\bar{W}^*_{S'}$, we have isomorphisms $\boldsymbol{F}^* \cong \bar{W}^*_{S'}$ and $\boldsymbol{F}_* \cong (\bar{W}_S)_*$. In virtue of (2.6), $\boldsymbol{F}$ is a subgroup of $\boldsymbol{F}^*$, and hence

$$\boldsymbol{F}_* \subset \boldsymbol{F} \subset \boldsymbol{F}^*.$$

Isomorphisms (2.4) imply

(2.7) $$\bar{W}_S \cong \boldsymbol{F}_* \times Z, \quad Z \cong \boldsymbol{Z}^{r_0}.$$

Note that from (2.2) we have

$$\boldsymbol{F}^* \cong \boldsymbol{F}_1{}^* \times \cdots \times \boldsymbol{F}_s{}^*, \tag{2.8}$$

where $\boldsymbol{F}_k{}^*$ is the corresponding group for the k-th irreducible factor $(\mathfrak{g}_k, \mathfrak{k}_k)$ $(1 \leqslant k \leqslant s)$. We define an injective map $\gamma \longmapsto \gamma^\natural$ from Π' into $\mathfrak{a}'$ by the correspondence (1.15), and denote its image by $\Pi^\natural$. Define

$$\mathrm{Aut}(\Pi_k{}^\natural) = \{s \in O(\mathfrak{a}_k);\ s\Pi_k{}^\natural = \Pi_k{}^\natural\} \quad (1 \leqslant k \leqslant s),$$

and then define a subgroup $\mathrm{Aut}(\Pi^\natural)$ of $O(\mathfrak{a}')$ by

$$\mathrm{Aut}(\Pi^\natural) = \mathrm{Aut}(\Pi_1{}^\natural) \times \cdots \times \mathrm{Aut}(\Pi_s{}^\natural).$$

We can prove the following lemma in the same way as in Takeuchi [10].

LEMMA 2.3. *Assume that* $\mathfrak{g}$ *is semi-simple and* $(\mathfrak{g}, \mathfrak{k})$ *is irreducible. Then, under the notation in the proof of Lemma* 1.4:

1) $\bar{S} \cap \Gamma^*$ *is a subset of the set* $\{P_\gamma;\ \gamma \in \Pi^\natural\}$ *of vertices of* $\bar{S}$, *given by*

$$\bar{S} \cap \Gamma^* = \{P_{\gamma_i};\ n_i = 1\}.$$

2) *For* $\tau \in \bar{W}_S^*$ *let* $\tau^\natural$ *be the permutatiom of* $\Pi^\natural$ *defined by*

$$\tau P_\gamma = P_{\tau^\natural \gamma} \quad \text{for } \gamma \in \Pi^\natural.$$

Then

$$\pi_W(\tau)\gamma = \tau^\natural \gamma \quad \text{for each } \gamma \in \Pi^\natural.$$

If $\tau^\natural \gamma_0 = \gamma_i$ $(0 \leqslant i \leqslant r)$, *then* $\pi_W(\tau) \in W$ *is characterized by*

$$\{\gamma \in \Sigma;\ \gamma > 0,\ \pi_W(\tau)^{-1}\gamma < 0\} = \{\gamma \in \Sigma; (\gamma, \varepsilon_i) > 0\}.$$

COROLLARY. *We have* $\boldsymbol{F}_k{}^* \subset \mathrm{Aut}(\Pi_k{}^\natural)$ *for each* k. *Therefore* $\boldsymbol{F}^* \subset \mathrm{Aut}(\Pi^\natural)$.

LEMMA 2.4. 1) *We have the folllowing commutative diagram.*

$$\begin{array}{ccccccc}
Z \times \boldsymbol{F} & \xleftarrow[\text{inj.}]{\pi_{\mathfrak{c}} \times \pi_W} & \bar{W}_S & \xrightarrow[\text{bij.}]{\pi_{\Gamma^*}} & \bar{S} \cap \Gamma \\
\cap\big\downarrow & & \text{inj.}\big\downarrow \pi_{\mathfrak{c}} \times \pi' & & \text{inj.}\big\downarrow p_{\mathfrak{c}} \times p_{\mathfrak{a}'} \\
Z \times \boldsymbol{F}^* & \xleftarrow[1 \times \pi_W]{\text{bij.}} & Z \times \bar{W}_{S'}^*, & \xrightarrow[1 \times \pi_{\Gamma^*}]{\text{bij.}} & Z \times (\bar{S}' \cap \Gamma^*)
\end{array}$$

Thus Z *is given by*

$$Z = p_{\mathfrak{c}}(\bar{S} \cap \Gamma). \tag{2.9}$$

2) *As for groups* $\boldsymbol{F}_*$, $\boldsymbol{F}$ *and* $\boldsymbol{F}^*$, *we have the following commutative diagram.*

$$\begin{array}{ccccc}
\boldsymbol{F}_* & \xleftarrow[\text{bij.}]{\pi_W} & (\bar{W}_S)_* & \xrightarrow[\text{bij.}]{\pi_{\Gamma^*}} & \bar{S}'\cap\Gamma \\
\cap\downarrow & & \cap\downarrow & & \cap\downarrow \\
\boldsymbol{F}^* & \xleftarrow[\pi_W]{\text{bij.}} & \bar{W}^*_{S'} & \xrightarrow[\pi_{\Gamma^*}]{\text{bij.}} & \bar{S}'\cap\Gamma^* \\
\cup\uparrow & & & & \cup\uparrow \\
\boldsymbol{F} & & \xleftarrow[\pi_W\circ\pi_{\Gamma^*}^{-1}]{\text{bij.}} & & p_{\mathfrak{a}'}(\bar{S}\cap\Gamma)
\end{array}$$

