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# FUNDAMENTAL SOLUTION OF CAUCHY PROBLEM FOR HYPERBOLIC SYSTEMS AND GEVREY CLASS

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## §1. Introduction

We consider a first order partial differential operator $L_{t,x}=\frac{\partial}{\partial t}+\sum_{j=1}^{n}A_j(t,x)\frac{\partial}{\partial x_j}+B(t,x)$ in $\Omega=[0,T]\times R^n$, whose coefficients are $m\times m$-matrices. We call a fundamental solution corresponding to the operator $L_{t,x}$, a distribution satisfying the following, $\tau\in[0,T)$, fixed,

$$(1.1)\qquad \begin{cases} L_{t,x}K(t,x.\tau,y)=0, & t\in(0,T) \\ K(\tau,x,\tau,y)=\delta(x-y)I, & \end{cases}$$

here $\delta(x)$ denotes the $n$-dimentional Dirac distribution and $I$ the indentity matrix. We require that the multiplicity of each characteristic remains constant in a region $\Omega=[0,T]\times R^n$ and that the characteristic matrix $A(t,x,\xi)=\Sigma A_j(t,x)\xi_j$ is diagonalizable for $(t,x)$ in $\Omega$ and $\xi$ in $R^n\setminus 0$. Moreover we suppose that the coefficients $A_j(t,x)$ and $B(t,x)$ are in Gevrey class $\gamma_s(\Omega)(s\geq 1)$.

Our aim is to construct globally in $\Omega$ a fundamental solution for the operator $L_{t,x}$ of this type. When $T$ is small, Lax [12] treated this problem. In the case of analytic coefficients, Leray [13] and Mizohata [19] analyzed locally a fundamental solution of hyperbolic systems. When $T$ is large, Ludwig [15] extended the interval of existence for a fundamental solution by use of it's semi-group property. We shall give a more precise expression of a fundamental solution than these of Ludwig. It should be remarked that Duistermaat [3] has recently constructed globally a fundamental solution of the Cauchy problem, applying the theory of Fourier integral operators of Hörmander and Duistermaat [4], [9].

In the first step we shall construct asymptotically a fundamental solution and in the second step we shall obtain successive estimates of it's expansion by use of the method of Mizohata [18], [19] and Hamada [7], [8]. We shall determine the wave front set in Gevrey class of a fundamental solution following the definition of Hörmander [10].

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The work presented here leans heavily Mizohata's results in [18], and I thank him sincerely.

I announce that we shall construct in the ultra distribution a fundamental solution for non diagonalizable hyperbolic systems in the forthcoming paper.

## 2. Results

We consider a operator $L_{t,x}=\partial/\partial t+\Sigma A_j(t,x)\partial/\partial x_j+B(t,x)$ under the following assumptions;

(A.I) each eigen value of $A(t,x,\xi)=\Sigma A_j(t,x)\xi_j$ is real for $(t,x,\xi)\in\Omega\times R^n\setminus 0$ and it's multiplicity is constant, that is, $\det(\lambda+A(t,x,\xi))=\prod_{p=1}^{l}(\lambda+\lambda^{(p)}(t,x,\xi))^{\nu_p}, (\Sigma\nu_p=m)$, here $\nu_p$ $(p=1\cdots l)$ is constant.

(A, II) there exists a positive constant $c_0$ such that

$$\sup_{\substack{(t,x)\\|\xi|=1\\p\neq q}}|\lambda^{(p)}(t,x,\xi)-\lambda^{(q)}(t,x,\xi)|\geq c_0$$

(A.III) the characteristic matrix $A(t,x,\xi)$ is diagonalizable.

A function $f\in C^\infty(\Omega)$ is said to be of Gevrey class $\gamma_s(\Omega)$ $(s\geq 1)$, if there exist constants $C,A$ such that for any $(t,x)\in\Omega$ and for any multi-index $\alpha=(\alpha_0,\alpha_1,\cdots,\alpha_n)$, the following inequality be true;

$$|D^\alpha f(t,x)|\leqslant CA^{|\alpha|}|\alpha|!^s,$$

here we have set $D^\alpha=(\partial/\partial t)^{\alpha_0}(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$, $|\alpha|=\sum_{i=0}^{n}\alpha_i$.

We suppose that the coefficients $A_j(t,x), B(t,x)$ of $L_{t,x}$ are in Gevrey class $\gamma_s(\Omega)$. Then all eigen values $\lambda^{(p)}(t,x,\xi)$ are in $\gamma_s(\Omega\times R^n\setminus 0)$.

We denote by $l^{(p)}(t,x,\tau,y,\xi)$ the phase function associated to $\lambda^{(p)}(t,x,\xi)$, that is, a solution satisfying the following non-linear equation;

$$\begin{cases} l_t{}^{(p)}+\lambda^{(p)}(t,x,l_x{}^{(p)})=0 \\ l^{(p)}|_{t=\tau}=\langle x-y,\xi\rangle\,, \end{cases} \tag{2.1}$$

here $\langle x,\xi\rangle=\sum_{i=1}^{n}x_i\xi_i$. To solve this equation, we consider the Hamiltonian system,

$$\begin{cases} \dfrac{d}{dt}\hat{x}^{(p)}(t)=\lambda^{(p)}(t,\hat{x}^{(p)},\hat{\xi}^{(p)}), & \dfrac{d}{dt}\hat{\xi}^{(p)}(t)=-\lambda_x{}^{(p)}(t,\hat{x}^{(p)},\hat{\xi}^{(p)}) \\ \hat{x}^{(p)}(\tau)=z\,, & \hat{\xi}^{(p)}(\tau)=\xi, (\xi\neq 0)\,. \end{cases}$$

We write $(\hat{x}^{(p)}(t),\hat{\xi}^{(p)}(t))=(\hat{x}^{(p)}(t,z,\tau,\xi),\hat{\xi}^{(p)}(t,z,\tau,\xi))$. We can solve globally this system, for $\lambda^{(p)}(t,x,\xi)$ is a homogeneous function in $\xi$. We note that $(\hat{x}^{(p)}(t),\hat{\xi}^{(p)}(t))$ is in Gevrey class $\gamma_s(\Omega\times R^n\setminus 0)$ with respect to $(t,z,\xi)$. We put $\Delta^{(p)}(t)=D(\hat{x}^{(p)}(t))/D(z)$.

Then there exists a positive constant $\delta>0$ such that $\Delta^{(p)}(t)\neq 0$ for $|t-\tau|\leqslant\delta$, because of $\Delta^{(p)}(\tau)=1$. Hence we can solve the equation $\hat{x}^{(p)}(t, z, \tau, \xi)=x$ with respect to $z$ for $|t-\tau|\leqslant\delta$. We denote this solution by $\tilde{z}^{(p)}(t, x, \tau, \xi)$. Then we can express the solution of (2.1) as follows,

$$l^{(p)}(t, x, \tau, y, \xi)=\langle\tilde{z}^{(p)}(t, x, \tau, \xi)-y, \xi\rangle . \tag{2.2}$$

We note that $\tilde{z}^{(p)}(t, x, \tau, \xi)$ and therefore $l^{(p)}(t, x, \tau, y, \xi)$ are in $\gamma_s([\tau-\delta, \tau+\delta]\times R^n\times R^n\setminus 0)$ with respect to $(t, x, \xi)$. We denote

$$\Lambda^{(p)}(t, \tau ; y)=\bigcup_{\xi\in R^n\setminus 0}\{(\hat{x}^{(p)}(t, y, \tau, \xi), \hat{\xi}^{(p)}(t, y, \tau, \xi))\}$$

Now we analyze the fundamental solution of $L_{t,x}$. As well known (c.f. [12], [15] and [19]), if $\delta$ is small, for $|t-\tau|\leqslant\delta$ we can express the fundamental solution $K(t, x, \tau, y)$ as follows,

$$K(t, x, \tau, y)=\sum_{p=1}^{l} K^{(p)}(t, x, \tau, y)+K^{(0)}(t, x, \tau, y) ,$$

here

$$K^{(p)}(t, x, \tau, y) =\int\{exp\ il^{(p)}(t, x, \tau, y, \xi)\}w^{(p)}(t, x, \tau, \xi)d\xi, p=1, \cdots, l .$$

Then we obtain

THEOREM 2.1. *Let $(\tau, y)$ be fixed. For $|t-\tau|\leqslant\delta$, we can compute the wave front sets of $K^{(p)}(t, x, \tau, y)$ in Gevrey class as follows,* ($s\geq 1$),

$$WF_s(K^{(p)}(t, \cdot, \tau, y))=\Lambda^{(p)}(t, \tau; y) ,$$

$$WF_{2s-1}(K^{(0)}(t, \cdot, \tau, y))=\phi .$$

*Here the definition of the wave front sets in Gevrey class followed from Hörmander* [10].

REMARK. *In the case of analytic cofficients* ($i, e, s=1$), *the propagation of the analytic wave front sets is studied in* [10] *and* [21]. *When* $s>1$, *Friedman* [23] *showed that the fundamental solution is in $\gamma_{3s-1}$ except the characteristic conoids.*

We decompose the interval $(0, T)$ such that $0=t_0<t_1<\cdots<t_{d+1}=T, t_j-t_{j-1}=\delta$. Then it follows from the semi-group property of a fundamental solution that we can write for $|t-t_j|\leqslant\delta$,

$$\begin{aligned}K(t, x, t_0, y)&=K(t, x, t_j, \cdot)K(t_j, \cdot, t_{j-1}, \cdot)\cdots K(t_1, \cdot, t_0, y)\\&=\sum_{p=1}^{l}K_j^{(p)}(t, x, t_0, y)+K_j^{(0)}(t, x, t_0, y) ,\end{aligned}$$

where we put

$$K_j^{(p)}(t, x, t_0, y) = K^{(p)}(t, x, t_j, \cdot)K^{(p)}(t_j, \cdot, t_{j-1}, \cdot)\cdots K^{(p)}(t_1, \cdot, t_0, y)$$

for $j=1, \cdots, d$, $|t-t_j| \leqslant \delta$ and $p=1, \cdots, l$.

THEOREM 2.2. *For $|t-t_j| \leqslant \delta$, we have*

$$WF_s(K_j^{(p)}(t, \cdot, t_0, y)) = \Lambda^{(p)}(t, t_0; y), p=1, \cdots, l,$$

*and*

$$WF_{2s-1}(K_j^{(0)}(t, \cdot, t_0, y)) = \phi,$$

*for $j=1, 2, \cdots, d$.*

REMARK. *For example, when $j=1$, Theorem 2.2 implies that the singularity of the summation $\sum_{p \neq q} K^{(p)}(t, x, t_1, \cdot)K^{(q)}(t_1, \cdot, t_0, y)$ disappears in the Gevrey class $\gamma_{2s-1}$.*

## § 3. Preliminaries

Let $\lambda(t, x, \xi)$ be a function in $\gamma_s(\Omega \times R^n \setminus 0)$ and homogeneous degree one in $\xi$. We consider the following equation;

$$\text{(3.1)} \qquad \begin{aligned} &l_t + \lambda(t, x, l_x) = 0, \\ &l|_{t=\tau} = \langle x-y, \xi \rangle, \xi \neq 0. \end{aligned}$$

To solve this nonlinear equation, we consider

$$\begin{cases} \dfrac{d\hat{x}(t)}{dt} = \lambda_\xi(t, \hat{x}, \hat{\xi}), & \dfrac{d\hat{\xi}(t)}{dt} = -\lambda_x(t, \hat{x}, \hat{\xi}) \\ \hat{x}(\tau) = z, & \hat{\xi}(t) = \xi. \end{cases}$$

We write the solution $(\hat{x}(t), \hat{\xi}(t)) = (\hat{x}(t, z, \tau, \xi), \hat{\xi}(t, z, \tau, \xi))$.

