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ON THE NILPOTENCY INDICES OF THE RADICALS OF GROUP ALGEBRAS OF P-GROUPS WHICH HAVE CYCLIC SUBGROUPS OF INDEX P

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Let K be a field with characteristic $p>0$, G a finite group, KG the group algebra of G over K and $J(KG)$ the radical of KG. We are interested in relations between ring-theoretical properties of KG and the structure of G. Particularly, in the present paper we shall study the nilpotency index $t(G)$ of $J(KG)$, which is the least positive integer $t(G)$ such that $J(KG)^{t(G)}=0$.

For a finite p-group P of order p^r, S. A. Jennings [3] showed that $r(p-1)+1\leqq t(P)\leqq p^r$. Recently K. Motose and Y. Ninomiya [7] determined all p-groups P of order p^r such that $t(P)$ are the lower bound $r(p-1)+1$ or the upper bound p^r. In fact they proved that for a p-group P of order p^r with $r\geqq 1$, $t(P)=r(p-1)+1$ if and only if P is elementary abelian and that $t(P)=p^r$ if and only if P is cyclic. So in this paper we shall investigate p-groups P of order p^r such that $t(P)$ are not necessarily equal to the lower bound $r(p-1)+1$ or the upper bound p^r. By the results of K. Motose [6, Theorem], K. Motose and Y. Ninomiya [7, Theorem 1] it follows that when P is an abelian p-group of order p^r with $r\geqq 2$, the secondarily highest nilpotency index $t(P)$ of $J(KP)$ is $p^{r-1}+p-1$ and in this case P is not cyclic and has a cyclic subgroup of index p. Our main result of §1 is a generalization of the above fact. This can be stated as follows: For an arbitrary p-group P of order p^r with $r\geqq 2$, the next conditions are equivalent;

(i) $t(P)=p^{r-1}+p-1$.

(ii) $p^{r-1}<t(P)<p^r$.

(iii) P is not cyclic and has a cyclic subgroup of index p.

There is a problem that when the value of $t(G)$ is given, what type is G? About this there are some solutions ([9], [7]). D. A. R. Wallace [9] determined all finite groups G with the property $t(G)=2$. Further, K. Motose and Y. Ninomiya [7] determined all finite p-solvable groups G such that $t(G)=3$. In connection with this in §2 we shall have all p-groups P such that $t(P)=4, 5$ or 6 by calculating

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$t(Q)$ for all p-groups Q of orders at most p^4.

1. p-Groups which have cyclic subgroups of index p

To begin with we shall study $t(P)$ for metacyclic p-groups P.

LEMMA 1.1. *Let P be a metacyclic p-group containing a cyclic normal subgroup $Q=\langle b\rangle$ of order p^n and with a cyclic factor group $P/Q=\langle aQ\rangle$ $(a\in P)$ of order p^m. Put $x=a-1$ and $y=b-1$ in KP. Then*

$$y^t x^s \in \sum_{\substack{i+j\geqq s+t\\ 0\leqq i\leqq s}} Kx^i y^j, \qquad \text{for all} \quad s, t\geqq 0.$$

PROOF. We may assume $n\geqq 1$. There is a positive integer h such that

$$(1) \qquad a^{-1}ba=b^h.$$

Since $a^{p^m}\in Q$, $h^{p^m}\equiv 1 \pmod{p^n}$, and so

$$(1') \qquad h\equiv 1 \pmod{p}.$$

At first we shall prove this lemma for $s=1$ and $t=1$ (cf. the proof of [4, Lemma]). Put $\binom{i}{j}=0$ if $i<j$. By (1) and (1′),

$$\begin{aligned} yx &= ab^h-a-b+1=(x+1)\Big(\sum_{j\geqq 2}\tbinom{h}{j}y^j+y+1\Big)-x-y-1 \\ &= xy+\sum_{j\geqq 2}\tbinom{h}{j}(x+1)y^j. \end{aligned}$$

This shows

$$(2) \qquad yx\in \sum_{\substack{i+j\geqq 2\\ 0\leqq i\leqq 1}} Kx^i y^j.$$

From (2), we can prove

$$(3) \qquad y^t x\in \sum_{\substack{i+j\geqq t+1\\ 0\leqq i\leqq 1}} Kx^i y^j, \qquad \text{for all} \quad t\geqq 0$$

by induction on t. Using (3) we can verify this lemma by induction on s.