Proof. 1) Let $\tau=t(A''+A')s\in\bar{W}_S$, where $A''\in\mathfrak{c}_{\mathfrak{m}}$, $A'\in\mathfrak{a}'$ and $s\in W$. Then $\pi_{\mathfrak{c}}(\tau)=A''$, $\pi'(\tau)=t(A')s$ and $\pi_W(\tau)=s$. If $\pi_{\mathfrak{c}}(\tau)=0$, $\pi'(\tau)=1$, then $A''=0$, $A'=0$, $s=1$ and hence $\tau=1$. If $\pi_{\mathfrak{c}}(\tau)=0$, $\pi_W(\tau)=0$, then $A''=0$, $s=1$ and hence $\tau=t(A')$ with $A'\in\mathfrak{a}'$. Since $\tau S=S$, we get $A'=0$, and hence $\tau=1$. Thus both $\pi_{\mathfrak{c}}\times\pi_W$ and $\pi_{\mathfrak{c}}\times\pi'$ are injective on $\bar{W}_S$. The commutativity of the left square follows from (2.5).

Note that π_{Γ^*}: $\bar{W}_S\longrightarrow\Gamma^*$ is injective, since $sS=S$ implies $s=1$ for $s\in W$. Each $\tau\in\bar{W}_S$ leaves also $\bar{S}$ invariant. Recalling $0\in\bar{S}$, we get $\pi_{\Gamma^*}(\tau)=\tau(0)\in\bar{S}\cap\Gamma$ for each $\tau\in\bar{W}_S$. Take an arbitrary $A\in\bar{S}\cap\Gamma$ and let $A=A''+A'$, where $A''\in\mathfrak{c}_{\mathfrak{m}}$ and $A'\in\mathfrak{a}'$. Then $p_{\mathfrak{a}'}(A)=A'\in\bar{S}'\cap\Gamma^*$. Now $t(A)^{-1}S=\mathfrak{c}_{\mathfrak{m}}\times t(A')^{-1}S'$, where $t(A')^{-1}S'$ is a fundamental cell for $(\mathfrak{g}', \mathfrak{k}')$ whose closure contains 0. Hence there exists $s\in W$ such that $s^{-1}t(A')^{-1}S'=S'$. Putting $\tau=t(A)s\in\bar{W}$, we get $\tau^{-1}S=\mathfrak{c}_{\mathfrak{m}}\times S'=S$, and thus $\tau\in\bar{W}_S$. We have $\pi_{\Gamma^*}(\tau)=A$, and hence $p_{\mathfrak{c}}(A)=A''=p_{\mathfrak{c}}\pi_{\Gamma^*}(\tau)=\pi_{\mathfrak{c}}(\tau)\in Z$. These show that π_{Γ^*}: $\bar{W}_S\longrightarrow\bar{S}\cap\Gamma$ is surjective and that $p_{\mathfrak{c}}\times p_{\mathfrak{a}'}$ maps $\bar{S}\cap\Gamma$ into $Z\times(\bar{S}'\cap\Gamma^*)$. Thus the map π_{Γ^*}: $\bar{W}_S\longrightarrow\bar{S}\cap\Gamma$ is bijective. The map $p_{\mathfrak{c}}\times p_{\mathfrak{a}'}$ is clearly injective.

Applying the same argument for the symmetric pair (G^*, K^*) of $G^*=\mathrm{Ad}G$ and $K^*=\{x\in G^*;\ \theta x=x\theta\}$, we see that π_{Γ^*}: $\bar{W}^*_{S'}\longrightarrow\bar{S}'\cap\Gamma^*$ is bijective. This implies the bijectivity of $1\times\pi_{\Gamma^*}$: $Z\times\bar{W}^*_{S'}\longrightarrow Z\times(\bar{S}'\cap\Gamma^*)$.

The commutativity of the right square follows from definitions.

2) The bijection π_{Γ^*}: $\bar{W}_S\longrightarrow\bar{S}'\cap\Gamma$ induces bijections $(\bar{W}_S)_*\longrightarrow\bar{S}'\cap\Gamma$ and $\bar{W}^*_{S'}\longrightarrow\bar{S}'\cap\Gamma^*$. The lower square follows from the diagram 1). q.e.d.

Corollary. *The groups* $\bar{W}_S$, $(\bar{W}_S)_*$ *and* $\bar{W}^*_{S'}$ *act simply transitively on* $\bar{S}\cap\Gamma$, $\bar{S}'\cap\Gamma$ *and* $\bar{S}'\cap\Gamma^*$ *respectively.*

Remark. 1) We can determine the torsion part $\boldsymbol{F}_*$ of $\bar{W}_S$ and the group $\boldsymbol{F}$ by making use of Lemmas 2.3 and 2.4. In fact, each $\boldsymbol{F}_k^*\subset\mathrm{Aut}(\Pi_k^{\natural})$ is determined by Lemma 2.3, and hence $\boldsymbol{F}^*=\boldsymbol{F}_1^*\times\cdots\times\boldsymbol{F}_s^*\subset\mathrm{Aut}(\Pi^{\natural})$ is determined. Finding subsets $\bar{S}'\cap\Gamma$ and $p_{\mathfrak{a}'}(\bar{S}\cap\Gamma)$ of $\bar{S}'\cap\Gamma^*$, we get subgroups $\boldsymbol{F}_*$ and $\boldsymbol{F}$ of $\boldsymbol{F}^*$ by means of Lemma 2.4.

On the other hand, the free part Z of $\bar{W}_S$ is obtained by (2.9).

Thus we get $\bar{W}_S$ as a subgroup of $Z\times \boldsymbol{F}$ by means of the diagram 1).

2) Let π: $\tilde{M}\longrightarrow M$ be the Riemannian universal covering of M. Then $\bar{S}\cap\Gamma$ and $\bar{W}_S$ are identified with $\pi^{-1}(o)$ and the covering transformation group $G(\pi)$ respectively, in such a way that the action of $\bar{W}_S$ on $\bar{S}\cap\Gamma$ corresponds to that of $G(\pi)$ on $\pi^{-1}(o)$.