Then we have,

LEMMA 3.1. *Let $\tau$ be fixed in $[0, T]$. For $z \in R^n$ and $\xi \in R^n \setminus 0$, (3.2) has a unique solution $(\hat{x}(t), \hat{\xi}(t))$ which is in $\gamma_s(\Omega \times R^n \setminus 0)$ with respect to $(t, z, \xi)$.*

Since the Jacobian $D(\hat{x})/D(z) = 1$ at $t = \tau$, there exists a positive number $\delta$ such that $D(x)/D(z) \neq 0$ for $|t-\tau| \leqslant \delta$. Hence we can solve an equation $\hat{x}(t, z, \tau, \xi) = x$ with respect to $z$ by an implicit function theorem. We denote this by $\tilde{z}(t, x, \tau, \xi)$. Then we obtain,

LEMMA 3.2. [2]. *For $|t-\tau| \leqslant \delta$, we can express a solution of* (3.1),

$$\text{(3.3)} \qquad l(t, x, \tau, y, \xi) = \langle \tilde{z}(t, x, \tau, \xi) - y, \xi \rangle,$$

(3.4) $$l_x=\hat{\xi}(t, \hat{z}(t, x, \tau, \xi), \tau, \xi)\ .$$

We denote the Jacobian $D(\hat{x}(t))/D(z)$ by $\Delta(t)$. Then we have as well known, (c.f. [5]),

LEMMA 3.3. *For $|t-\tau|\leqslant\delta$, we have*

(3.5) $$\frac{d}{dt}\Delta(t)=\Delta(t)\left\{\sum_{i,j}\lambda_{\xi_i\xi_i}(t, x, l_x)l_{x_ix_j}+\sum_i\lambda_{\xi_ix_i}(t, x, l_x)\right\}_{x=\hat{x}(t)}$$
$$=\Delta(t)\left\{\sum_i\frac{\partial}{\partial x_i}(\lambda_{\xi_i}(t, x, l_x))\right\}_{x=\hat{x}(t)}$$

*here $l$ is a solution of* (3.1)

Let $A(t, x, \xi)=\Sigma A_j(t, x)\xi_j$ be a matrix and $\lambda(t, x, \xi)$ be an eigenvalue of $A(t, x, \xi)$. We denote the right eigenvectors and the left eigenvectors by $h_1, \cdots, h_\nu$ and $g_1, \cdots, g_\nu$ respectively. We write $H=(h_1, \cdots, h_\nu)$ and $G={}^t(g_1, \cdots, g_\nu)$. Then simple calculations imply

LEMMA 3.4. *For $j=1, \cdots, n$, we have*

(1) $GA_{\xi_j}H=\lambda_{\xi_j}GH,\ GA_{x_j}H=\lambda_{x_j}GH$

(2) $\sum_{i,j}GA_{\xi_j}H_{\xi_j}z_{ij}=\sum_{i,j}\lambda_{\xi_i}GH_{\xi_i}z_{ij}+\frac{1}{2}\Sigma\lambda_{\xi_i\xi_j}z_{ij}GH$ for $z_{ij}=z_{ji}$

(3) $G_{\xi_j}A_{x_j}H-G_{x_j}A_{\xi_j}H=GA_{\xi_j}H_{x_i}-GA_{x_j}H_{\xi_j}$

(4) $GA_{x_j}H_{\xi_j}-G_{x_j}A_{\xi_i}H=GH_{\xi_j}\lambda_{x_j}+GH_{x_j}\lambda_{\xi_j}\ .$

## § 4. Asymptotic construction of fundamental solution

We shall construct asymptotically a fundamental solution $K(t, x, \tau, y)$. We note that the distribution $\delta(x-y)$ is represented by

$$\delta(x-y)=\frac{1}{(2\pi)^n}\int exp\ i\langle x-y, \xi\rangle d\xi\ .$$

Let $w(t, x, \tau, y, \xi)$ be a function satisfying following equation,

(4.1) $$\begin{cases}L_{t,x}w(t, x, \tau, y, \xi)=0\\ w(t, x, \tau, y, \xi)=\dfrac{1}{(2\pi)^n}\{exp\ i\langle x-y, \xi\rangle\}I\ .\end{cases}$$

Then we have a fundamental solution $K(t, x, \tau, y)$ as follows,

$$K(t, x, \tau, y)=\int_{R^n}w(t, x, \tau, y, \xi)d\xi\ .$$

We can construct asymptotically $w(t, x, \tau, y, \xi)$ with respect to $\xi$, provided that the system $L_{t,x}$ satisfies the algebraic conditions (A.I), (A.II) and (A.III) in § 2.

We seek $w$ as the following form;

$$w(t, x, \tau, y, \xi)=\sum_{j=0}^{\infty}\sum_{k=1}^{l}\{exp\ il^{(k)}(t, x, \tau, y, \omega)\rho\}\rho^{-j}w_j^{(k)}(t, x, \tau, \omega)\,, \tag{4.2}$$

here

$$l^{(k)}(t, x, \tau, y, \omega)=\langle\tilde{z}^{(k)}(t, x, \tau, \omega)-y, \omega\rangle,\ \omega=\xi/|\xi| \text{ and } \rho=|\xi|\,.$$

Applying $L_{t,x}$ to $w$, we obtain

$$L_{t,x}[w]=\sum_{j=0}^{\infty}\rho^{-j}\sum_{k=1}^{l}(exp\ il^{(k)}\rho)\{i(l_t^{(k)}+A(t, x, l_x^{(k)})w_j^{(k)}+L_{t,x}w_{j-1}^{(k)}\}$$
$$=0\,.$$

Hence we have

$$(\lambda^{(k)}(t, x, l_x^{(k)})-A(t, x, l_x^{(k)}))w_j^{(k)}+iL_{t,x}(w_{j-1}^{(k)})=0 \quad j=0, 1, 2, \cdots, (w_{-1}^{(k)}\equiv 0)\,. \tag{4.3}_j$$

We put

$$H^{(k)}(t, x, \xi)=(h_1^{(k)}(t, x, \xi), \cdots, h_{\nu_k}^{(k)}(t, x, \xi))\,,$$
$$G^{(k)}(t, x, \xi)={}^t(g_1^{(k)}, \cdots, g_{\nu_k}^{(k)})\,,$$

here $h_j^{(k)}(t, x, \xi)$ (resp. $g_j^{(k)}$) is a right (resp. left) eigenvector of $A(t, x, \xi)$ corresponding to $\lambda^{(k)}(t, x, \xi)$.

For $j=0$, we obtain

$$w_0^{(k)}(t, x, w)=H^{(k)}(t, x, l_x^{(k)})\sigma_0^{(k,k)}(t, x, \omega)\,, \tag{4.4}$$

where $\sigma_0^{(k,k)}(t, x, \omega)$ is a $\nu_k\times m$ matrix which is determined later on. In general, to solve $(4.3)_j$ $(j\geq 1)$, it is necessary that

$$G^{(k)}(t, x, l_x^{(k)})L_{t,x}(w_{j-1}^{(k)})=0\,. \tag{4.5}_{j-1}$$

Then we obtain as a solution of $(4.3)_j$

$$w_j^{(k)}(t, x, \omega)=\sum_{p=1}^{l}H^{(p)}(t, x, l_x^{(k)})\sigma_j^{(p,k)}(t, x, \omega) \tag{4.6}$$

where $\sigma_j^{(p,k)}(t, x, \omega)$ is a $\nu_p\times m$ matrix, and for $p\neq k$,

$$\sigma_j^{(p,k)}=\{i(\lambda^{(k)}-\lambda^{(p)})^{-1}G^{(p)}|_{\xi=l_x^{(k)}}\}L_{t,x}(w_{j-1}^{(k)}) \tag{4.7}$$

We can rewrite $(4.5)_j$ as an equation of $\sigma_j^{(k,k)}$, that is,

$$\left\{\frac{\partial}{\partial t}+\sum_j\lambda_{\xi_j}^{(k)}\frac{\partial}{\partial x_j}+\frac{1}{2}(\Sigma\lambda_{\xi_i\xi_j}^{(k)}l_{x_ix_j}^{(k)}+\Sigma\lambda_{x_i\xi_j}^{(k)})+j^{(k)}\right\}_{\xi=l_x^{(k)}}\sigma_j^{(k,k)} \tag{4.8}$$
$$-iG^{(k)}(t, x, l_x^{(k)})L_{t,x}(\tilde{w}_j^{(k)})=0\,,$$

here we used Lemma 3.4 and $G^{(k)}H^{(k)}=I_{\nu_k}$, ($\nu_k\times\nu_k$-identity matrix),

$$j^{(k)}(t,x,\xi)=G^{(k)}L_{t,x}H^{(k)}-\sum_{j=1}^{n}\left(G^{(k)}H_{\xi_j}^{(k)}\lambda_{x_j}^{(k)}-\frac{1}{2}\lambda_{x_j\xi_j}^{(k)}G^{(k)}H^{(k)}\right), \tag{4.9}$$

$$\tilde{w}_j^{(k)}=\sum_{p\neq k}H^{(p)}(t,x,l_x^{(k)})\sigma_j^{(p,k)}(t,x,\omega). \tag{4.10}$$

We note that $j^{(k)}$ is invariant under the transformation of variables. For we can rewrite, by virtue of Lemma 3.4, (for simplicity, abbreviating an index $k$),

$$j=\frac{1}{2}(G_{\xi_j}A_{x_j}H-G_{x_j}A_{\xi_j}H+GH_{x_j}\lambda_{\xi_j}-GH_{\xi_j}\lambda_{x_j})$$

$$+2GH_t-\Sigma\frac{1}{2}GA_{jx_j}H+GBH,$$

here we put $L=\xi_0I+A$, $f=\xi_0+\lambda^{(k)}$ and $t=x_0$. Then we have

$$j=\frac{1}{2}\sum_{j=0}^{n}\{G_{\xi_j}L_{x_j}-G_{x_j}L_{\xi_j})H+G(H_{x_j}f_{\xi_j}-H_{\xi_j}f_{x_j})\}$$

$$+GBH-\frac{1}{2}\sum_{j=0}^{n}GL_{x_j\xi_j}H.$$

which is evidently invariant under the transformation of variables.

Now we return to the equation (4.8). We transform the variables $x$ into $\hat{x}^{(k)}(t,z,\tau,\omega)$. Then by use of Lemma 3.3, we can rewrite (4.8) as following,

$$\left(\frac{\partial}{\partial t}+\frac{1}{2}\Delta^{(k)}(t)+j^{(k)}(t)\right)\sigma_j^{(k,k)}(t,\hat{x}^{(k)}(t),\omega)-i\{G^{(k)}L_{t,x}(\tilde{w}_j^{(k)})\}_{x=\hat{x}^{(k)}(t)}=0 \tag{4.11}$$

We denote by $J^{(k)}(t)=J^{(k)}(t,\tau)$ a solution of the following equation

$$\frac{d}{dt}J^{(k)}(t)=-j^{(k)}(t)J^{(k)},\ J^{(k)}(\tau)=I_{\nu_k}.$$

We put

$$\sigma_j^{(k)}(t)=\sigma_j^{(k)}(t,z,\omega)=\Delta^{(k)}(t)^{1/2}J^{(k)}(t)\sigma_j^{(k,k)}(t,\hat{x}^{(k)}(t),\omega).$$

Then we obtain from (4.11)

$$\frac{d}{dt}\sigma_j^{(k)}(t)=i(G^{(k)}L_{t,x}\tilde{w}_j^{(k)})_{x=\hat{x}^{(k)}(t)}=M^{(k)}\{(\tilde{w}_j^{(k)})_{x=\hat{x}^{(k)}(t)}\} \tag{4.12}$$

here $M^{(k)}$ is a first order differential operator in $(t,z)$ and $\tilde{w}_j^{(k)}$ is given by (4.10) and (4.7). As an initial condition of (4.12), we obtain from (4.1)

$$\sum_{k=1}^{l}H^{(k)}\sigma_0^{(k)}=\frac{1}{(2\pi)^n}I$$

and

$$\sum_{k=1}^{l}(\tilde{w}_j^{(k)}+H^{(k)}\sigma_j^{(k)})=0, \quad (j\geq 1)$$

for $t=\tau$, that is

$$\sigma_0^{(k)}(\tau)=G^{(k)}(\tau, z, \omega)$$

and

$$\sigma_j^{(k)}(\tau)=-G^{(k)}(\tau, z, \omega)\sum_{p=1}^{l}\tilde{w}_j^{(p)}(\tau, z, \omega), \quad (j\geq 1).$$

Summarizing, we have obtained,

$$\sigma_0^{(k)}(t, z, \omega)=\frac{1}{(2\pi)^n}G^{(k)}(\tau, z, \omega) \tag{4.13$_0$}$$

and for $j\geq 1$ and $k=1, \cdots, l$,

$$\begin{cases} \dfrac{d}{dt}\sigma_j^{(k)}(t)=M^{(k)}\tilde{w}_j^{(k)} \\ \tilde{w}_j^{(k)}=N_1^{(k)}\sigma_{j-1}^{(k)}+N_2^{(k)}\tilde{w}_{j-1}^{(k)} \\ \sigma_j^{(k)}(\tau)=G^{(k)}(\tau, z, \omega)\sum_{p=1}^{l}\tilde{w}_j^{(p)}|_{t=\tau} \end{cases} \tag{4.13$_j$}$$

here $M^{(k)}, N_1^{(k)}$ and $N_2^{(k)}$ are first order differential operators in $(t, z)$.

Then we have the following theorem which will be proved in the next section,

THEOREM 4.1. *Let $\tau$ be fixed in $[0, T)$. For $|t-\tau|\leqslant\delta$ and for $x\in R^n$, we have*

$$|D_{t,z}^{\alpha}D_{\omega}^{\beta}\sigma_j^{(k)}|_{|\omega|=1}\leqslant C_1A_1^{|\alpha|+|\beta|+j}(|\alpha|+|\beta|)!^s j!^{2s-1}$$

*and*

$$|D_{t,z}^{\alpha}D_{\omega}^{\beta}\tilde{w}_j^{(k)}|_{|\omega|=1}\leqslant C_1A_1^{|\alpha|+|\beta|+j}(|\alpha|+|\beta|)!^s j!^{2s-1}$$

*here $C_1$ and $A_1$ are positive constants independent of $\alpha, \beta$ and $j$.*

Therfore we obtain

THEOREM 4.2. *$w_j^{(k)}(t, x, \tau, \omega)$ the terms of the expantion* (4.2) *are homogeneous functions of degree zero with respect to $\omega$ and are estimated by,*

$$|D_{t,x}^{\alpha}D_{\omega}^{\beta}w_j^{(k)}|_{|\omega|=1}\leqslant C_2A_2^{|\alpha|+|\beta|+j}(|\beta|+|\beta|)!^s j!^{2s-1},\ j=0, 1, 2, \cdots,$$

*for $k=1, \cdots, l$, and for $(t, x)\in[\tau-\delta, \tau+\delta]\times R^n$.*

## § 5. Successive estimate in Gevrey class

We start with a lemma which will be often used in our reasonning (c.f. [6], [18]).