Put $J(KP)^0=KP$ for a p-group P.

THEOREM 1.2. *Let P be a metacyclic p-group containing a cyclic normal subgroup Q of order p^n and with a cyclic factor group $P/Q=\langle aQ\rangle$ $(a\in P)$ of order p^m and k an integer such that $|a|=p^{m+n-k}$. Put*

$$h=\begin{cases} m, & \text{if } m\leqq k\\ k, & \text{if } m>k. \end{cases}$$

Then we have $t(P)=p^{m+n-h}+p^h-1$.

PROOF. Put $Q=\langle b\rangle$. We can assume $a^{p^m}=b^{p^k}$. Set x and y as in Lemma 1.1.

Case 1. $m\leqq k$: We shall claim that $C_i=\{x^s y^t \mid 0\leqq s\leqq p^m-1, 0\leqq t\leqq p^n-1, s+t\geqq i\}$ is a K-basis of $J(KP)^i$ by induction on i. Every $g\in P$ can be written as $g=a^s b^t$, $0\leqq s\leqq p^m-1$, $0\leqq t\leqq p^n-1$ and the number of elements of C_0 is p^{m+n}. Thus C_0 is a K-basis of KP. By [3, Theorem 1.2], C_1 is a K-basis of $J(KP)$. Assume $i\geqq 2$. Since $x, y\in J(KP)$, we have $C_i\subseteqq J(KP)^i$. Since $J(KP)^i=J(KP)J(KP)^{i-1}$, it suffices to prove that if $0\leqq s, s'\leqq p^m-1$, $0\leqq t, t'\leqq p^n-1$, $s+t\geqq 1$ and $s'+t'\geqq i-1$, then $(x^s y^t)(x^{s'}y^{t'})$ can be written as a K-linear combination of C_i. From Lemma 1.1,

$$(4)\qquad (x^s y^t)(x^{s'}y^{t'})=\sum_{\substack{i'+j'\geqq s'+t\\ 0\leqq i'\leqq s'}} a_{i'j'}x^{s+i'}y^{j'+t'},\qquad a_{i'j'}\in K.$$

Consider each term of (4). Put $s+i'=up^m+u'$, where u, u' are integers with $0\leqq u'\leqq p^m-1$. Since $x^{p^m}=y^{p^k}$, it is seen that $x^{s+i'}y^{j'+t'}=x^{u'}(x^{p^m})^u y^{j'+t'}=x^{u'}y^{up^k+j'+t'}$. Since $y^{p^n}=0$, we can put $up^k+j'+t'\leqq p^n-1$. We also have $u'+(up^k+j'+t')\geqq i$ since $k\geqq m$ and $i'+j'\geqq s'+t$. Hence (4) can be written as a K-linear combination of C_i. This shows $J(KP)^{p^m+p^n-2}$ is of K-dimension one, and so $t(P)=p^m+p^n-1$.

Case 2. $m>k$: As in Case 1 we can show that $C_i=\{x^s y^t \mid 0\leqq s\leqq p^{m+n-k}-1, 0\leqq t\leqq p^k-1, s+t\geqq i\}$ is a K-basis of $J(KP)^i$. Thus $t(P)=p^{m+n-k}+p^k-1$. This completes the proof of Theorem 1.2.