3) If we identify $\bar{W}_S$ and $\bar{S}\cap\Gamma$ with a subgroup and a subset of $Z\times\boldsymbol{F}$ by means of bijections in the diagram 1), then the action of $\bar{W}_S$ on $\bar{S}\cap\Gamma$ is nothing but the left translation in the group $Z\times\boldsymbol{F}$.

We define an action $\gamma\longmapsto\tau\cdot\gamma$ of $\bar{W}_S$ on the set Π' by

$$(\tau\cdot\gamma)^{\natural}=\pi_w(\tau)\gamma^{\natural}\quad\text{for } \tau\in\bar{W}_S,\ \gamma\in\Pi'.$$

With these definitions we have

Lemma 2.5. *Let Δ be an admissible subset of Π'. Then:*

1) $\tau S^{\Delta}=S^{\tau\cdot\Delta}$ *for each* $\tau\in\bar{W}_S$;

2) $\bar{N}^{\Delta}=\{\tau\in\bar{W}_S;\ \tau\cdot\Delta=\Delta\}$, *and* $\bar{Z}^{\Delta}=\{\tau\in(\bar{W}_S)_*;\ \tau\cdot|\Delta=\mathrm{id}\}$.

Proof. 1) We may assume that $\mathfrak{g}$ is semi-simple and $(\mathfrak{g},\mathfrak{k})$ is irreducible. In this case, under the notation in Lemmas 1.4 and 2.3, S^{Δ} is given by

$$S^{\Delta}=\{\sum_{\gamma\in\Delta^{\natural}}h_{\gamma}P_{\gamma};\ 0<h_{\gamma}<1,\ \sum_{\gamma\in\Delta^{\natural}}h_{\gamma}=1\},\tag{1.16}$$

and τ is given by

$$\tau(\sum_{\gamma\in\Delta^{\natural}}h_{\gamma}P_{\gamma})=\sum_{\gamma\in\Delta^{\natural}}h_{\gamma}P\pi_w(\tau)\gamma.\tag{2.10}$$

These imply the assertion 1).

2) The assertion for $\bar{N}^{\Delta}$ follows from 1). If $\mathfrak{g}$ is semi-simple and $(\mathfrak{g},\mathfrak{k})$ is irreducible, then from (1.16) and (2.10) we have

$$\bar{Z}^{\Delta}=\{\tau\in\bar{W}_S;\ \tau\cdot\gamma=\gamma\ \text{ for each } \gamma\in\Delta\}.$$

This implies also the assertion for $\bar{Z}^{\Delta}$ in general case. q.e.d.

Let Δ_1 and Δ_2 be admissible subsets of Π'. They are said to be *equivalent* if their exists $s\in\boldsymbol{F}$ such that $s\Delta_1^{\natural}=\Delta_2^{\natural}$. We denote by $\Delta_1\succ\Delta_2$ if there exists $s\in\boldsymbol{F}$ such that $s\Delta_1^{\natural}\supset\Delta_2^{\natural}$. With these definitions, by Theorem 1.1, its Corollary 3 and Lemma 2.5, 1), we have the following theorem.

Theorem 2.2. 1) *Let $\mathscr{D}^*$ be a set of complete representatives of equivalence classes of admissible subsets in Π'. Then*

$$M = \bigcup_{\Delta \in \mathcal{D}^*} M^{\Delta} \text{ (disjoint union)},$$

where $\bar{M}^{\Delta_1} \supset M^{\Delta_2}$ *if and only if* $\Delta_1 > \Delta_2$.

2) *Let* $\mathcal{D} = \mathcal{D}^* - \{\Pi'\}$, *and let* $\mathcal{F}$ *be the subset of* $\mathcal{D}$ *consisting of all* $\Delta \in \mathcal{D}$ *which is equivalent to some* Δ' *with* $\Delta' \not\ni \Sigma'$. *Then*

$$Q = \bigcup_{\Delta \in \mathcal{D}} M^{\Delta} \text{ (disjoint union)},$$

$$F = \bigcup_{\Delta \in \mathcal{F}} M^{\Delta} \text{ (disjoint union)}.$$

REMARK. Note that the set $\mathcal{D}$ as well as the set $\mathcal{F}$ is a finite set.

§ 3. Cut loci of compact symmetric spaces

In this section, we shall study the structure of cut loci of compact symmetric spaces and give stratifications of them by a refinement of methods for conjugate loci.

For $H \in \mathfrak{a}$, the norm $\sqrt{(H, H)}$ of H with respect to the inner product $(\ ,\ )$ will be denoted by $|H|$. For a subset Γ' of Γ with $\Gamma' - \{0\} \neq \phi$, we define functions $m_{\Gamma'}$ and $M_{\Gamma'}$ on $\mathfrak{a}$ by

$$m_{\Gamma'}(H) = \operatorname*{Min}_{A \in \Gamma' - \{0\}} |H - A|,$$

$$M_{\Gamma'}(H) = \operatorname*{Max}_{A \in \Gamma' - \{0\}} 2(H, A)/(A, A).$$

An elementary calculation shows

$$\text{(3.1)} \qquad \begin{cases} |H| < |H - A| \Longleftrightarrow 2(H, A)/(A, A) < 1, \\ |H| = |H - A| \Longleftrightarrow 2(H, A)/(A, A) = 1, \\ |H| > |H - A| \Longleftrightarrow 2(H, A)/(A, A) > 1. \end{cases}$$

Thus we have

$$\text{(3.2)} \qquad m_{\Gamma'}(H) = |H| \Longleftrightarrow M_{\Gamma'}(H) = 1.$$

Let $\tilde{C}$ and $\tilde{Q}$ be the tangenital cut locus and the tangenital conjugate locus of (M, g) in $\mathfrak{m} = T_o(M)$ respectively. Let $C = \operatorname{Exp} \tilde{C}$ be the cut locus of (M, g) with respect to o. Now Sakai characterized $\tilde{C}$ as follows.