LEMMA 5.1. *Let $p_1$ and $p_2$ be non negative integers and $\alpha=(\alpha_1, \cdots, \alpha_m)$ a multi integer. For any $k>1$ and $s\geq 1$, we have*

$$\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}k^{-|\alpha|}(|\alpha'|+p_1)!^s(|\alpha''|+p_2)!^s \tag{5.1}$$

$$\leqslant \frac{k}{k-1}(|\alpha|+p_1+p_2)!^s\binom{p_1+p_2}{p_1}^{-1}$$

PROOF. Noting that $\prod_{i=1}^{m}(t+1)^{\alpha_i}=(t+1)^{|\alpha|}$,
we have

$$\sum_{|\alpha'|=j}\binom{\alpha}{\alpha'}=\binom{|\alpha|}{j}, \quad j=0, 1, \cdots, |\alpha|.$$

In paticular for $m=2$,

$$\binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\leqslant\binom{\alpha_1+\alpha_2}{\beta_1+\beta_2}.$$

Hence

$$\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}k^{-|\alpha'|}(|\alpha'|+p_1)!^s(|\alpha'|+p_2)!^s$$

$$\leqslant\sum_{j=0}^{|\alpha|}\sum_{|\alpha'|=j}\binom{\alpha}{\alpha'}k^{-j}(j+p_1)!^s(|\alpha|-j+p_2)!^s$$

$$\leqslant\sum_{j=0}^{|\alpha|}k^{-j}\binom{|\alpha|}{j}\binom{|\alpha|+p_1+p_2}{j+p_1}^{-1}(|\alpha|+p_1+p_2)!^s$$

$$\leqslant\sum_{j=0}^{\infty}k^{-j}(|\alpha|+p_1+p_2)!^s\binom{p_1+p_2}{p_1}^{-1}$$

which implies (5.1).

Let $G$ be an open set in $R^m$ and $\bar{G}$ a closure of $G$.

LEMMA 5.2. *Let $P(x, D)=\sum_{|\beta|\leq d}a_\beta(x)D^\beta$ be a differential operator, $p_1, p_2$ non negative integers and $k$ a positive number$>1$. Assume*

$$|D^\alpha a_\beta(x)|\leqslant C_0(k^{-1}A)^{|\alpha|}(|\alpha|+p_1)!^s, \ |\beta|\leqslant d,$$

$$|D^\alpha u(x)|\leqslant CA^{|\alpha|}(|\alpha|+p_2)!^s$$

*for any multi integer* $\alpha$ *and for* $x\in\bar{G}$. *Then*

$$(5.2) \qquad |D^{\alpha}P(x, D)u(x)|\leqslant C_0C\,\bar{m}_dA^{d+|\alpha|}(|\alpha|+p_1+p_2+d)!^s$$

*for* $x\in\bar{G}$, *where* $\bar{m}_d=(m^{d+1}-1)(m-1)^{-1}(k-1)^{-1}k$.

PROOF. Leibniz formula implies

$$|D^{\alpha}Pu|\leqslant\sum_{|\beta|\leqslant d}\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}|D^{\alpha'}a||D^{\alpha''+\beta}u|$$

$$\leqslant C_0CA^{|\alpha|+d}\sum_{|\beta|\leqslant d}\sum_{\alpha'}\binom{\alpha}{\alpha'}k^{-|\alpha'|}(|\alpha'|+p_1)!^s(|\alpha''|+p_2+d)!^s$$

which implies (5.2) with (5.1), where we used that

$$\sum_{|\beta|\leqslant d}1\leqslant\sum_{j=0}^{d}m^j=(m^{d+1}-1)(m-1)^{-1}.$$

LEMMA 5.3. *Let* $X_j(x, D)=\sum_{i=1}^{m}a_{ji}(x)\frac{\partial}{\partial x_i}+a_{j0}(x)$, $(j=1, \cdots, N)$ *be first order differential operators. Assume*

$$|D^{\alpha}a_{ji}(x)|\leqslant C_0(k^{-1}A)^{|\alpha|}|\alpha|!^s, j=1, \cdots, N, i=0, \cdots, m,$$

$$|D^{\alpha}u(x)|\leqslant CA^{|\alpha|}(|\alpha|+p)!^s,$$

*for* $x\in G$. *Then*

$$(5,3) \qquad |D^{\alpha}X_{j_1}X_{j_2}\cdots X_{j_l}u|\leqslant C(C_0\bar{m}_1)^lA^{|\alpha|+l}(|\alpha|+l+p)!^s,$$

*for* $x\in G$ *and for* $(j_1, \cdots, j_l)\subset(1, \cdots, N)$, *where* $m_1=(m+1)(k-1)^{-1}k, k>1$.

PROOF. We shall prove our statement by induction with respect to $l$. For $l=1$ it follows from lemma 5.2. In general

$$|D^{\alpha}X_{j_1}(X_{j_2}\cdots X_{j_l}u)|\leqslant\sum_{i=1}^{m}\sum_{\alpha'}\binom{\alpha}{\alpha'}|D^{\alpha'}a_{j_1i}|D^{\alpha''}\frac{\partial}{\partial x_i}(X_{j_2}\cdots X_{j_l}u)|$$

$$+\sum_{\alpha'}\binom{\alpha}{\alpha'}|D^{\alpha'}a_{j_10}||D^{\alpha''}X_{j_2}\cdots X_{j_l}u|$$

$$\leqslant C_0A^{|\alpha|+l}C(C_0m_1)^{l-1}(m+1)\sum_{\alpha'}\binom{\alpha}{\alpha'}k^{-|\alpha'|}(|\alpha'|)!^s(|\alpha'|+l+p)!^s$$

which implies (5.3) with (5.1).

LEMMA 5.4. *Let* $G_1$ *and* $G_2$ *be an open set in* $R^{m_1}$ *and in* $R^{m_2}$ *respectively and* $\varphi$ *be a mapping from* $G_2$ *to* $G_1$ *satisfying*

$$|D_y{}^\alpha\varphi(y)|\leqslant C_0A_0^{|\alpha|}|\alpha|!^s$$

*for* $y\in\bar{G}_2$. *Then for any* $u(x)$ *satisfying for* $x\in\bar{G}_1$,

$$|D_x{}^\alpha u(x)|\leqslant CA^{|\alpha|}(|\alpha|+p)!^s\,,$$

*if* $A>A_0$, *we have*

$$|D_y{}^\alpha(u\circ\varphi)(y)|\leqslant C(2^sC_0\bar{m}_1A_0)^{|\alpha|}A^{|\alpha|}(|\alpha|+p)!^s \tag{5.4}$$

*for* $y\in\bar{G}_2$, *here* $\bar{m}_1=(m_1+1)(k-1)^{-1}k,\ k=A/A_0>1$.

PROOF. Denote $\varphi(y)=(\varphi_1(y),\cdots,\varphi_{m_1}(y))$. Then we have

$$\begin{aligned}D_{y_j}(u\circ\varphi)(y)&=\sum_{l=1}^{m_1}\frac{\partial\varphi_l}{\partial y_j}\left(\frac{\partial}{\partial x_l}u\right)(\varphi(y))\\&=\left(\sum_{l=1}^{m_1}\frac{\partial\varphi_l(y)}{\partial y_j}\frac{\partial}{\partial x_l}+\frac{\partial}{\partial y_j}\right)u(x)\Big|_{x=\varphi(y)}.\end{aligned}$$

We put

$$X_j=\sum_{k=1}^{m_1}a_{jk}(y)\frac{\partial}{\partial x_k}+\frac{\partial}{\partial y_j},\quad a_{jk}=\frac{\partial}{\partial y_j}\varphi_k(y)\,.$$

Nothing that

$$|D^\alpha a_{jk}(y)|\leqslant(2^sC_0A_0)(k^{-1}A)^{|\alpha|}|\alpha|!^s,\ k=A/A_0>1\,,$$

$$D_y{}^\alpha(u\circ\varphi)(y)=(X_1{}^{\alpha_1}X_2{}^{\alpha_2}\cdots X_{m_2}^{\alpha m_2}u(x))|_{x=\varphi(y)}\,,$$

we obtain (5.4) by virtue of Lemma 5.3.

COROLLARY 5.5. *Let* $\varphi$ *be given by Lemma* 5.4. *Then if* $u\in\gamma_s(G_1)$, $u\circ\varphi\in\gamma_s(G_2)$.

PROOF. It is obious from (5.4).

Let $G=(\tau-\delta,\tau+\delta)\times R^{m-1}$ be a band in $R^m$, $P(x,D)=\sum_{|\beta|\leqslant d}a_\beta(x)D^\beta$ and $Q(x,D)=\sum_{|\rho|\leqslant d-1}b_\beta(x)D^\beta$, of which coefficients are $m_1\times m_2$ matrices and satisfy

$$\begin{aligned}&|D^\alpha a_\beta(x)|\leqslant C_0(k^{-1}A)^{|\alpha|}|\alpha|!^s,\ |\beta|\leqslant d\,,\\&|D^\alpha b_\beta(x)|\leqslant C_0(k^{-1}A)^{|\alpha|}!^s,\ |\beta|\leqslant d-1\,,\end{aligned}\tag{5.5}$$

for $x\in\bar{G}$, where $k>1$.

We consider the following equations

$$\begin{cases}D_1F_j=P(x,D)F_{j-1}\ \text{in}\ G\\ F_j|_{x_1=\tau}=Q(x,D)F_{j-1}|_{x_1=\tau}\,,\end{cases}\tag{5.6$_j$}$$

for $j=0,1,2,\cdots$, where $F_j(x)$ are $m_2\times m_3$ matrices.

PROPOSITION 5.6. *Let $P(x, D)$ and $Q(x, D)$ be differential operators of order $d$ and $d-1$ respectively, of which coefficients satisfy* (5.5). *Assume that $F_0(x)$ is estimated by*

$$(5.7)_0 \qquad |D^\alpha F_0(x)| \leqslant CA^{|\alpha|}|\alpha|!^s, x \in \bar{G}.$$

*Then for every $j$, $F_j(x)$ satisfying* $(5.6)_j$, *can be estimated by*

$$(5.7)_j \qquad |D^\alpha F_j(x)| \leqslant C(C_0\bar{m}_d)^j A^{|\alpha|+(d-1)j} \sum_{l=0}^{j} \frac{(|x_1-\tau|A)^l}{l!}(|\alpha|+j(d-1)+l)!^s$$

*for $x \in \bar{G}$, where $\bar{m}_d=(m^{d+1}-1)(m-1)^{-1}(k-1)^{-1}k$, $k>1$.*

PROOF. We shall prove $(5.7)_j$ by induction. For $j=0$ it is trivial. Assume that $(5.7)_{j-1}$ is valid. For $\alpha=(\alpha_1, \cdots, \alpha_m)=(\alpha_1, \tilde{\alpha}), \alpha_1 \neq 0$, we have from $(5.6)_j$,

$$D^\alpha F_j = D_1^{\alpha_1-1}D^{\tilde{\alpha}}PF_{j-1} = D^\gamma PF_{j-1}, \gamma=(\alpha_1-1, \tilde{\alpha}).$$

Hence

$$|D^\alpha F_j| \leqslant \sum_{|\beta|\leqslant d} \sum_{\alpha'+\alpha''=\gamma} \binom{\gamma}{\alpha'} |D^{\alpha'}a_\beta||D^{\beta+\alpha''}F_{j-1}|$$

$$\leqslant C_0 \sum_{|\beta|\leqslant d} \sum_{\alpha'} \binom{\gamma}{\alpha'} k^{-|\alpha'|}A^{|\alpha'|}|\alpha'|!^s C(C_0\bar{m}_d)^{j-1}$$

$$\times A^{|\alpha''|+|\beta|+(d-1)(j-1)} \sum_{l=0}^{j-1} \frac{(|x_1-\tau|A)^l}{l!}(|\alpha''|+(j-1)(d-1)+l+|\beta|)!^s$$

$$\leqslant (C_0\bar{m}_d)^j \left(\frac{k-1}{k}\right) A^{|\alpha|+(d-1)j} \sum_{l=0}^{j-1} \frac{(|x-\tau|A)^l}{l!} \sum_{\alpha'} \binom{\gamma}{\alpha'} k^{-|\alpha'|}|\alpha'|!^s(|\alpha''|$$

$$+j(d-1)-1+l)!^s$$

which implies $(5.7)_j$ with (5.1).

For $\alpha=(0, \tilde{\alpha})$, we have

$$D^\alpha F_j = D^\alpha F_j(\tau, x') + \int_\tau^{x_1} (D^\alpha PF_j)(t, x')dt$$

here $x'=(x_2, \cdots, x_m)$. Hence

$$|D^\alpha F_j(x)| \leqslant |D^\alpha QF_{j-1}(\tau, x')| + \int_0^{|x_1-\tau|} |D^\alpha PF_{j-1}(t+\tau, x')|dt.$$

Since it follows from $(5.7)_{j-1}$ that

$$|D^\alpha F_{j-1}(\tau, x')| \leqslant C(C_0\bar{m}_d)^{j-1}A^{|\alpha|+(j-1)(d-1)}(|\alpha|+(j-1)(d-1))!^s,$$

we obtain by use of Lemma 5.2.