Put that

$$D_r=\langle a, b \mid a^2=b^{2^{r-1}}=1, a^{-1}ba=b^{-1}\rangle \qquad \text{for} \quad r\geqq 3,$$

$$Q_r=\langle a, b \mid a^2=b^{2^{r-2}}, a^4=1, a^{-1}ba=b^{-1}\rangle \qquad \text{for} \quad r\geqq 3,$$

$$S_r=\langle a, b \mid a^2=b^{2^{r-1}}=1, a^{-1}ba=b^{2^{r-2}-1}\rangle \qquad \text{for} \quad r\geqq 4,$$

$$M_r(p)=\langle a, b \mid a^p=b^{p^{r-1}}=1, a^{-1}ba=b^{p^{r-2}+1}\rangle$$
$$\text{for} \quad r\geqq 4 \text{ if } p=2, \qquad \text{and for} \quad r\geqq 3 \text{ if } p\geqq 3,$$

$$M(p)=\langle a, b, c \mid a^p=b^p=c^p=1, a^{-1}ba=bc, a^{-1}ca=c, b^{-1}cb=c\rangle \qquad \text{for} \quad p\geqq 3.$$

LEMMA 1.3. *Let P be a p-group of order p^r. If P is not cyclic and has a cyclic subgroup of index p, $t(P)=p^{r-1}+p-1$.*

PROOF. This follows from [2, I 14.9 Satz] and Theorem 1.2.

Next, we shall compute $t(M(p))$ whose calculation is very fundamental in calculating $t(P)$ for the other p-groups P.

LEMMA 1.4. *For $p\geqq 3$, $t(M(p))=4p-3$.*

PROOF. Put $P=M(p)$. As in Lemma 1.1 set that $x=a-1$, $y=b-1$ and $z=$

$c-1$ in KP. Note that $x^p=y^p=z^p=0$ and $x, y, z\in J(KP)$. We have $zx=xz, zy=yz$ and $yx=xyz+xy+yz+xz+z$. Hence we know

$$(5)\qquad z\in J(KP)^2,$$

$$(6)\qquad yx\in \sum_{\substack{i+j+2k\geqq 2\\ 0\leqq i\leqq 1}} Kx^iy^jz^k.$$

Using (6) we can show

$$(7)\qquad y^tx\in \sum_{\substack{i+j+2k\geqq t+1\\ 0\leqq i\leqq 1}} Kx^iy^jz^k, \quad \text{for all } t\geqq 0$$

by induction on t as in the proof of (3). From (7) we obtain

$$(8)\qquad y^tx^s\in \sum_{\substack{i+j+2k\geqq s+t\\ 0\leqq i\leqq s}} Kx^iy^jz^k, \quad \text{for all } s, t\geqq 0$$

by induction on s. Next, we shall show that $C_i=\{x^sy^tz^u|0\leqq s, t, u\leqq p-1, s+t+2u\geqq i\}$ is a K-basis of $J(KP)^i$ by induction on i. For $i=0$ or 1, it is easy as in the proof of Theorem 1.2. Assume $i\geqq 2$. By (5), $C_i\subseteqq J(KP)^i$. As in the proof of Theorem 1.2 it is sufficient to prove that $(x^sy^tz^u)(x^{s'}y^{t'}z^{u'})$ can be written as a K-linear combination of C_i when $0\leqq s, s', t, t', u, u'\leqq p-1, s+t+2u\geqq 1$ and $s'+t'+2u'\geqq i-1$. By (8),

$$(*)\qquad (x^sy^tz^u)(x^{s'}y^{t'}z^{u'})=\sum_{\substack{i'+j'+2k'\geqq s'+t\\ 0\leqq i'\leqq s'}} a_{i'j'k'}x^{s+i'}y^{j'+t'}z^{k'+u+u'},$$

$a_{i'j'k'}\in K$. Since $x^p=y^p=z^p=0$, we can assume that $0\leqq s+i', j'+t', k'+u+u'\leqq p-1$. We have $(s+i')+(j'+t')+2(k'+u+u')\geqq i$. Thus C_i is a K-basis of $J(KP)^i$, and so $t(P)=(p-1)+(p-1)+2(p-1)+1=4p-3$.