THEOREM 3.1. (Sakai [7]) *We have*

$$\tilde{C} = \operatorname{Ad} K (\tilde{C} \cap \mathfrak{a}),$$

where $\tilde{C} \cap \mathfrak{a}$ *is given by*

$$\tilde{C}\cap\mathfrak{a}=\{H\in\mathfrak{a};\ m_{\Gamma}(H)=|H|\},$$

or equivalently, by

$$\tilde{C}\cap\mathfrak{a}=\{H\in\mathfrak{a};\ M_{\Gamma}(H)=1\}.$$

REMARK. Let $\hat{A}=Ao$. It is a maximal totally geodesic flat submanifold of (M,g). Then $\tilde{C}\cap\mathfrak{a}$ coincides with the tangenital cut locus of $\hat{A}$ in $\mathfrak{a}=T_o(\hat{A})$.

In the course of the proof of Theorem 3.1, Sakai [7] proved the following result.

LEMMA 3.1. *Let $H\in\tilde{C}\cap\bar{S}$ and $H\notin\tilde{Q}$. Then any $A\in\Gamma-\{0\}$ with $|H|=|H-A|$ belongs to $\bar{S}$. Thus, we have $\bar{S}\cap\Gamma-\{0\}\neq\phi$ and*

$$m_{\bar{S}\cap\Gamma}(H)=|H|.$$

In Theorem 3.1, it is not easy to compute $m_{\Gamma}(H)$, since Γ is an infinite set. So we will try to replace Γ by a finite subset of Γ.

Define subsets $\mathscr{K}$ and $\mathscr{L}$ of $\bar{S}$ by

$$\mathscr{K}=\{H\in\bar{S};\ 2(H,A)/(A,A)<1 \quad \text{for each } A\in\bar{S}\cap\Gamma-\{0\}\},$$

$$\mathscr{L}=\begin{cases}\{H\in\bar{S};\ M_{\bar{S}\cap\Gamma}(H)=1\} & \text{if } \bar{S}\cap\Gamma-\{0\}\neq\phi\\ \phi & \text{if } \bar{S}\cap\Gamma-\{0\}=\phi,\end{cases}$$

and then define

$$\Theta=\{\gamma\in\Sigma';\ \mathscr{K}\cap S_{\gamma}^{1}\neq\phi\},$$

$$\Lambda=\{A\in\bar{S}\cap\Gamma-\{0\};\ 2(H,A)/(A,A)=1 \quad \text{for some } H\in\mathscr{L}\}.$$

Put

$$A(\Theta)=\{A_{\gamma};\ \gamma\in\Theta\}.$$

It should be noted that both Λ and $A(\Theta)$ are finite subsets of Γ.

THEOREM 3.2. 1) *We have $\tilde{C}=\mathrm{Ad}K(\tilde{C}\cap\bar{S})$. Therefore $C=K\mathrm{Exp}(\tilde{C}\cap\bar{S})$.*

2) *The set $\Lambda\cup A(\Theta)-\{0\}=\Lambda\cup A(\Theta)$ is not empty, and $\tilde{C}\cap\bar{S}$ is given by*

$$\tilde{C}\cap\bar{S}=\{H\in\bar{S};\ M_{\Lambda\cup A(\Theta)}(H)=1\},$$

or equivalently, by

$$\tilde{C}\cap\bar{S}=\{H\in\bar{S};\ m_{\Lambda\cup A(\Theta)}(H)=|H|\}.$$

PROOF. 1) We know $\mathfrak{m}=\mathrm{Ad}K\mathfrak{a}_{+}$, and hence $\tilde{C}=\mathrm{Ad}K(\tilde{C}\cap\mathfrak{a}_{+})$. Therefore it suffices to show $\tilde{C}\cap\mathfrak{a}_+\subset\tilde{C}\cap\bar{S}$. Take an arbitrary $H\in\tilde{C}\cap\mathfrak{a}_+$. Then, by Theorem 3.1, $2(\gamma, H)=2(H, A_\gamma)/(A_\gamma, A_\gamma)\leqslant 1$ for each $\gamma\in\Sigma'$, and hence $H\in\bar{S}$. This proves the required inclusion.

2) Let $H\in\tilde{C}\cap\bar{S}$. Then $M_\Gamma(H)=1$ by Theorem 3.1. Put

$$\Omega_H=\{A\in\bar{S}\cap\Gamma-\{0\};\ 2(H, A)/(A, A)=1\}.$$

Case 1: $\Omega_H\neq\phi$. We have $H\in\mathscr{L}$, and hence $\Omega_H\subset\Lambda$.

Case 2: $\Omega_H=\phi$. We have $H\in\mathscr{K}$. Moreover, Lemma 3.1 implies $H\in\tilde{Q}$. Therefore there exists $\gamma\in\Sigma'$ such that $2(\gamma, H)=1$ so that $H\in S_\gamma{}^1$. Thus we have

$$2(H, A_\gamma)/(A_\gamma, A_\gamma)=1 \quad \text{with } \gamma\in\Theta.$$

These prove that $\Lambda\cap A(\Theta)\neq\phi$ always and that $M_{\Lambda\cup A(\Theta)}(H)=1$.