(5.8) $$|D^\alpha QF_{j-1}(\tau, x')| \leqslant CC_0\bar{m}_{d-1}(C_0\bar{m}_d)^{j+1}A^{|\alpha|+j(d+1)}(|\alpha|+j(d-1))!^s$$

On the other hand, we have by Leibniz' formula

$$|D^\alpha PF_{j-1}(t+\tau, x')| \leqslant \sum_{|\beta|\leqslant d}\binom{\alpha}{\alpha'}|D^{\alpha'}a(t+\tau, x')||D^{\beta+\alpha''}F_{j-1}(t+\tau, x')|$$

$$\leqslant C\left(\frac{k-1}{k}\right)(C_0 m_d)^j A^{|\alpha|+(d-1)j-1}\sum_{l=0}^{j-1}\frac{(tA)^l}{l!}\sum_{\alpha'}\binom{\alpha}{\alpha'}k^{-|\alpha'|}|\alpha'|!^s(|\alpha''|$$
$$+j(d-1)+l+1)!^s$$

$$\leqslant C(C_0\bar{m}_d)^j A^{|\alpha|+(d-1)j}\sum_{l=1}^{i}\frac{t^{l-1}A^l}{(l-1)!}(|\alpha|+j(d-1)+l)!^s$$

of which integration with respect to $t$ implies $(5.7)_j$ with (5.8).

Now we can prove Theorem 4.1 and 4.2. Let $G=(\tau-\delta, \tau+\delta)\times R^n\times V$, where $V$ is a neigbourhood of a sphere $S^{n-1}$. We put in $(4.13)_j$,

$$F_j=\begin{bmatrix} \sigma_j^{(1)}, \cdots, \sigma_j^{(l)} \\ W_j^{(1)}, \cdots, W_j^{(l)} \end{bmatrix},$$

$$P=\begin{bmatrix} M^{(1)}N_1^{(1)}, M^{(1)}N_2^{(1)}, \cdots, M^{(l)}N_1^{(l)}, M^{(l)}N_2^{(l)} \\ D_tN_1^{(1)}, D_tN_2^{(1)}, \cdots, D_tN_1^{(l)}, D_tN_2^{(l)} \end{bmatrix}$$

$$Q=\begin{bmatrix} G^{(1)}N_1^{(1)}, G^{(1)}N_2^{(1)}, \cdots, G^{(l)}N_1^{(l)}, G^{(l)}N_2^{(l)} \\ N_1^{(1)}, \quad N_2^{(1)}, \cdots, N_1^{(l)}, \quad N_2^{(l)} \end{bmatrix}$$

Then we obtain by virtue of Proposition 5. 6 with $d=2$,

$$|D_{t,z}^\alpha D_\omega^\beta F_j| \leqslant C(C_0\bar{m}_2)^i A^{|\alpha|+|\beta|+j}\sum_{l=0}^{j}\frac{(A\delta)^l}{l!}(|\alpha|+|\beta|+j+l)!^s$$

Noting that

$$(|\alpha|+|\beta|+j+l)! \leqslant 2^{|\alpha|+|\beta|+j+l}(|\alpha|+|\beta|)!\ (j+l)!,$$

we have

$$|D_{t,z}^\alpha D_\omega^\beta F_j| \leqslant C(4^s C_0\bar{m}_2\delta A^2)^j(2^s A)^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^s\sum_{l=0}^{j}\frac{(j+l)!^s}{l!^s j!^s}l!^{s-1}j!^s$$
$$\leqslant CA_1^{|\alpha|+|\beta|+j}(|\alpha|+|\beta|)!^s j!^{2s-1}$$

where

$$A_1=\max\{8^s C_0\bar{m}_2\delta A^2, 2^s A\}.$$

Theorem 4.2 is an immediate result of Theorem 4.1 and Lemma 5.4. For, it follows from Lemma 3.1 that the mapping $(t, x, \omega)\rightarrow(t, \tilde{z}^{(p)}(t, x, \tau, \omega), \omega)$ is in the class

$\gamma_s(G)$.

REMARK. *Friadman [23] showed that when $s>1$, the fundamental solution belongs to $\gamma_{3s-1}$ except the characteristic conoids. Our theorem implies that it belongs to $\gamma_{2s-1}$ except the characteristic conoids.*

## § 6. Wave front sets of fundamental solution in Gevrey class

In the term of (4.2), we denote $|\xi|^{-j}w_j^{(p)}(t,x,\tau,\omega)$ by $w_j^{(p)}(t,x,\tau,\xi)$. Theorem 4.2 implies

$$|D_{t,x}^{\alpha}D_{\xi}^{\beta}w_j^{(p)}(t,x,\tau,\xi)|\leqslant CA^{|\alpha|+|\beta|+j}(|\alpha|+|\beta|)!^s j!^{2s-1}|\xi|^{-j-|\beta|}\,, \tag{6.1}$$

for $(t,x)\in[\tau-\delta,\tau+\delta]\times R^n, \xi\in R^n\setminus 0, j=0,1,\cdots$. Then it follows from the article of Boutet de Monvel and Kree [1] that there exist $w^{(p)}(t,x,\tau,\xi)\in C^\infty([\tau-\delta,\tau+\delta]\times R^n\times(R^n\cap|\xi|\geq 1))$ such that

$$\left|D_{t,x}^{\alpha}D_{\xi}^{\beta}\left(w^{(p)}(t,x,\tau,\xi)-\sum_{j=0}^{N-1}w_j^{(p)}(t,x,\tau,\xi)\right)\right| \tag{6.2}$$
$$\leqslant C_1A_1^{|\alpha|+|\beta|+N}(|\alpha|+|\beta|)!^s N!^{2s-1}|\xi|^{-N-|\beta|}$$

for any positive integer $N$, $(t,x)\in[\tau-\delta,\tau+\delta]\times R^n$, and $\xi\in R^n$, $|\xi|\geq 1, p=1,\cdots,l$.

We define distributions $W^{(p)}(t,x,\tau,y)$ by

$$W^{(p)}(t,x,\tau,y)=\int(exp\ il^{(p)}(t,x,\tau,y,\xi))\theta(\xi)w^{(p)}(t,x,\tau,\xi)d\xi\,, \tag{6.3}$$

where $\theta(\xi)$ is a $C^\infty$ function in $R^n$, which is equal to zero for $|\xi|\leqslant 1$ and 1 for $|\xi|\geq 2$.

In this section our aim is to examine the wave front sets of $W^{(p)}(t,x,\tau,y)$ as a distribution in $x$ or $(x,y)$.

We shall describe the definition of the wave frount sets in Gevrey class, given by Hörmander [10]. We start with

LEMMA 6.1, [10]. *Let $K$ be a compact set in $R^n, \varepsilon>0$ and $N$ a positive integer. Then there exists a function $\chi_N^{K_\varepsilon}(x)\in C_0^\infty(R^n)$ equal to 1 on $K$ such that supp $\chi_N^{K_\varepsilon}$ is contained in $K_\varepsilon$, an $\varepsilon$-neigborhood of $K$, and satisfies*

$$|D^{\alpha+\beta}\chi_N^{K_\varepsilon}(x)|\leqslant C_\alpha\varepsilon^{-|\alpha|}(CN\varepsilon^{-1})^{|\beta|},\ |\beta|\leqslant N, \tag{6.4}$$

*where $C$ depends only on $n$ and $C_\alpha$ depends only on $n$ and $\alpha$.*

REMARK. It follows from Stirling's formula that we have

$$C_0{}^j(j+1)^j\leqslant j!\leqslant C_1{}^j(j+1)^j\,. \tag{6.5}$$

Hence, noting that $N^{|\beta|}|\beta|!^{-1}\leqslant N^N N!^{-1}$, $|\beta|\leqslant N$, we have

$$|D^\beta\chi_N^{K_\varepsilon}(x)|\leqslant CA_0{}^N A^{|\beta|}|\beta|!,\ |\beta|\leqslant N. \tag{6.6}$$

It follows from Lemma 5.3 that we obtain

LEMMA 6.2. *Let* $X_j=\sum_{i=1}^n a_{ji}(x)\dfrac{\partial}{\partial x_i}+a_{j0}, j=1,\cdots,n$ *and* $a_{ji}(x)$ *satisfy*

$$|D^\alpha a_{ji}(x)|\leqslant C_0(k^{-1}A)^{|\alpha|}|\alpha|!^s$$

*for* $x\in K_\varepsilon, k>1$. *Then we have*

$$|D^\alpha X_{j_1}\cdots X_{j_p}\chi_N^{K_\varepsilon}(x)|\leqslant C(C_0n_1)^p A^{|\alpha|+p}A_0{}^N(|\alpha|+p)!^s \tag{6.7}$$

*for* $|\alpha|+p\leqslant N$, *where* $n_1=(n+1)(k-1)^{-1}k$.

DEFINITION 6.3, [10]. *Let* $x_0\in R^n, \xi_0\in R^n\diagdown 0$ *and* $u\in\mathcal{D}'(R^n)$. *Then we say that* $(x_0,\xi_0)$ *is in the complement of the wave front sets* $WF_s(u)$ *of* $u$ *in the class* $\gamma_s$, *if there exist a neigborhood* $U$ *of* $x_0$ *and a conic neigborhood* $F$ *of* $\xi_0$ *such that for* $\xi\in F$

$$|\mathcal{F}(\chi_N^{\bar U_\varepsilon}u)(\xi)|\leqslant CA^N N!^s|\xi|^{-N}, N=1,2,\cdots, \tag{6.8}$$

*are valid for some constants* $\varepsilon, C$ *and* $A$ *independent of* $N$. *Here* $\bar U_\varepsilon$ *is an* $\varepsilon$*-neigborhood of the closure of* $U$ *and* $\mathcal{F}$ *stands for the Fourier transform.*

We note that we can replace $\chi_{N+p}^{\bar U_\varepsilon}(x)$ instead of $\chi_N^{\bar U_\varepsilon}(x)$. Then the constant $A$ must be replaced $A'$ dependent of $p$.

We denote by $\Lambda^{(p)}(t,\tau;y)$ the sets of Hamiltonian flows corresponding to $\lambda^{(p)}(t,x,\xi)$, that is,

$$\Lambda^{(p)}(t,\tau;y)=\bigcup_{\xi\in R^n\diagdown 0}\{(\hat x^{(p)}(t,y,\tau,\xi),\hat\xi^{(p)}(t,y,\tau,\xi))\}$$

here $(\hat x^{(p)},\hat\xi^{(p)})$ is a solution of (3.2) with $\lambda=\lambda^{(p)}(t,x,\xi), p=1,\cdots,l$.

THEOREM 6.4. *Let* $(t,\tau,y)$ *be fixed,* $\delta$ *a small coxstant*$>0$, *and regard* $W^{(p)}(t,x,\tau,y)$ *defined in* (6.3) *as a distribution in* $R_x{}^n$. *Then we have*

$$WF_s(W^{(p)}(t,\cdot,\tau,y))=\Lambda^{(p)}(t,\tau;y)$$

*for* $|t-\tau|\leqslant\delta, p=1,\cdots,l$.

PROOF. We show at first that

$$WF_s(W^{(p)}(t,\cdot,\tau,y))\subset\Lambda^{(p)}(t,\tau;y).$$

Let $(\hat x,\hat\xi)$ be not in $\Lambda^{(p)}(t,\tau;y)$. Then there exist a neigborhood $U$ of $\hat x$ and a conic neigborhood $F$ of $\hat\xi$ such that

$$(\bar U_\varepsilon\times F)\cap\Lambda^{(p)}(t,\tau;y)=\phi \tag{6.9}$$

for some $\varepsilon>0$. It is sufficient to prove that

$$I_N^{(p)}(\zeta)=\iint\{exp\ (il^{(p)}(t,x,\tau,y,\xi)-i\langle x,\zeta\rangle)\}\theta(\xi)\chi_{N+2n+1}^{\bar{U}_\varepsilon}(x)w^{(p)}(t,x,\tau,\xi)dxd\xi$$

satisfies (6.8) for sufficiently large $|\zeta|, \zeta\in F$. We can write

$$I_N^{(p)}(\zeta)=\rho^n\iint exp\ i\rho\varphi^{(p)}\chi_N(x)\theta(\rho\xi)w^{(p)}(t,x,\tau,\rho\xi)dxd\xi\,,$$

here, for simplicity we put $\chi_N=\chi_{N+n}^{\bar{U}_\varepsilon}$ and $\varphi^{(p)}=l^{(p)}(t,x,\tau,y,\xi)-\langle x,\tilde{\zeta}\rangle$, $\tilde{\zeta}=\zeta|\zeta|^{-1}$, $\rho=|\zeta|$. In order to anihilate the singularity of $w^{(p)}(t,x,\tau;\rho\xi)\theta(\rho\xi)$ with respect to $\xi$, we decompose

$$\begin{aligned}I_N^{(p)}(\zeta)&=\rho^n\iint_{\rho|\xi|\geq 1}(exp\ i\rho\varphi^{(p)})\{\chi_N\theta_N(w^{(p)}\theta)+\chi_N(1-\theta_N)w^{(p)}\theta\}d\xi dx\\&=I_{N1}^{(p)}(\zeta)+I_{N2}^{(p)}(\zeta)\end{aligned}$$

where $\theta_N=\chi_{N+n}^{B_\varepsilon}(\xi)$, $B=\{\xi\in R^n;\ |\xi|\leqslant\varepsilon_1\}$. If $\varepsilon$ and $\varepsilon_1$ are sufficiently small, $\text{grad}_x\varphi^{(p)}=l_x^{(p)}-\tilde{\zeta}$ does not vanish for $\xi\in B_\varepsilon$. For, $l_x^{(p)}$ is homogeneous degree one in $\xi$ from Lemma 3.2 and $\tilde{\zeta}=\zeta|\zeta|^{-1}\neq 0$. Hence we may assume that $\varphi_{x_1}^{(p)}\neq 0$ for $x\in\bar{U}_\varepsilon$ and $\xi\in B_\varepsilon$. Then we obtain from an integration by part, for $\rho\geq\varepsilon_1^{-1}$,

$$I_{N1}^{(p)}(\zeta)=\rho^n\iint(exp i\varphi^{(p)}\rho)\left(\frac{\partial}{\partial x_1}\frac{1}{\rho\varphi_{x_1}^{(p)}}\right)^{N+n}(\chi_N w^{(p)})\theta_N(\xi)\theta(\rho\xi)d\xi dx\,.$$