LEMMA 1.5. *Let P be a p-group of order p^r with $r\geqq 1$. If $t(P)>p^{r-1}$, then P has an element of order p^{r-1}.*

PROOF. We use induction on r. It is clear for $r=1$ or 2. Assume $r=3$. When P is abelian, it follows from [6, Theorem]. When P is nonabelian, by [2, I 14.10 Satz], P is one of the following types;

(i) $p=2$ and $P\cong D_3$ or Q_3,

(ii) $p\geqq 3$ and $P\cong M_3(p)$ or $M(p)$.

By Lemma 1.4 and $t(P)>p^2, P\ncong M(p)$. Thus the assertion is proved for $r=3$. Assume $r\geqq 4$. There is an element $c\in Z(P)$ of order p, where $Z(P)$ is the center of P. $C=\langle c\rangle$ is normal in P. By [10, Theorem 2.4] and $t(P)>p^{r-1}$, it follows that $t(P/C)>p^{r-2}$. Thus, from the hypothesis of induction, P/C has an element bC ($b\in P$) of order p^{r-2}. Now, suppose that P has no elements of order p^{r-1}. Hence $B=\langle b\rangle$

has order p^{r-2}. By [2, I 14.9 Satz], P/C is one of the following types;

Case 1. P/C is an abelian group of type (p^{r-2}, p),

Case 2. $p=2$ and $P/C\cong D_{r-1}$,

Case 3. $p=2$ and $P/C\cong Q_{r-1}$,

Case 4. $p=2$, $r\geqq 5$ and $P/C\cong S_{r-1}$,

Case 5. $r\geqq 5$ if $p=2$, and $P/C\cong M_{r-1}(p)$.

Case 1: Put $P/C=\langle aC, bC\mid (aC)^p=(bC)^{p^{r-2}}=C, abC=baC\rangle$ and $A=\langle a\rangle$. Clearly $|a|=p$ or p^2. If $|a|=p^2$, we may put $a^p=c$. Since P/C is abelian, A is normal in P. Thus P is a semi-direct product of A by B, and so $t(P)=p^{r-2}+p^2-1$ from Theorem 1.2. This is a contradiction, and so $|a|=p$. If $b^{-1}a^{-1}ba=1$, P is an abelian group of type (p^{r-2}, p, p). Hence, by [6, Theorem], $t(P)=p^{r-2}+2p-2$, a contradiction. This shows that $b^{-1}a^{-1}ba\neq 1$, and so we may put $b^{-1}a^{-1}ba=c$. Thus $P=\langle a, b, c\mid a^p =b^{p^{r-2}}=c^p=1, a^{-1}ba=bc, a^{-1}ca=c, b^{-1}cb=c\rangle$. Just as in the proof of Lemma 1.4, it is seen $t(P)=(p-1)+(p^{r-2}-1)+2(p-1)+1=p^{r-2}+3p-3$, a contradiction.

Case 2: Put $p=2, P/C=\langle aC, bC\mid (aC)^2=(bC)^{2^{r-2}}=C,\ a^{-1}baC=b^{-1}C\rangle$ and $A=\langle a\rangle$. We know $|a|=2$ or 4. Put x, y and z as in the proof of Lemma 1.4.

(i) Put $|a|=2$ and $ba^{-1}ba=1$. Then P is a direct product of $AB\cong D_{r-1}$ and a cyclic group of order 2. It follows from [6, Theorem] and Lemma 1.3 that $t(P) =2^{r-2}+2$, a contradiction.

(ii) Put $|a|=4$ and $ba^{-1}ba=1$. Since $a^2=c, P=\langle a, b\mid a^4=b^{2^{r-2}}=1, a^{-1}ba=b^{-1}\rangle$. Thus, by Theorem 1.2, $t(P)=2^{r-2}+3$. This is a contradiction.