Conversely, assume that $H\in\bar{S}$ satisfies $M_{\Lambda\cup A(\Theta)}(H)=1$. Suppose $H\notin\tilde{C}$. If $H'=s_0H\in\tilde{C}$ with $s_0>1$, then $M_\Gamma(H')>1$, which contradicts to Theorem 3.1. Thus there exists s_0 with $0<s_0<1$ such that $H'=s_0H\in\tilde{C}$. But $H'\notin\tilde{Q}$ since $H\in\bar{S}$. Now Lemma 3.1 implies that $\bar{S}\cap\Gamma-\{0\}\neq\phi$ and $M_{\bar{S}\cap\Gamma}(H')=1$, and hence $H'\in\mathscr{L}$. Therefore there exists $A\in\bar{S}\cap\Gamma-\{0\}$ such that $2(H', A)/(A,A)=1$. From the definition, we have $A\in\Lambda$. But $H=(1/s_0)H'$ implies $2(H, A)/(A, A)>1$, which contradicts to $M_{\Lambda\cup A(\Theta)}(H)=1$. This shows $H\in\tilde{C}\cap\bar{S}$. q.e.d.

REMARK. By Theorem 3.2 we can show a well known fact that M is simply connected if and only if $F=C$ (cf. Crittenden[2], Sakai[8]).

We have defined in §1 a cellular decomposition of $\bar{S}$ closely related to the conjugate locus. Now we shall define another cellular decomposition of $\bar{S}$ closely related to the cut locus.

Let Φ be a subset of $\bar{S}\cap\Gamma$. The complement $\bar{S}\cap\Gamma-\Phi$ of Φ in $\bar{S}\cap\Gamma$ will be denoted by Φ^c. Let T^Φ be the subset of $\bar{S}$ consisting of all $H\in\bar{S}$ satisfying the conditions:

$$\begin{cases} |H-A|=|H-A'| & \text{for each } A, A'\in\Phi^c, \\ |H-A|<|H-A'| & \text{for each } A\in\Phi^c,\ A'\in\Phi. \end{cases}$$

It is easily seen that T^Φ is a convex subset of $\bar{S}$. A subset Φ of $\bar{S}\cap\Gamma$ is said to be *admissible* if $\Phi\subsetneqq\bar{S}\cap\Gamma$ and $T^\Phi\neq\phi$. Note that $|\Phi^c|<\infty$ for any admissible subset Φ of $\bar{S}\cap\Gamma$.

LEMMA 3.2. 1) $\bar{S}=\bigcup_\Phi T^\Phi$ *(disjoint union), where Φ ranges over the admissible subsets of* $\bar{S}\cap\Gamma$.

2) *The group $\bar{W}_S$ acts on the set of all T^Φ with Φ admissible. More precisely, we have $\tau T^\Phi=T^{\tau\Phi}$ for $\tau\in\bar{W}_S$ and an admissible subset Φ of $\bar{S}\cap\Gamma$.*

3) *For admissible subsets* Φ_1 *and* Φ_2 *of* $\bar{S}\cap\Gamma$,

$$\bar{T}^{\Phi_1}\supset T^{\Phi_2}\Longleftrightarrow \Phi_1\supset\Phi_2.$$

In this case, for $H_1\in T^{\Phi_1}$ *and* $H_2\in T^{\Phi_2}$, *we have*

$$tH_1+(1-t)H_2\in T^{\Phi_1} \quad \text{for each } t \text{ with } 0<t\leqq 1.$$

PROOF. 1) Let $H\in\bar{S}$. We define a function ρ_H on $\bar{S}\cap\Gamma$ by

$$\rho_H(A)=|H-A| \qquad \text{for } A\in\bar{S}\cap\Gamma.$$

Put

$$\Phi'_H=\{A\in\bar{S}\cap\Gamma;\ \rho_H(A)=\min_{A'\in\bar{S}\cap\Gamma}\rho_H(A')\}$$

and let $\Phi_H=\Phi'^{\,C}_H$. Then Φ'_H is a finite non-empty subset of $\bar{S}\cap\Gamma$, and Φ_H is an admissible subset of $\bar{S}\cap\Gamma$ such that $H\in T^{\Phi_H}$. This shows the assertion 1).

2) follows from that $\bar{W}_S$ preserves the Euclidean distance $|H-H'|$ on $\mathfrak{a}$.

3) By a translation, we may assume $0\in\Phi_1^C$. Then, $H\in T^{\Phi_1}$ if and only if

$$\begin{cases} |H|=|H-A| & \text{for each } A\in\Phi_1^C,\\ |H|<|H-A'| & \text{for each } A'\in\Phi_1.\end{cases}$$

Assume that $\bar{T}^{\Phi_1}\supset T^{\Phi_2}$. Then there exists a sequence $H_n\in T^{\Phi_1}$ converging to $H_0\in T^{\Phi_2}$. The conditions:

$$\begin{cases} |H_n|=|H_n-A| & \text{for each } A\in\Phi_1^C,\\ |H_n|<|H_n-A'| & \text{for each } A'\in\Phi_1\end{cases}$$

imply

$$\begin{cases} |H_0|=|H_0-A| & \text{for each } A\in\Phi_1^C,\\ |H_0|\leqq|H_0-A'| & \text{for each } A'\in\Phi_1.\end{cases}$$

This shows $\Phi_1^C\subset\Phi_2^C$, and hence $\Phi_1\supset\Phi_2$. Coversely, assume $\Phi_1\supset\Phi_2$. It follows from (3.1) that $H\in T^{\Phi_i}$ $(i=1,2)$ if and only if

$$\begin{cases} 2(H,A)/(A,A)=1 & \text{for each } A\in\Phi_i^C-\{0\},\\ 2(H,A')/(A',A')<1 & \text{for each } A'\in\Phi_i.\end{cases}$$

Take $H_1\in T^{\Phi_1}$ and $H_2\in T^{\Phi_2}$ and put

$$H=tH_1+(1-t)H_2 \qquad 0<t\leqq 1.$$

Then the equality:

$$2(H,A)/(A,A)=t\cdot 2(H_1,A)/(A,A)+(1-t)\cdot 2(H_2'\,A)/(A,A)$$

implies

$$\begin{cases} 2(H, A)/(A, A)=1 & \text{for each } A\in \Phi_1{}^C-\{0\}, \\ 2(H, A)/(A, A)<1 & \text{for each } A\in \Phi_1, \end{cases}$$

and hence $H\in T^{\Phi_1}$. This shows also $\bar{T}^{\Phi_1}\supset T^{\Phi_2}$. q.e.d.