Hence it follows from Lemma 6.2 that $I_{N1}^{(p)}(\zeta)$ satisfies (6.8). Next we estimate $I_{N2}^{(p)}(\zeta)$. It follows from (6.9) that $\text{grad}_{x,\xi}\varphi^{(p)}\neq 0$ for $x\in\bar{U}, \zeta\in F$ and $|\xi|=1$. Then we can find a first order differential operator $M$ such that $\rho^{-1}M(exp\ i\rho\varphi^{(p)})=exp\ i\rho\varphi^{(p)}$, that is

$$M=\left\{\sum_{j=1}^n i((\varphi_{x_j}^{(p)})^2|\xi|^{-2}+(\varphi_{\xi_j}^{(p)})^2)\right\}^{-1}\sum_{j=1}^n\left(|\xi|^{-2}\varphi_{x_j}^{(p)}\frac{\partial}{\partial x_j}+\varphi_{\xi_j}^{(p)}\frac{\partial}{\partial\xi_j}\right),$$

of which coefficients are in Gevrey class $\gamma_s$ for $x\in\bar{U}_\varepsilon$ and for $|\xi|\geq\varepsilon_1$. Hence we obtain

$$I_{N2}^{(p)}(\zeta)=\rho^n\int_{|\xi|\geq 2\rho^{-1}}(exp\ i\rho\varphi^{(p)})(\rho^{-1}M)^{N+n}(\chi_N(1-\theta_N)w^{(p)})dxd\xi\,.$$

Applying Lemma 6.2, we have for some $C$ and $A$,

$$\begin{aligned}|{}^tM^{N+n}(\chi_N(1-\theta_N)w^{(p)}(t,x,\tau,\rho\xi)|&\leqslant C(A|\xi|^{-1})^{N+n}(N+n)!^s\\&\leqslant CA_1^{N+n}(|\xi|+1)^{-n-1}N!^s\end{aligned}$$

for $x\in\bar{U}_\varepsilon$ and $|\xi|\geq\varepsilon_1\geq 2\rho^{-1}$. This implies (6.8) for $I_{N2}^{(p)}(\zeta)$. The fact that $\Lambda^{(p)}(t,\tau;y)\subset WF_s(W^{(p)}(t,\cdot,\tau,y))$ follows from the method of stationary phase. Let $(\hat{x},\hat{\zeta})$ be in $\Lambda^{(p)}(t,\tau;y)$, that is, there exists $\hat{\xi}\in R^n\setminus 0$, such that $\hat{x}=\hat{x}^{(p)}(t,y,\tau,\hat{\xi})$, $\hat{\zeta}=\hat{\xi}^{(p)}(t,y,\tau,\hat{\xi})$. Then it follows from Lemma 3.2 that $\text{grad}_{x,\xi}\varphi^{(p)}=0$ for $x=\hat{x}$ and $\xi=\hat{\xi}$. On the other hand, the Hessian of $\varphi^{(p)}$ with respect to $(x,\xi)$ (denote by $Q^{(p)}$) is non singular, for $|t-\tau|\leqslant\delta$, because of $Q^{(p)}|_{t=\tau}=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$. Hence we can apply the method of stationary phase to $I_N^{(p)}(\zeta)$ (c.f. [3]). It follows,

$$I_N^{(p)}(\rho\hat{\zeta})=(2\pi)^n(exp\ -i\rho\langle x,\hat{\zeta}\rangle)|detQ^{(p)}|^{-1/2}w_0{}^{(p)}(t,\hat{x},\tau,\hat{\xi})$$
$$+0(\rho^{-1}),\ \rho\longrightarrow\infty\,.$$

By virtue of (4.4) and $(4.13)_o$ we have

$$w_0{}^{(p)}(t,\hat{x},\tau,\hat{\xi})=(2\pi)^{-n}H^{(p)}(t,\hat{x},\hat{\zeta})G^{(p)}(\tau,y,\hat{\xi})\Delta^{(p)}(t)^{1/2}J^{(p)}(t)^{-1}\,,$$

which does not vanish, because of $G^{(p)}(\tau,y,\xi)H^{(p)}(\tau,y,\xi)=I$. This completes the proof of our theorem.

REMARK. *We can regard the integral form* (6.3) *as the kernel of Fourier integral operator,* (*c.f.* [9]). *When* $s=1$, *K. Nishiwada* [19] *investigates the wave fromt sets of Fourier integral operators in terms of boundary values of holomorphic functions.*

As a corollary of Theorem 6.4 we have

THEOREM 6.5. *Let* $(t,\tau)$ *be fixed, a small constant* $\delta>0$, *and regard* $W^{(p)}(t,\tau)=W^{(p)}(t,x,\tau,y)$ *as a distribution in* $R_x{}^n\times R_y{}^n$. *Then for* $|t-\tau|\leqslant\delta$,

$$WF_s(W^{(p)}(t,\tau))=\bigcup_{(y,\xi)\in R^n\times R^n\setminus 0}\{(\hat{x}^{(p)}(t,y,\tau,\xi),y,\hat{\xi}^{(p)}(t,y,\tau,\xi),-\xi)\}$$

We next consider the remainder term $L_{t,x}W^{(p)}(t,x,\tau,y)=R^{(p)}(t,x,\tau,y)$ as a distrubution in $R_x{}^n$. It follows evident from Theorem 6.3 that $WF_s(R^{(p)}(t,\cdot,\tau,y))\subset\Lambda^{(p)}(t,\tau\,;\,y)$. Moreover we can see from the asymptotic expanssion that $WF_{2s-1}(R^{(p)}(t,\cdot,\tau,y))$ is empty. In fact, we can write

(6.10) $$R^{(p)}(t,x,\tau,y)=\int\{exp\ il^{(p)}(t,x,\tau,y,\xi)\}r^{(p)}(t,x,\tau,\xi)d\xi$$

where

$$r^{(p)}(t,x,\tau,\xi)=\{i(l_t^{(p)}+A(t,x,l_x^{(p)}))w^{(p)}+L_{t,x}w^{(p)}\}\theta(\xi)$$

satisfies

(6.11) $$|D_{t,x}^{\alpha}D_{\xi}{}^{\beta}r^{(p)}(t,x,\tau,\xi)|\leqslant CA^{|\alpha|+|\beta|+N}(|\alpha|+|\beta|)!^sN!^{2s-1}|\xi|^{-N-|\beta|}$$

for $(t,x)\in[\tau-\delta,\tau+\delta]\times R^n$, $\xi\in R^n$, $|\xi|\geq 2$, and for any positive integer $N$.

Thus we have proved

THEOREM 6.6. *Let* $R^{(p)}(t,x,\tau,y)$ *be the remainder terms defined by* (6.10). *Then* $WF_s(R^{(p)}(t,\cdot,\tau,y))\subset\Lambda^{(p)}(t,\tau;y)$, *and* $WF_{2s-1}(R^{(p)}(t,\cdot,\tau,y))=\phi$ *for* $|t-\tau|\leqslant\delta,p=1,\cdots l$.

Now we turn to prove Theorem 2.1. To anihilate the remainder terms $R^{(p)}(t,x,\tau,y)$, we reduce our problem to an integral equation of Voltera's type, following the method of Kumano-go [11] and Tsutsumi [22].

We denote

$$W(t, x, \tau, y)=\sum_{p=0}^{l} W^{(p)}(t, x, \tau, y), \tag{6.12}$$

where $W^{(p)}(t, x, \tau, t)$ is defined by (6.3), $p=1, \cdots, l$, and

$$W^{(0)}(t, x, \tau, y)=\int exp\, i<x-z, \xi> w^{(0)}(\tau, x, \tau, \xi)d\xi,$$

here

$$w^{(0)}(t, x, \tau, \xi)=(2\pi)^{-n}(1-\theta(\xi))I-\left\{\sum_{p=1}^{l} w^{(p)}(\tau, x, \tau, \xi)-(2\pi)^{-n}I\right\}\theta(\xi)$$

It follows evidently from (6.2) that

$$|D_x^{\alpha}D_{\xi}^{\beta}w^{(0)}(t, x, \tau, \xi)|\leqslant CA^{|\alpha|+|\beta|+N}N!^{2s-1}|\xi|^{-N-|\beta|}(|\alpha|+|\beta|)!^{s} \tag{6.13}$$

for $|\xi|\geq 2$, and therefore

$$WF_{2s-1}(W^{(0)}(t, \cdot, \tau, y))=\phi,$$

and that we have

$$W(\tau, x, \tau, y)=\delta(x-y). \tag{6.14}$$

We shall seek a fundamental solution of $L_{t,x}$ as the following form

$$K(t, x, \tau, y)=W(t, x, \tau, y)+\int_{\tau}^{t} d\sigma \int W(t, x, \sigma, z)F(\sigma, z, y)dz. \tag{6.15}$$

Then noting (6.14), we have

$$L_{t,x}K(t, x, \tau, y)=L_{t,x}W+F(t, x, \tau, y)+\int_{\tau}^{t} d\sigma\int(L_{t,x}W)(t, x, \sigma, z)F(\sigma, z, \tau, y)dz$$
$$=0.$$

Hence we obtain an integral equation

$$F(t, x, \tau, y)=R(t, x, \tau, y)+\int_{\tau}^{t}\int d\sigma R(t, x, \sigma, z)F(\sigma, z, \tau, y)dz \tag{6.16}$$

where we denote

$$R(t, x, \tau, y)=-L_{t,x}W(t, x, \tau, y) \tag{6.17}$$
$$=-\sum_{p=1}^{l}\int\{exp\, il^{(p)}(t, x, \tau, y, \xi)\}r^{(p)}(t, x, \tau, \xi)d\xi-\int\{exp\, i\langle x-y, \xi\rangle\}r^{(0)}(t, x, \tau, \xi)d\xi,$$
$$=\sum_{p=0}^{l} R^{(p)}(t, x, \tau, y)$$

where from (6.11) and (6.13) we have

(6.18) $$|D_x^\alpha D_\xi^\beta r^{(p)}(t, x, \tau, \xi)| \leqslant C_0 A^{|\alpha|+|\beta|+N} N!^{2s-1} |\xi|^{-N-|\beta|}$$

for $|\xi| \geq 2, p=0, 1, \cdots l$.

PROPOSITION 6.7. *Let $R(t, x, \tau, y)$ be the remainder term given by* (6.17). *There exist positive constants $C_0$ and $A_1$ such that*

(6.19) $$|D_x^\alpha D_y^\beta R(t, x, \tau, y)| \leqslant C_1 A_1^{|\alpha|+|\beta|} |\alpha|!^{2s-1} |\beta|!^{2s-1}$$

*for $(x, y) \in R^n \times R^n$, $|t-\tau| \leqslant \delta$.*

PROOF. We have

$$D_x^\alpha D_y^\beta R(t, x, \tau, y) = \sum_{p=0}^{l} \int_{|\xi|\geq 1} D_x^\alpha D_y^\beta \{(exp\, il^{(p)}) r^{(p)}(t, x, \tau, \xi)\} d\xi\,.$$

It follows from (6.17) that we have

$$\left(\frac{1}{|\xi|} D_x\right)^\alpha \left(\frac{1}{|\xi|} D_y\right)^\beta \{(exp\, il^{(p)}) r^{(p)} |\xi|^{|\alpha|+|\beta|}\}$$

$$\leqslant C' A_1^{|\alpha|+|\beta|} |\alpha|!^{2s-1} |\beta|!^{2s-1} |\xi|^{-n-1}\,,$$

which implies (6.19).