(iii) Put $|a|=2$ and $ba^{-1}ba\neq 1$. Then $ba^{-1}ba=c$. So $P=\langle a, b, c\mid a^2=b^{2^{r-2}}=c^2= 1, a^{-1}ba=b^{-1}c, a^{-1}ca=c, b^{-1}cb=c\rangle$. We have $zx=xz$ and $zy=yz$. Set $f=2^{r-2}-1$. Since $f\equiv 1 \pmod 2$, $yx=(x+1)(y+1)^f(z+1)-x-y-1=(x+1)\{\sum_{j=2}^{f}\binom{f}{j}y^j\}(z+1)+xyz+xy+yz +xz+z$. Hence we have (5) and (6), and so we have (7) and

$$(8')\qquad y^t x^s \in \sum_{\substack{i+j+2k\geqq s+t\\ 0\leqq i\leqq s}} Kx^i y^j z^k,\quad \text{for all } t\geqq 0 \text{ and } s=0,1.$$

As in the proof of Lemma 1.4, $t(P)=1+(2^{r-2}-1)+2+1=2^{r-2}+3$, a contradiction.

(iv) Put $|a|=4$ and $ba^{-1}ba\neq 1$. Then $a^2=c$ and $ba^{-1}ba=c$. Hence $P=\langle a, b, c\mid a^2 =c, b^{2^{r-2}}=c^2=1, a^{-1}ba=b^{-1}c, a^{-1}ca=c, b^{-1}cb=c\rangle$. Note $x^2=z\neq 0$ and $y^{2^{r-2}}=z^2=0$. As (iii) we obtain (5) and (6), and so (7) and (8′) hold. We shall show that $C_i=\{x^s y^t z^u\mid 0\leqq s, u\leqq 1, 0\leqq t\leqq 2^{r-2}-1, s+t+2u\geqq i\}$ is a K-basis of $J(KP)^i$ by induction on i. It is clear for $i=0$ or 1. Assume $i\geqq 2$. By (5), $C_i\subseteq J(KP)^i$. As usual it suffices to show that $(x^s y^t z^u)(x^{s'} y^{t'} z^{u'})$ can be written as a K-linear combination of C_i if $0\leqq s, s', u, u' \leqq 1, 0\leqq t, t'\leqq 2^{r-2}-1, s+t+2u\geqq 1$ and $s'+t'+2u'\geqq i-1$. From (8′), we have (*). Consider each term of (*). Put $s+i'=2v+v'$, where v, v' are integers with $0\leqq v'\leqq 1$. Since $x^2=z, x^{s+i'}y^{j'+t'}z^{k'+u+u'}=x^{v'}y^{j'+t'}z^{v+k'+u+u'}$. We may assume $j'+t'\leqq 2^{r-2}-1$ and

$v+k'+u+u'\leqq 1$ since $y^{2^{r-2}}=z^2=0$. We also have $v'+(j'+t')+2(v+k'+u+u')\geqq i$. This implies that C_i is a K-basis of $J(KP)^i$, and so $t(P)=1+(2^{r-2}-1)+2+1=2^{r-2}+3$, a contradiction.

Case 3: Put $p=2$, $P/C=\langle aC, bC|(aC)^2=(bC)^{2^{r-3}}, (aC)^4=C, a^{-1}baC=b^{-1}C\rangle$ and $A=\langle a\rangle$. We can put $a^2=b^{2^{r-3}}c^i$ for some i, and so $a^4=1$. This implies $|a|=4$. Put x, y and z as in the proof of Lemma 1.4.

(i) Put $ba^{-1}ba=1$ and $a^2=b^{2^{r-3}}$. Then P is a direct product of $AB\cong Q_{r-1}$ and a cyclic group of order 2. Thus we have a contradiction as in (i) of Case 2.