Let (Δ, Φ) be a pair of subsets $\Delta\subset \Pi'$ and $\Phi\subsetneqq \bar{S}\cap \Gamma$. We define a subset $S^{\Delta,\Phi}$ of $\bar{S}$ by

$$S^{\Delta,\Phi}=S^{\Delta}\cap T^{\Phi}.$$

A pair (Δ, Φ) is said to be *admissible* if $S^{\Delta,\Phi}\neq\phi$. Note that for an admissible pair (Δ, Φ), $S^{\Delta,\Phi}$ is homeomorphic with a cell, since it is an open convex polyhedren in an affine subspace of $\mathfrak{a}$.

LEMMA 3.3. 1) *$\bar{S}=\bigcup_{(\Delta,\Phi)} S^{\Delta,\Phi}$ (disjoint union), where (Δ, Φ) ranges over the admissible pairs.*

2) *The group $\bar{W}_S$ acts on the set of all $S^{\Delta,\Phi}$ with (Δ, Φ) admissible. More precisely, we have $\tau S^{\Delta,\Phi}=S^{\tau\cdot\Delta,\tau\Phi}$ for $\tau\in\bar{W}_S$ and an admissible pair (Δ, Φ).*

3) *For admissible pairs (Δ_1, Φ_1) and (Δ_2, Φ_2),*

$$\bar{S}^{\Delta_1,\Phi_1}\supset S^{\Delta_2,\Phi_2}\Longleftrightarrow \Delta_1\supset\Delta_2 \text{ and } \Phi_1\supset\Phi_2.$$

PROOF. 1) and 2) follow from Lemmas 1.4, 2.5 and 3.2.

3) Assume $\bar{S}^{\Delta_1,\Phi_1}\supset S^{\Delta_2,\Phi_2}$. Then, Lemma 1. 4, 3) and Lemma 3.2, 3) imply $\Delta_1\supset\Delta_2$ and $\Phi_1\supset\Phi_2$. Assume conversely $\Delta_1\supset\Delta_2$ and $\Phi_1\supset\Phi_2$. Then it follows from the same lemmas that for $H_1\in S^{\Delta_1,\Phi_1}$ and $H_2\in S^{\Delta_2\Phi_2}$ we have

$$tH_1+(1-t)H_2\in S^{\Delta_1,\Phi_1} \quad \text{for each } t \text{ with } 0<t\leqslant 1.$$

This implies $\bar{S}^{\Delta_1,\Phi_1}\supset S^{\Delta_2,\Phi_2}$. q.e.d.

We can also extend the above decomposition of $\bar{S}$ to a $\bar{W}$-invariant cellular decomposition of $\mathfrak{a}$ as in §1.

A pair (Δ, Φ) of subsets $\Delta\supset\Pi'$ and $\Phi\subsetneqq\bar{S}\cap\Gamma$ is called a *c-pair* if it satisfies the following conditions.

(i) (Δ, Φ) is admissible.

(ii) $0\in\Phi^C$.

(iii) $\Phi^C-\{0\}\subset\Delta$.

(iv) $\Delta\not\supset\Phi$ if $\Phi^C=\{0\}$.

LEMMA 3.4. *$\tilde{C}\cap\bar{S}=\bigcup'_{(\Delta,\Phi)} S^{\Delta,\Phi}$ (disjoint union), where $\bigcup'$ means the union over* *the all c-pairs* (Δ, Φ).

Proof. Let $H\in\tilde{C}\cap\bar{S}$. Let Δ be the unique admissible subset of Π' with $H\in S^{\Delta}$. Put

$$\Omega_H=\{A\in\bar{S}\cap\Gamma-\{0\};\ |H|=|H-A|\}.$$

In the proof of Theorem 3.2, we have showed the following:

Case 1: $\Omega_H\neq\phi$. $\Omega_H\subset\Lambda$.

Case 2: $\Omega_H=\phi$. $H\in\mathscr{K}$ and there exists $\gamma\in\Theta$ with $2(\gamma, H)=1$.

In Case 1, put $\Phi=\bar{S}\cap\Gamma-(\Omega_H\cup\{0\})$. Then $H\in T^{\Phi}$, $\Phi^C=\Omega_H\cup\{0\}\neq\{0\}$ and $\Phi^C-\{0\}=\Omega_H$. Hence, $H\in S^{\Delta,\Phi}$ and (Δ, Φ) is a c-pair. In Case 2, put $\Phi=\bar{S}\cap\Gamma-\{0\}$. Then $H\in T^{\Phi}=\mathscr{K}$, $\Phi^C=\{0\}$, $\Phi^C-\{0\}=\phi$ and $\Theta\not\subset\Delta$. Hence, $H\in S^{\Delta,\Phi}$ and (Δ, Φ) is a c-pair.

Conversely, let $H\in S^{\Delta,\Phi}$ with (Δ, Φ) a c-pair. In virtue of $0\in\Phi^C$, we have $|H|\leqslant|H-A|$ for each $A\in\bar{S}\cap\Gamma-\{0\}$. In particular, we have

$$|H|\leqslant|H-A|\quad\text{for each } A\in\Lambda.$$

On the other hand, $H\in\bar{S}$ implies that $2(\gamma, H)\leqslant 1$ for each $\gamma\in\Sigma'$, or equivalently, $2(H, A_\gamma)/(A_\gamma, A_\gamma)\leqslant 1$ for each $\gamma\in\Sigma'$. In particular, we have

$$|H|\leqslant|H-A|\quad\text{for each } A\in A(\Theta).$$

Therefore we get

$$|H|\leqslant m_{\Lambda\cup A(\Theta)}(H).$$

Case 1: $\Phi^C\neq\{0\}$. In this case, we have

$$|H|=|H-A|\quad\text{for each } A\in\Phi^C-\{0\}\,(\neq\phi)\subset\Lambda.$$

Thus we get $|H|=m_{\Lambda\cup A(\Theta)}(H)$, which implies $H\in\tilde{C}\cap\bar{S}$ by Theorem 3.2.