We define

$$R(t, \tau) u(x) = \int R(t, x, \tau, y) u(y) \mathrm{d}y.$$

Then we have

PROPOSITION 6.8. *Let $R(t, x, \tau, y)$ be the remainder term, $A_1 \geq 2A$, given in* (6.18) *and $u(x)$ satisfy*

(6.20) $$|D_x^\alpha u(x)| \leqslant C_1 A_1^{|\alpha|} |\alpha|!^{2s-1}\,,$$

*for $x \in R^n$. Then there exists a positive constant $C$ such that*

(6.21) $$|D_x^\alpha R(t, \tau) u(x)| \leqslant C_1 C A_1^{|\alpha|} |\alpha|!^{2s-1}\,,$$

*for $x \in R^n$, $|t-\tau| \leq \delta$.*

PROOF. We note that

$$l^{(p)}(t, x, \tau, y, \xi) = \langle \tilde{z}^{(p)}(t, x, \tau, \xi), \xi \rangle - \langle y, \xi \rangle$$

$$= \langle x-y, \xi \rangle + \langle \varphi^{(p)}(t, x, \tau, \xi), \xi \rangle$$

Then we have

(6.22) $$|D_x^\alpha D_\xi^\beta \varphi^{(p)}| \leqslant |t-\tau| C_0 A^{|\alpha|+|\beta|} (|\alpha|+|\beta|)!^s,\ \beta \neq 0\,,$$

for $x \in R^n$, $|\xi| \geq 1$. We write

$$R(t,\tau)u(x)=-\sum_{p=0}^{l}\iint(exp\, il^{(p)})r^{(p)}(t,x,\tau,\xi)u(y)dyd\xi$$

$$=-\sum\iint(exp\, i\langle x-y,\xi\rangle)r^{(p)}u(\varphi^{(p)}+y)dyd\xi\,.$$

Hence we have

$$D_x{}^\alpha R(t,\tau)u(x)=-\sum_p\iint\sum_{\alpha'}\binom{\alpha}{\alpha'}D_x^{\alpha'}(exp\, i\langle x-y,\xi\rangle)D_x^{\alpha''}(r^{(p)}u(\varphi^{(p)}+y))dyd\xi$$

$$=-\sum_p\iint(exp\, i\langle x-y,\xi\rangle)\sum_{\alpha'}\binom{\alpha}{\alpha'}D_x^{\alpha''}((i\xi)^{\alpha'}r^{(p)}u(\varphi^{(p)}+y))dyd\xi$$

$$=-\sum_p\iint_{|\xi|\geq1}(1+|x-y|^2)^{-n}\,(exp\, i\langle x-y,\xi\rangle)$$

$$\times\sum_{\alpha'}\binom{\alpha}{\alpha'}(1-\Delta_\xi)^n D_x^{\alpha''}((i\xi)^{\alpha'}r^{(p)}u(\varphi^{(p)}+y))dyd\xi\,.$$

It follows from Lemma 5.4, (6.20) and (6.22) that we have

$$D_x{}^\alpha D_\xi{}^\beta u(\varphi^{(p)}+y)|\leqslant C_1A_1^{|\alpha|+|\beta|}(2^sC_0|t-\tau|\bar{n}A)^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^{2s-1}$$

where we have put $k=A_1/A$, $\bar{n}=(n+1)(k-1)^{-1}k$.

Hence

(6.23) $$|D_x{}^\alpha D_\xi{}^\beta u(\varphi^{(p)}+y)|\leqslant C_1A_1^{2n}(2n)!^{2s-1}A_1^{|\alpha|}|\alpha|!^{2s-1},\ |\beta|\leqslant 2n$$

if $|t-\tau|\leqslant\delta$ is sufficiently small, that is,

$$2^{3s-1}C_0\delta(n+1)k(k-1)^{-1}A\leqslant 2^{3s}C_0\delta(n+1)\leqslant 1\,,$$

here we used $k\geq 2$. Moreover we have from (6.18),

(6.24) $$|D_x{}^\alpha D_\xi{}^\beta(i\xi)^\alpha r^{(p)}|\leqslant C_0'(k^{-1}A_1)^{|\alpha|+|\beta|}|\alpha|!^s|\alpha'|!^{2s-1}|\xi|^{-n-1}$$

for $|\beta|\leqslant 2n$, $|\xi|\geq 1$, $k=A_1/A$. Hence we obtain from (6.23) and (6.24),

$$\sum_{\alpha'}\binom{\alpha}{\alpha'}|(1-\Delta_\xi)^n D_x^{\alpha''}((i\xi)^{\alpha'}r^{(p)}u(\varphi^{(p)}+y))|$$

$$\leqslant\sum_{\alpha'}\binom{\alpha}{\alpha'}\sum_{|\beta+\beta'|\leq 2n}\sum_{\gamma'}\binom{\alpha''}{\gamma'}|D_x^{\gamma'}D_\xi{}^\beta(i\xi)^{\alpha'}r^{(p)}D_x^{\gamma''}D^{\beta'}u(\varphi^{(p)}+y)|$$

$$\leqslant C_1C_2|\xi|^{-n-1}\sum_{\alpha'}\binom{\alpha}{\alpha'}\binom{\alpha''}{\gamma'}(k^{-1}A_1)^{|\gamma'|+|\alpha'|}|\gamma'|!^s|\alpha'|!^{2s-1}A_1^{|\alpha''|}|\gamma''|!^{2s-1}$$

$$\leqslant C_1C_3|\xi|^{-n-1}A_1^{|\alpha|}\sum_{\alpha'}\binom{\alpha}{\alpha'}k^{-|\alpha''|}|\alpha'|!^{2s-1}\sum_{\gamma'}\binom{\alpha''}{\gamma'}k^{-|\gamma'|}(|\gamma'|!|\gamma''|!)^{2s-1}$$

$$\leqslant C_1C_4|\xi|^{-n-1}A_1^{|\alpha|}|\alpha|!^{2s-1},\ \text{(by (5.1))}\,,$$

which implies (6.21).

Now we shall construct a solution $F$ of the integral equation (6.16). We define inductively

$$F_0(t, x, \tau, y)=R(t, x, \tau, y)$$

$$F_j(t, x, \tau, y)=\int_\tau^t \int R(t, x, \sigma, z)F_{j-1}(\sigma, z, \tau, y)d\tau dz$$

$$=\int_\tau^t R(t, \sigma)F_{j-1}(\sigma, \tau)d\sigma .$$

Then we can estimate

$$(6.25)_j \qquad |D_x^\alpha D_y^\beta F_j(t, x, \tau, y)|\leqslant C_1C^j \frac{|t-\tau|^j}{j!} A_1^{|\alpha|+|\beta|}|\alpha|!^{2s-1}|\beta|!^{2s-1},$$

for $|t-\tau|\leqslant\delta, (x, y)\in R^n\times R^n$. $(6.25)_0$ follows from Proposition 6.7. Assume that $(6.25)_{j-1}$ is valid. Then we have from Proposition 6.8

$$|D_x^\alpha D_x^\beta R(t, \sigma)F_{j-1}(\sigma, \tau)|\leqslant C_1C^j \frac{|\sigma-\tau|^{j-1}}{(j-1)!} A_1^{|\alpha|+|\beta|}|\alpha|!^{2s-1}|\beta|!^{2s-1} .$$

Integrating this with respect to $\sigma$, we obtain $(6.25)_j$. We difine

$$F(t, x, \tau, y)=\sum_{j=0}^\infty F_j(t, x, \tau, y)$$

which is a solution of (6.16) and satisfies

$$(6.26) \qquad |D_x^\alpha D_y^\beta F(t, x, \tau, y)|\leqslant C_1 (exp\ |t-\tau|C)A_1^{|\alpha|+|\beta|}|\alpha|!^{2s-1}|\beta|!^{2s-1} .$$

PROPOSITION 6.9. *Let $W(t, x, \tau, y)$ be given by* (6.12), *and $u(x)$ satisfied with*

$$(6.27) \qquad |D_x^\alpha u(x)|\leqslant C_1A_1^{|\alpha|}|\alpha|!^{s_1}, x\in R^n .$$

*If $s_1\geq s$, then there exist positive constants $C_2$ and $A_2$ such that*

$$(6.28) \qquad |D_x^\alpha W(t, \tau)u(x)|\leqslant C_1C_2A_2^{|\alpha|}|\alpha|!^{s_1}$$

*for $|t-\tau|\leqslant\delta, x\in R^n$.*

PROOF. We have

$$W(t, \tau)u(x)=\sum_{p=0}^l \iint(exp\ i\langle\tilde{z}^{(p)}-y, \xi\rangle)w^{(p)}(t, x, \tau, \xi)u(y)d\xi dy$$

$$=\sum_p \iint(exp\ -i\langle y, \xi\rangle)w^{(p)}u(\tilde{z}^{(p)}+y)d\xi dy$$

Hence we have

$$D_x^\alpha W(t,\tau)u(x)=\sum_p\iint(exp\, i\langle y,\xi\rangle)D_x^\alpha(w^{(p)}u(\tilde{z}^{(p)}+y))d\xi dy$$

$$=\sum\iint_{|\xi|\geq 1}(exp\, -i\langle y,\xi\rangle)(1+|y|^2)^{-n}(1+|\xi|^2)^{-n}(1-\Delta_y)^n$$

$$\times(1-\Delta_\xi)^n D_x^\alpha(w^{(p)}u(\tilde{z}^{(p)}+y))d\xi dy$$

From (6.2), (6.13) and (6.27) we obtain

$$|(1-\Delta_y)^n(1-\Delta_\xi)^n D_x^\alpha(w^{(p)}u(\tilde{z}^{(p)}+y))|\leqslant C_1C_2A_2^{|\alpha|}|\alpha|!^{s_1}$$

which implies (6.28).

Thus it follows from Proposition 6.8 and 6.9 that we can obtain a fundamental solution such that, $|t-\tau|\leqslant\delta$,

$$K(t,x,\tau,y)=W(t,x,\tau,y)+\int_\tau^t\int W(t,x,\tau,z)F(\sigma,z,\tau,y)d\sigma dz\,,$$

of which second term belogs to $\gamma_{2s-1}(R_x^n\times R_y^n)$. Thus we have proved Theorem 2.1.

## §7. Global construction of fundamental solution

In the previous section we have construct the fundamental solution $K(t,x,\tau,y)$ for $|t-\tau|\leqslant\delta$, if $\delta$ is sufficiently small. In the present section we shall give an expression of the fundamental solution for any interval $[0,T]$, $T>0$.

We decompose the interval $[0,T]$ such that $0=t_0<t_1<\cdots<t_{d+1}=T, t_j-t_{j-1}=\delta$. Then it follows from semigroup property that we obtain

$$K(t,x,t_0,y)=K(t,x,t_j,\cdot)K(t_j,\cdot,t_{j-1},\cdot)\cdots K(t_1,\cdot,t_0,y)$$

for $|t-t_j|\leqslant\delta$. We put

$$K_j^{(p)}(t,x,t_0,y)=W^{(p)}(t,x,t_j,\cdot)W^{(p)}(t,\cdot,t_{j-1},\cdot)\cdots W^{(p)}(t_1,\cdot,t_0,y)$$

for $|t-t_j|\leqslant\delta, j=0,1,\cdots,d$, and $p=1,\cdots,l$, where $W^{(p)}(t,x,\tau,y)$ is given by (6.3) for $|t-\tau|\leqslant\delta$. Then we can express

$$K(t,x,t_0,y)=\sum_{p=1}^{l}K_j^{(p)}(t,x,t_0,y)+K_j^{(o)}(t,x,t_0,y) \tag{7.1}$$

for $|t-t_j|\leqslant\delta$. Our purpose is to prove that

$$WF_s(K_j^{(p)}(t,\cdot,t_0,y))=\Lambda^{(p)}(t,t_0;y), p=1,\cdots,l\,, \tag{7.2}$$

$$WF_{2s-1}(K_j^{(o)}(t,\cdot,t_0,y))=\phi \tag{7.3}$$

for $|t-t_j|\leqslant\delta$. $j=0,\cdots,d$.

We define $\Lambda^{(p)}(t,\tau)$ by

$$\Lambda^{(p)}(t,\tau)(y,\xi)=(\hat{x}^{(p)}(t,y,\tau,\xi),\hat{\xi}^{(p)}(t,y,\tau,\xi)).$$

Let $F$ be a set in $R^n\times R^n\setminus 0$. We write

$$\Lambda^{(p)}(t,\tau)F=\{\hat{x}^{(p)}(t,y,\tau,\xi),\hat{\xi}^{(p)}(t,y,\tau,\xi));\ (y,\xi)\in F\},$$

where $(\hat{x}^{(p)},\hat{\xi}^{(p)})$ is a solution of (3.2) with $\lambda=\lambda^{(p)}, p=1,\cdots,l$. Then we have

$$\Lambda^{(p)}(t,\tau)\Lambda^{(p)}(\tau,\sigma)=\Lambda^{(p)}(t,\sigma)$$
$$\Lambda^{(p)}(t,\tau)\Lambda^{(p)}(\tau,t)=I$$

for any $(t,\tau,\sigma)$.

THEOREM 7.1. *Let $u$ be in $S'(R^n)$ and $s'\geq s$. Then*

$$WF_{s'}(W^{(p)}(t,\tau)u)\subset\Lambda^{(p)}(t,\tau)WF_{s'}(u).$$

*for* $|t-\tau|\leqslant\delta, p=1,\cdots,l$.

Since $W^{(p)}(t,x,\tau,y)$ is in $S'(R^n)$ with respect to $x$ for $|t-\tau|\leqslant\delta$, we obtain

COROLLARY 7.2.

$$WF_s(K_j^{(p)}(t,\cdot,t_0,y))\subset\Lambda^{(p)}(t,t_0;y),$$

*for* $|t-t_j|\leqslant\delta, j=0,1,\cdots,d,p=1,\cdots,l$.

PROOF OF THEOREM 7.1. Let $K$ be a neigborhood of $x_0$ and $\chi_N(x)=\chi_N^K(x)$. Put

$$I_N(\zeta,y)=\int(exp\ -i\langle x,\zeta\rangle)\chi_N(x)w^{(p)}(t,x,\tau,y)dx$$
$$=\iint\{exp(-i\langle x,\zeta\rangle+i\langle\tilde{z}^{(p)}(t,x,\tau,\xi)-y,\xi\rangle)\}\chi_N(x)w^{(p)}(t,x,\tau,\xi)d\xi dx.$$

Then there exists a positive constant $r$ such that for any positive integer $m$ and for $|y|\geq r$

$$\sum_{|\alpha|\leq m}|D_y{}^\alpha I_N(\zeta,y)|\leqslant c_m(1+|y|)^{-m}A^N|\zeta|^{-N}N!,\ N=1,2,\cdots, \tag{7.4}$$

where $c_m$ depends only on $m$. For, $\text{grad}_\xi\langle\tilde{z}^{(p)}(t,x,\tau,\xi)-y,\xi\rangle=\tilde{z}^{(p)}(t,x,\tau,\xi)-y\neq 0$ and $x\in\text{supp}\,\chi_N(x)$, if $r$ is sufficiently large. Let $\chi_N^1(y)=\chi_N^{B_r}(y)$, where $B_r=\{y,|y|\leqslant 2r\}$. Then we have

$$\mathcal{F}(\chi_N(x)W^{(p)}(t,\tau)u)(\zeta)=\langle I_N(\zeta,\cdot),\chi_N^1u\rangle+\langle I_N(\zeta,\cdot),(1-\chi_N^1)u\rangle.$$

Then the second term can be estimated by $c_m|\zeta|^{-N}A^NN!^{s'}$ by use of (7.4), where $m$ is the order of the distribution $u$. Let $K_1$ be the intersection of $B_r$ and a neigborhood of the projection of $WF_s(u)$ into $R^n_x$ and $\chi^2_N(y)=\chi^{K_1}_N(y)$. Then we have

$$|\mathcal{F}((1-\chi^2_N)\chi^1_N u)(\xi)|\leqslant C|\xi|^{-N}A^N N!^{s'} N=1, 2, \cdots,$$

for any $\xi\neq 0$. Hence we have

$$|\langle I_N(\zeta, \cdot), (1-\chi^2_N)\chi^1_N u\rangle|\leqslant C|\zeta|^{-N}A^N N!^{s'}$$

Moreover for $(y, \xi)\notin WF_{s'}(u)$, $y\in\operatorname{supp}\chi^2_N$, we have

$$|\mathcal{F}(\chi^2_N\chi^1_N u)(\xi)|\leqslant C|\xi|^{-N}A^N N!^{s'}$$

and for $(y, \xi)\in WF_s(u)$ and $(x, \zeta/|\zeta|)\notin \Lambda^{(p)}(t, \tau)WF_s(u)$,

$$d_{(x,\xi)}(\langle \tilde{z}^{(p)}-y, \xi\rangle-\langle x, \zeta/|\zeta|\rangle)\neq 0.$$

Hence we obtain

$$|\langle I_N(\zeta, \cdot), \chi^1_N\chi^2_N u\rangle|\leqslant C|\zeta|^{-N}A^N N!^{s'}$$

Thus we have proved our theorem.

Denote by $WF(u)$ the wave front sets with respect to $C^\infty$ functions. Then it holds that (c.f. [10]),

$$WF(u)\subset WF_s(u).$$

Hence to prove that

$$WF_s(K^{(p)}_j(t, \cdot, t_0, y))\supset \Lambda^{(p)}(t, t_0; y)$$

it suffices to indicate

$$WF(K^{(p)}_j(t, \cdot, t_0, y))\supset \Lambda^{(p)}(t, t_0; y) \tag{7.4}$$

for $|t-t_j|\leqslant\delta, j=1, \cdots, d, p=1, \cdots, l.$

LEMMA 7.3. [3]. *Let $u$ be in $\mathcal{D}'(R^n)$. Then $(x_0, \xi_0)\notin WF(u)$ if and only if for any real valued $C^\infty$ function $\psi(x)$ with $d_x\psi(x_0)=\xi_0$ there exists an open neigborhood $U_0$ of $x_0$ such that for any $\chi(x)\in C^\infty_0(U_0)$ we have*

$$\langle(\exp -i\rho\psi)\chi, u\rangle=0(\rho^{-N}) \quad \textit{for } \rho\to\infty$$

*uniformly with respect to $\psi$.*

We can express

$$K^{(p)}_j(t, x, t_0, y)=\int(\exp i\varphi^{(p)}_j(t, x, y, t_0, \theta))a^{(p)}_j(t, x, t_0, \theta)d\theta,$$

where

$$\theta=((\xi^{(j)}, y^{(j)}, \xi^{(j-1)}, y^{(j-1)}, \cdots, y^{(1)}, \xi^{(0)})\in R^{(2j+1)n},$$

$$\varphi^{(p)}(t, x, t_0, \theta, y)=\langle \tilde{z}^{(p)}(t, x, t_j, \xi^{(j)})-y^{(j)}, \xi^{(j)}\rangle$$

$$+\sum_{k=2}^{j}\langle \tilde{z}^{(p)}(t_k, y^{(k)}, t_{k-1}, \xi^{(k-1)})-y^{(k-1)}, \xi^{(k-1)}\rangle$$

$$+\langle \tilde{z}^{(p)}(t_1, y^{(1)}, t_0, \xi^{(0)})-y, \xi^{(0)}\rangle$$

and $a_j^{(p)}(t, x, t_0, \theta) = w^{(p)}(t, x, t_j, \xi^{(j)})w^{(p)}(t_j, y^{(j)}, t_{j-1}, \xi^{(j-1)})\cdots w^{(p)}(t_1, y^{(1)}, t_0, \xi^{(0)})$ It is obious that $(x, d_x\varphi_j^{(p)})\in \Lambda^{(p)}(t, t_0; y)$ if and only if $d_\theta\varphi_j^{(p)}=0$. Hence we have

$$\Lambda^{(p)}(t, t_0; y)=\{(x, d_x\varphi_j^{(p)});\ d_\theta\varphi_j^{(p)}=0\}. \tag{7.5}$$

We note that the rank of $d_{(x,\theta)}d_\theta\varphi_j^{(p)}=(2j+1)n$. For,

$$d_{(x,\theta)}d_\theta\varphi_j^{(p)}=\begin{pmatrix} A_{j+1} & & & & & & \\ & -I & & & & & \\ & & A_j & & & 0 & \\ & & & -I & & & \\ & & & & \ddots & & \\ & * & & & & A_1 & \\ & & & & & & -I \\ * & & & & & \cdots & * \end{pmatrix},$$

where $A_{j+1}=d_x\tilde{z}^{(p)}(t, x, t_j, \xi^{(j)})$, $A_k=d_{y^{(k)}}\tilde{z}^{(p)}(t_k, y^{(k)}, t_{k-1}, \xi^{(k-1)})$ $(k=1, \cdots, j)$ and $I$ the $n\times n$ identity matrix, all are non singular.

We shall prove (7.4) by use of the method of stationary phase. To do so, we need

Lemma 7.4. [3]. *Let $\psi(x)$ be a real valued function with $(\hat{x}, \hat{\xi})\in\Lambda^{(p)}(t, t_0, y)$, $\hat{\xi}=d_x\psi(\hat{x})$. Then the matrix $d^2_{(x,\theta)}(\varphi_j^{(p)}-\psi)$ is non singular at $(\hat{x}, \hat{\xi})$, if and only if*

(i) *the rank of $d_{(x,\theta)}d_\theta\varphi_j^{(p)}=(2j+1)n$*

(ii) *the graph $(x, d_x\psi)$ and $\{(x, d_x\varphi_j^{(p)}), d_\theta\varphi_j^{(p)}=0\}$*

*intersect transversally at $(\hat{x}, \hat{\xi})$.*

Lemma 7.5. *Let $(\hat{x}, \hat{\xi})$ be in $\Lambda^{(p)}(t, t_0; y)$. There exists a non symmetrix matrix $R$ such that $\psi(x)=\langle x-\hat{x}, \hat{\xi}\rangle+1/2\langle R(x-\hat{x}), x-\hat{x}\rangle$ and $d^2_{(x,\theta)}(\varphi_j^{(p)}-\psi)$ is non singular when $d_{(x,\theta)}(\varphi_j^{(p)}-\psi)=0$.*

Proof. It follows from Lemma 7.4 and (7.5) that $d^2_{(x,\theta)}(\varphi_j^{(p)}-\psi)$ is non singular if and only if the graph $(x, d_x\psi)$ and $\Lambda^{(p)}(t, t_0, y)$ intersect transversally at $(\hat{x}, \hat{\xi})$. The transversality means that

$$T_{(\hat{x},\hat{\xi})}(x, d_x\psi)\cap T_{(\hat{x},\hat{\xi})}(\Lambda^{(p)}(t, t_0; y))=\{0\},$$

where

$$T_{(\hat{x},\hat{\xi})}(x, d_x\psi)=\{(\delta_x, R\delta_x);\ \delta_x\in R^n\},$$

$$T_{(\hat{x},\hat{\xi})}(\Lambda^{(p)}(t, t_0; y))=\{(d_\xi\hat{x}^{(p)}(t, y, t_0, \xi)\delta_\xi, d_\xi\hat{\xi}^{(p)}(t, y, t_0, \xi)\delta_\xi), \delta_\xi\in R^n\}.$$

Hence the transversality implies that $Rd_\xi x^{(p)}-d_\xi\hat{\xi}^{(p)}$ is non singular. Since the rank of $(\hat{x}_\xi^{(p)}, \hat{\xi}_\xi^{(p)})$ is equal to $n$, we can find $R$ such that det $(Rd_\xi x^{(p)}-d_\xi\hat{\xi}^{(p)})\neq 0$.

Now we prove (7.4). Denote by $Q_j^{(p)}$ the matrix $d^2_{(x,\theta)}(\varphi_j^{(p)}-\psi)$. Let $(\hat{x}, \hat{\xi})=(\hat{x}^{(p)}(t, y, t_0, w), \hat{\xi}^{(p)}(t, y, t_0, w))$ and $\chi(x)\in C_0^\infty$, it's support contained in a neigborhood of $\hat{x}$. Then by virtue of the method of stationary phase, we obtain

$$\begin{aligned}
&\langle(exp -i\rho\psi)\chi, K_j^{(p)}(t, \cdot, t_0, y)\rangle\\
&=\iint\{exp\, i(\varphi_j^{(p)}(t, x, t_0, \theta, y)-\rho\psi(x))\}\chi(x)a_j^{(p)}(t, x, t_0, \theta)dxd\theta\\
&=\rho^{(j+1)n}\iint\{exp\, i(\varphi_j^{(p)}-\psi)\rho\}\chi(x)a_j^{(p)}(t, x, t_0, \rho\xi^{(j)}, y^{(j)}, \cdots, \rho\xi^{(0)})dxd\theta\\
&=\rho^{(j+1)n}\left\{\left(\frac{2\pi}{\rho}\right)^{2(j+1)n/2}|detQ_j^{(p)}|^{-1/2}(exp\, i(\pi/4)sgnQ_j^{(p)})a_j^{(p)}\right\}_{d_{(x,\theta)}(\varphi_j^{(p)}-\psi)=0}\\
&\quad+0(\rho^{-1}).
\end{aligned}$$

For $(x, \theta)$ such that $d_{(x,\theta)}(\varphi_j^{(p)}-\psi)=0$, that is, $(x, \psi_x)=\Lambda^{(p)}(t, t_0)(y, \omega)$, $(y^{(k)})=\Lambda^{(p)}(t_k, t_0)(y, \omega)(k=1, \cdots, j)$ and $\xi^{(0)}=\omega$, we have from (4.4) and $(4.13)_0$,

$$\begin{aligned}
a_j^{(p)}&=\{exp\, i(\pi/4)sgn\, Q_j^{(p)}\}\left|det\, Q_j^{(p)}\Delta^{(p)}(t)\prod_{k=1}^{j}\Delta^{(p)}(t_k)\right|^{-1/2}\\
&\times H^{(p)}(t, \hat{x}, \hat{\xi})J^{(p)}(t, t_0)G^{(p)}(t_0, y, \omega)\left(\frac{1}{2\pi}\right)^{(j+1)n}\\
&\neq 0.
\end{aligned}$$

Hence $(\hat{x}, \hat{\xi})\in WF(K_j^{(p)}(t, \cdot, t_0, y))$. Thus we have proved (7.2).