Case 2: $\Phi^C=\{0\}$. In this case, we have $2(\gamma, H)=1$ for each $\gamma\in\Delta^C\cap\Theta$, where Δ^C denotes the complement $\Pi'-\Delta$ of Δ in Π'. In particular, we have

$$|H|=|H-A_\gamma|\quad\text{for each } \gamma\in\Delta^C\cap\Theta(\neq\phi)\subset\Theta.$$

Thus we get $H\in\tilde{C}\cap\bar{S}$ in the same way as Case 1. q.e.d.

Note that the dimension $k_{\Delta,\Phi}$ of $S^{\Delta,\Phi}$ for a c-pair (Δ, Φ) is given by

$$k_{\Delta,\Phi}=r-\dim\{(\Phi^C-\{0\})\cup\Delta^C\}_{\boldsymbol{R}},$$

where $\{*\}_{\boldsymbol{R}}$ means the subspace of $\mathfrak{a}$ spanned over $\boldsymbol{R}$ by $*$.

Now we shall proceed as in §1 to study the structure of the set $K\operatorname{Exp}S^{\Delta,\Phi}$. For an admissible pair (Δ, Φ), we define

$$N^{\Delta,\Phi}=\{k\in K;\ k\mathrm{Exp}S^{\Delta,\Phi}=\mathrm{Exp}S^{\Delta,\Phi}\},$$

$$Z^{\Delta,\Phi}=\{k\in N^{\Delta,\Phi};\ k|\mathrm{Exp}S^{\Delta,\Phi}=\mathrm{id}\},$$

$$W^{\Delta,\Phi}=N^{\Delta,\Phi}/Z^{\Delta,\Phi},$$

and

$$\bar{N}^{\Delta,\Phi}=\{\tau\in\bar{W}_S;\ \tau S^{\Delta,\Phi}=S^{\Delta,\Phi}\},$$

$$\bar{Z}^{\Delta,\Phi}=\{\tau\in\bar{N}^{\Delta,\Phi};\ \tau|S^{\Delta,\Phi}=\mathrm{id}\},$$

$$\bar{W}^{\Delta,\Phi}=\bar{N}^{\Delta,\Phi}/\bar{Z}^{\Delta,\Phi}.$$

Elements of $W^{\Delta,\Phi}$ and $\bar{W}^{\Delta,\Phi}$ will be denoted by $[k]$ with $k\in N^{\Delta,\Phi}$ and $[\tau]$ with $\tau\in\bar{N}^{\Delta,\Phi}$ respectively.

We define a C^∞ map $\Psi^{\Delta,\Phi}$: $K/Z^{\Delta,\Phi}\times S^{\Delta,\Phi}\longrightarrow M$ by

$$\Psi^{\Delta,\Phi}(kZ^{\Delta,\Phi},\ H)=k\mathrm{Exp}H \quad \text{for } k\in K,\ H\in S^{\Delta,\Phi},$$

and denote by $M^{\Delta,\Phi}$ the image of $\Psi^{\Delta,\Phi}$.

Lemma 1.7 implies $N^{\Delta,\Phi}\subset N^{\Delta}$, and hence

$$Z^{\Delta}\subset Z^{\Delta,\Phi}\subset N^{\Delta,\Phi}\subset N^{\Delta}\subset N_K(\mathfrak{m}^{\Delta}).$$

These groups are compact and have the same Lie algebra $\mathfrak{k}^{\Delta}$. In particular, the group $W^{\Delta,\Phi}$ is a finite group. Moreover, Corollary 2 of Lemma 1.6 implies that $\Psi^{\Delta,\Phi}$ is a C^∞ immersion and that

$$\dim K/Z^{\Delta,\Phi}=(1/2)(\dim\mathfrak{g}-\dim\mathfrak{g}^{\Delta}).$$

In the same way as the proof of Lemmas 1.7 and 1.8, we can show the following

LEMMA 3.5. 1) *Let* (Δ_1, Φ_1) *and* (Δ_2, Φ_2) *be admissible pairs,* $H_1\in S^{\Delta_1,\Phi_1}$, $H_2\in S^{\Delta_2,\Phi_2}$ *and* $k\in K$. *If* $k\,\mathrm{Exp}\,H_1=\mathrm{Exp}\,H_2$, *then there exists* $\tau\in\bar{W}_S$ *such that*:

i) $\tau S^{\Delta_1,\Phi_1}=S^{\Delta_2,\Phi_2}$;

ii) $k\,\mathrm{Exp}\,H=\mathrm{Exp}\,\tau H$ *for each* $H\in S^{\Delta_1,\Phi_1}$;

iii) $\tau H_1=H_2$,

and hence $k\,\mathrm{Exp}\,S^{\Delta_1,\Phi_1}=\mathrm{Exp}\,S^{\Delta_2,\Phi_2}$.