LEMMA 7.6. *Let $y$ be fixed in $R^n$ and $\delta>0$, small. Then for $p\neq q$ and $0<|\sigma-\tau|\leqslant\delta$, we have*

$$\Lambda^{(p)}(\sigma, \tau; y)\cap\Lambda^{(q)}(\sigma, \tau; y)=\phi \tag{7.6}$$

*and*

$$\Lambda^{(p)}(\sigma, \tau)\Lambda^{(q)}(\tau, \sigma; y)\cap\{(y, R^n\setminus 0)\}=\phi. \tag{7.7}$$

PROOF. Let $(\hat{x}, \hat{\xi})$ be in $\Lambda^{(p)}(\sigma, \tau, y)\cap\Lambda^{(q)}(\sigma, \tau, y)$, that is, $\hat{x}=x^{(p)}(\sigma, y, \tau, \omega)=\hat{x}^{(q)}(\sigma, y, \tau, \eta)$ and $\hat{\xi}=\hat{\xi}^{(p)}(\sigma, y, \tau, \omega)=\hat{\xi}^{(q)}(\sigma, y, \tau, \eta)$. On the other hand we have

$$\begin{aligned}
\frac{d}{dt}\hat{x}^{(p)}&=\lambda_\xi^{(p)}(t, \hat{x}^{(p)}(t), \hat{\xi}^{(p)}(t))\\
&=\lambda_\xi^{(p)}(\sigma, \hat{x}, \hat{\xi})+0(t-\sigma)
\end{aligned}$$

Hence $\hat{x}^{(p)}(\sigma)-y=\lambda_\xi^{(p)}(\sigma,\hat{x},\hat{\xi})(\sigma-\tau)+0(\sigma-\tau)^2$. Similarly we have

$$\hat{x}^{(q)}(\sigma)-y=\lambda_\xi^{(q)}(\sigma,\hat{x},\hat{\xi})(\sigma-\tau)+0(\sigma-\tau)^2$$

Since $\lambda_\xi^{(q)}(\sigma,\hat{x},\hat{\xi})\neq\lambda_\xi^{(p)}(\sigma,\hat{x},\hat{\xi})$, we have $\hat{x}^{(p)}(\sigma,y,\tau,\omega)\neq\hat{x}^{(q)}(\sigma,y,\tau,\eta)$ for $0<|\sigma-\tau|\leq\delta$, if $\delta$ is small. This is contradition. Put $\hat{x}^{(q)}(\tau)=\hat{x}^{(q)}(\tau,y,\sigma,\omega)$ and $\hat{\xi}^{(q)}(\tau)=\hat{\xi}^{(q)}(\tau,y,\sigma,\omega)$. Then we have

$$\begin{aligned}
&\hat{x}^{(p)}(\sigma,\hat{x}^{(q)}(\tau),\tau,\hat{\xi}^{(q)}(\tau))-y\\
&=\hat{x}^{(p)}(\sigma,\hat{x}^{(p)}(\tau),\tau,\hat{\xi}^{(q)}(\tau))-\hat{x}^{(q)}(\tau)+\hat{x}^{(q)}(\tau)-y\\
&=(\sigma-\tau)\lambda_\xi^{(p)}(\tau,\hat{x}^{(q)}(\tau),\hat{\xi}^{(q)}(\tau))+(\tau-\sigma)\lambda_\xi^{(q)}(\sigma,y,\omega)+0(\tau-\sigma)^2\\
&=(\sigma-\tau)(\lambda_\xi^{(p)}(\sigma,y,\omega)-\lambda_\xi^{(q)}(\sigma,y,\omega))+0(\sigma-\tau)^2\\
&\neq 0\,,
\end{aligned}$$

for $0<|\sigma-\tau|\leq\delta$, if $\delta$ is small. Thus we have proved (7.7).

PROPOSITION 7.7, ([14], [17]). *Let $u_0$ be is $\gamma_{2s-1}(R^n)$ and $f(t,x)$ be is $\gamma_{2s-1}$ with respect to $x$ and contineous with respect to $t$. Then a solution of the following equation is in $\gamma_{2s-1}(R^n)$ with respect to $x$,*

$$\begin{cases} L_{t,x}u=f\,,\\ u|_{t=\tau}=u_0\,.\end{cases}$$

PROOF. A solution $u$ can be written

$$u(t,x)=K(t,\tau)u_0(x)+\int_\tau^t K(t,\sigma)f(\sigma,x)d\sigma$$

which is in $\gamma_{2s-1}(R^n)$ with respect to $x$, from Proposition 6.8 and 6.9.

For $|t-\tau|\leqslant\delta$ and $|\tau-\sigma|\leqslant\delta$, we can write

$$\begin{aligned}
\text{(7.8)}\qquad K(t,x,\tau,y)&=K(t,x,\tau,\cdot)K(\tau,\cdot,\sigma,y)\,.\\
&=\sum_{p=1}^{l}\sum_{q=1}^{l}K^{(p)}(t,x,\tau,\cdot)K^{(q)}(\tau,\cdot,\sigma,y)\\
&\quad+K(t,x,\tau,\cdot)K^{(0)}(\tau,\cdot,\sigma,y)+K^{(0)}(t,x,\tau,\cdot)K(\tau,\cdot,\sigma,y).
\end{aligned}$$

here $K^{(p)}(t,\tau)=W^{(p)}(t,\tau), p=1,\cdots,l$ and $K^{(0)}(t,\tau)=W^{(0)}(t,\tau)+\int_\tau^t W(t,\sigma)F(\sigma,\tau)d\sigma$. Since $K^{(0)}(t,x,\tau,y)$ is in $\gamma_{2s-1}$ with respect to $x$ and $y$, it follows from Proposition 6.8 and 6.9 that the wave front sets in $\gamma_{2s-1}$ of $K^{(0)}(t,x,\tau,\cdot)K(\tau,\cdot,\sigma,y)$ and $K(t,x,\tau,\cdot)K^{(0)}(\tau,\cdot,\sigma,y)$ are empty. Hence we have

$$\text{(7.9)}\qquad K(t,x,\sigma,y)\equiv\sum_{p=1}^{l}\sum_{q=1}^{l}K^{(p)}(t,x,\tau,\cdot)K^{(q)}(\tau,\cdot,\sigma,y), (mod\,\gamma_{2s-1})$$

for $|t-\tau|\leqslant\delta$ and $|\tau-\sigma|\leqslant\delta$.

THEOREM 7.8. *For* $|t-\tau|\leqslant\delta$ *and* $|\tau-\sigma|\leqslant\delta$, *we have*

$$\tilde{K}_1^{(0)}(t, x, \sigma, y)=\sum_{p\neq q} K^{(p)}(t, x, \tau, \cdot)K^{(p)}(\tau, \cdot, \sigma, y)\equiv 0, (mod\ \gamma_{2s-1}). \tag{7.10}$$

PROOF. Since $L_{t,x}K^{(p)}(t, x, \tau, y)\equiv 0$ (mod $\gamma_{2s-1}$), we have

$$L_{t,x}(\tilde{K}_1^{(0)}(t, x, \sigma, y)\equiv 0 \ (\mathrm{mod}\ \gamma_{2s-1})$$

for $|t-\tau|\leqslant\delta$. Since $|\tau-\sigma|\leqslant\delta$, we can put $t=\sigma$. By virtue of Proposition 7.7 it suffices to prove

$$\tilde{K}_1^{(0)}(\sigma, x, \sigma, y)\equiv 0 \ (\mathrm{mod}\ \gamma_{2s-1}).$$

Then we have from (7.8)

$$\tilde{K}_1^{(0)}(\sigma, x, \sigma, y)\equiv\delta(x-y)-\sum_{p=1}^{l} K^{(p)}(\sigma, x, \tau, \cdot)K^{(p)}(\tau, \cdot, \sigma, y) \ (\mathrm{mod}\ \gamma_{2s-1})$$

Hence it follows from Corollary 7.2 that

$$WF_{2s-1}(\tilde{K}_1^{(0)}(\sigma, \cdot, \sigma, y))\subset\{(y, \xi);\ \xi\in R^n\diagdown 0\}.$$

On the other hand it follows from Theorem 7.1, that the wave front set in $\gamma_{2s-1}$ of $K^{(p)}(\sigma, x, \sigma, \cdot)K^{(q)}(\tau, \cdot, \sigma, y)$ is contained in $\Lambda^{(p)}(\sigma, \tau)\Lambda^{(q)}(\tau, \sigma; y)$. Hence we have

$$WF_{2s-1}(\tilde{K}_1^{(0)}(\sigma, \cdot, \sigma, y))\subset\bigcup_{p\neq q}\Lambda^{(p)}(\sigma, \tau)\Lambda^{(q)}(\tau, \sigma; y).$$

From Lemma 7.6 it follows that

$$\bigcup_{p\neq q}\Lambda^{(p)}(\sigma, \tau)\Lambda^{(q)}(\tau, \sigma; y)\cap\{(y, R^n\diagdown 0)\}=\phi.$$

Hence we obtain (7.10).

COROLLARY 7.9. *For* $0<\tau-\sigma\leqslant\delta$ *and* $\delta\leqslant t\leqslant\tau$, *we have*, $p=1, \cdots, l$,

$$K^{(p)}(t, x, \tau, \cdot)K^{(p)}(\tau, \cdot, \sigma, y)\equiv K^{(p)}(t, x, \sigma, y)(mod\ \gamma_{2s-1}). \tag{7.11}$$

PROOF. It suffices to prove (7.11) for $t=\tau$. Then from (7.9) and (7.10) we obtain

$$\sum_{q=1}^{l}(K^{(q)}(\tau, x, \tau, \cdot)K^{(q)}(\tau, \cdot, \sigma, y)\equiv\sum_{q=1}^{l}K^{(q)}(\tau, x, \sigma, y).$$

Hence

$$K^{(p)}(\tau, x, \tau, \cdot)K^{(p)}(\tau, \cdot, \sigma, y)-K^{(p)}(\tau, x, \sigma, y)$$
$$\equiv\sum_{q\neq p}K^{(q)}(\tau, x, \tau, \cdot)K^{(q)}(\tau, \cdot, \sigma, y)-K^{(q)}(\tau, x, \sigma, y).$$

It follows from (7.6) that

$$\Lambda^{(p)}(\tau, \sigma; y) \cap \{\bigcup_{q \neq p} \Lambda^{(q)}(\tau, \sigma; y)\} = \phi ,$$

which implies (7.11).

We put, $\sigma \leq t \leq \tau$,

$$S^{(p)}(t, x, \sigma, y) = K^{(p)}(t, x, \tau, \cdot) K^{(p)}(\tau, \cdot, \sigma, y) - K^{(p)}(t, x, \sigma, y)$$

$$= \iiint \{exp\, i\langle \tilde{z}^{(p)}(t, x, \tau, \xi) - z, \xi) - z\rangle + i\langle \tilde{z}^{(p)}(\tau, z, \sigma, \eta) - y, \eta\rangle\}$$

$$\times w^{(p)}(t, x, \tau, \xi) w^{(p)}(\tau, z, \sigma, \eta) d\xi dz d\eta$$

$$- \int \{exp\, i\langle \tilde{z}^{(p)}(t, x, \sigma, \xi) - y, \xi\rangle\} w^{(p)}(t, x, \sigma, \xi) d\xi ,$$

PROPOSITION 7.10. *Let $u$ be in $S'(R^n)$. Then $WF_{2s-1}(S^{(p)}(t, \sigma)u) = \phi, p = 1, \cdots, l$.*

PROOF. We put

$$I_N(y, \zeta) = \int (exp\ -i\langle x, \zeta\rangle) \chi_N(x) S^{(p)}(t, x, \sigma, y) dx .$$

which satisfies

$$|D_y{}^\alpha I_N(y, \zeta)| \leqslant C_{\alpha m} |\zeta|^{-N} A^N N!^{2s-1} (1 + |y|)^{-m},\ |\alpha| \leqslant m, \tag{7.12}$$

for any positive integer $m$. For, it is true for $|y| \leqslant r$, $r$ is a positive constant. If $r$ is suitablly large, for $|y| \geq r$ and for $x \in \operatorname{supp} \chi_N$, we have

$$d_{(x, \xi, z, \eta)}(\langle \tilde{z}^{(p)}(t, x, \tau, \xi) - z, \xi\rangle + \langle \tilde{z}^{(p)}(\tau, z, \sigma, \eta) - y, \eta\rangle - \langle x, \tilde{\zeta}\rangle) \neq 0 ,$$

$$d_{(x, \xi)}(\langle \tilde{z}^{(p)}(t, x, \sigma, \xi) - y, \xi\rangle - \langle x, \tilde{\zeta}\rangle) \neq 0 ,$$

where $\tilde{\zeta} = \zeta / |\zeta|$. So we obtain (7.12) by part of integration. Hence

$$|\langle I_N(\cdot, \zeta), u\rangle| \leq C_m \sup_y (1 + |y|)^m \sum_{|\alpha| \leq m} |D_y{}^\alpha I_N(y, \zeta)|$$

$$\leq C_m |\zeta|^{-N} A^N N^{2s-1} .$$

when $m$ is the order of the distribution $u$.

Now we turn to prove (7.3) by induction with respect to $j$. It is true for $j=1$ from Theorem 7.8. Assume that (7.3) is valid for $j-1$. By virtue of Proposition 7.7 it suffices to prove that (7.3) is valid for $t = t_j$. For, $L_{t,x} K_j{}^{(0)}(t, x, t_0, y) = 0$, for $t_j \leq t \leq t_{j+1}$. We have from (7.1),

$$K_j{}^{(0)}(t_j, t_0) = K(t_j, t_0) - \sum_{p=1}^{l} K_j{}^{(p)}(t, t_0)$$

$$\equiv \sum_{p=1}^{l} K_{j-1}^{(p)}(t_j, t_0) - K_j^{(p)}(t_j, t_0) \pmod{\gamma_{2s-1}}$$

$$= -\sum_{p=1}^{l} \{K^{(p)}(t_j, t_j) K^{(p)}(t_j, t_{j-1}) - K^{(p)}(t_j, t_{j-1})\} K_{j-2}^{(p)}(t_{j-1}, t_0)$$

$$= -\sum_{p=1}^{l} S^{(p)}(t_j, t_{j-1}) K_{j-2}^{(p)}(t_{j-1}, t_0)$$

of which wave front set in $\gamma_{2s-1}$ is empty from Proposition 7.10.

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