2) *For each admissible pair* (Δ, Φ), *there exists a unique homomorphism* $\pi^{\Delta,\Phi}$: $\bar{W}^{\Delta,\Phi}\longrightarrow W^{\Delta,\Phi}$ *such that if* $\pi^{\Delta,\Phi}[\tau]=[k]$ *with* $\tau\in\bar{N}^{\Delta,\Phi}$ *and* $k\in N^{\Delta,\Phi}$, *then*

$$k\,\mathrm{Exp}\,H=\mathrm{Exp}\,\tau H \quad \text{for each } H\in S^{\Delta,\Phi}.$$

LEMMA 3.6. *The homomorphism* $\pi^{\Delta,\Phi}$: $\bar{W}^{\Delta,\Phi}\longrightarrow W^{\Delta,\Phi}$ *is an isomorphism. Therefore* $\bar{W}^{\Delta,\Phi}$ *is also a finite group.*

PROOF. The surjectivity of $\pi^{\Delta,\Phi}$ follows from Lemma 3.5, 1). Assume $\pi^{\Delta,\Phi}[\tau]=1$ where $\tau\in\bar{W}^{\Delta,\Phi}$. Then, in the same way as in the proof of Lemma 1.8, we find $A\in\Gamma$ such that

$$\tau H=H+A \quad \text{for each } H\in S^{\Delta,\Phi}.$$

Since $S^{\Delta,\Phi}$ is bounded, we have $A=0$, and hence $[\tau]=1$. q.e.d.

From Lemma 3.3 we have the following

LEMMA 3.7. *For an admissible pair* (Δ, Φ), $\bar{N}^{\Delta,\Phi}$ *is given by*

$$\bar{N}^{\Delta,\Phi}=\{\tau\in\bar{W}_S;\ \tau\cdot\Delta=\Delta,\ \tau\Phi=\Phi\}.$$

We define a free C^∞ action of $\bar{W}^{\Delta,\Phi}$ on $K/Z^{\Delta,\Phi}\times S^{\Delta,\Phi}$ in the same way as for $K/Z^{\Delta}\times S^{\Delta}$. Let

$$E^{\Delta,\Phi}=K/Z^{\Delta,\Phi}\times_{\bar{W}^{\Phi,\Delta}}S^{\Delta,\Phi}$$

be the quotient manifold relative to this action. Put

$$B^{\Delta,\Phi}=K/N^{\Delta,\Phi}.$$

It is a compact connected C^∞ manifold. By Lemma 3.6, $K/Z^{\Delta,\Phi}$ is a C^∞ principal bundle over $B^{\Delta,\Phi}$ with the group $\bar{W}^{\Delta,\Phi}$, and $E^{\Delta,\Phi}$ is a fibre bundle over $B^{\Delta,\Phi}$ associated to $K/Z^{\Delta,\Phi}$ with the fibre $S^{\Delta,\Phi}$.

Let (Δ_1, Φ_1) and (Δ_2, Φ_2) be c-pairs. They are said to be *equivalent* if there exists $\tau\in\bar{W}_S$ such that $\tau\cdot\Delta_1=\Delta_2$ and $\tau\Phi_1=\Phi_2$. We denote by $(\Delta_1, \Phi_1)\succ(\Delta_2, \Phi_2)$ if there exists $\tau\in\bar{W}_S$ such that $\tau\cdot\Delta_1\supset\Delta_2$ and $\tau\Phi_1\supset\Phi_2$. Let $\mathscr{C}$ be a set of complete representatives of equivalance classes of c-pairs. Note that $\mathscr{C}$ is a finite set. Then in the same way as the proof of Theorems 1.1 and 2.2, we get the following theorem.

THEOREM 3.3. 1) *For each c-pair* (Δ, Φ), $M^{\Delta,\Phi}$ *is a connected regular submanifold of M with*

$$\dim M^{\Delta,\Phi}=(1/2)(\dim \mathfrak{g}-\dim \mathfrak{g}^{\Delta})+k_{\Delta,\Phi}.$$

Each $M^{\Delta,\Phi}$ *is diffeomorphic with* $E^{\Delta,\Phi}$ *by the diffeomorphism* $\psi^{\Delta,\Phi}$: $E^{\Delta,\Phi}\longrightarrow M^{\Delta,\Phi}$ *induced by the* C^∞ *map* $\Psi^{\Delta,\Phi}$: $K/Z^{\Delta,\Phi}\times S^{\Delta,\Phi}\longrightarrow M$.

2) *The cut locus C of a compact connected symmetric space* (M, g) *with respect to the origin o has a stratification*:

$$C=\bigcup_{(\Delta,\Phi)\in\mathscr{C}} M^{\Delta,\Phi} \quad \text{(disjoint union)},$$

where $\bar{M}^{\Delta_1,\Phi_1}\supset M^{\Delta_2,\Phi_2}$ *if and only if* $(\Delta_1, \Phi_1)\succ(\Delta_2, \Phi_2)$.

REMARK. Let dim $M=n$. Then M is homeomorphic with the space obtained from the cut locus C by attaching an n-cell M^0. In fact (cf. Kobayashi [4]), let

$$E^0=\{tX;\ 0\leqslant t<1,\ X\in\tilde{C}\},$$
$$\bar{S}^0=E^0\cap\bar{S}.$$

Then $E^0=\mathrm{Ad}K\bar{S}^0$ (cf. proof of Theorem 3.2, 1)), and the closure $\bar{E}^0$ of E^0 is given by $\bar{E}^0=E^0\cup\tilde{C}$. The subset

$$M^0=\mathrm{Exp}\,E^0$$

of M is called the *interior* of (M, g) with respect to the point o. Subsets $\bar{E}^0$, E^0 and $\tilde{C}$ of $\mathfrak{m}$ are homeomorphic with the closed n-disk $\bar{D}^n$, n-cell D^n and $(n-1)$-sphere S^{n-1} respectively. Thus the cut locus C is a closed subset of M. Moreover, Exp: $\bar{E}^0\longrightarrow M$ is surjective and the continuous map Exp: $(\bar{E}^0, \tilde{C})\longrightarrow(M, C)$ of pairs induces a relative diffeomorphism Exp: $E^0\longrightarrow M^0$.

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