(ii) Put $ba^{-1}ba=1$ and $a^2\neq b^{2^{r-3}}$. Then $A\cap B=1$. Hence $P=AB=\langle a, b|a^4=b^{2^{r-2}}=1, a^{-1}ba=b^{-1}\rangle$, and so we have a contradiction as in (ii) of Case 2.

(iii) Put $ba^{-1}ba\neq 1$ and $a^2=b^{2^{r-3}}$. Since $ba^{-1}ba=c$, $P=\langle a, b, c|a^2=b^{2^{r-3}}, b^{2^{r-2}}=c^2=1, a^{-1}ba=b^{-1}c, a^{-1}ca=c, b^{-1}cb=c\rangle$. As in (iii) of Case 2 we have (5), (6), (7) and (8′). Note $0\neq x^2=y^{2^{r-3}}$. By $2^{r-3}\geqq 2$, it is seen that $C_i=\{x^sy^tz^u|0\leqq s, u\leqq 1, 0\leqq t\leqq 2^{r-2}-1, s+t+2u\geqq i\}$ is a K-basis of $J(KP)^i$ as in (iv) of Case 2. Thus $t(P)=2^{r-2}+3$, a contradiction.

(iv) Put $ba^{-1}ba\neq 1$ and $a^2\neq b^{2^{r-3}}$. Hence $ba^{-1}ba=c$ and $a^2=b^{2^{r-3}}c$. Thus $P=\langle a, b, c|a^2=b^{2^{r-3}}c, a^4=b^{2^{r-2}}=c^2=1, a^{-1}ba=b^{-1}c, a^{-1}ca=c, b^{-1}cb=c\rangle$. We have $x^2=y^{2^{r-3}}(z+1)+z$. This implies (5) and

$$x^2\in\sum_{j+2k\geqq 2}Ky^jz^k. \tag{9}$$

As in (iii) of Case 2 we also have (6), (7) and (8′). Note $x^2\neq 0$. By (9), as in (iv) of Case 2, we know that $C_i=\{x^sy^tz^u|0\leqq s, u\leqq 1, 0\leqq t\leqq 2^{r-2}-1, s+t+2u\geqq i\}$ is a K-basis of $J(KP)^i$, and so we have a contradiction.

Case 4: As in Case 2 we have a contradiction.

Case 5: Put $r\geqq 5$ if $p=2$, and put $P/C=\langle aC, bC|(aC)^p=(bC)^{p^{r-2}}=C, a^{-1}baC=b^{p^{r-3}+1}C\rangle$ and $A=\langle a\rangle$. Set x, y and z as in the proof of Lemma 1.4. Put $f=p^{r-3}+1$, and so $f\equiv 1 \pmod p$.

(i) Assume $|a|=p$ and $b^{-f}a^{-1}ba=1$. So P is a direct product of $AB\cong M_{r-1}(p)$ and a cyclic group of order p, and so we have a contradiction by [6, Theorem] and Lemma 1.3.

(ii) Assume $|a|=p^2$ and $b^{-f}a^{-1}ba=1$. We may put $a^p=c$. So $P=\langle a, b|a^{p^2}=b^{p^{r-2}}=1, a^{-1}ba=b^f\rangle$, hence $t(P)=p^{r-2}+p^2-1$, by Theorem 1.2. This is a contradiction.

(iii) Assume $|a|=p$ and $b^{-f}a^{-1}ba\neq 1$. We can put $b^{-f}a^{-1}ba=c$. Hence $P=\langle a, b, c|a^p=b^{p^{r-2}}=c^p=1, a^{-1}ba=b^fc, a^{-1}ca=c, b^{-1}cb=c\rangle$. Since $f\equiv 1 \pmod p$, as (iii) of Case 2, we have (5), (6), (7) and (8). As in the proof of Lemma 1.4, $C_i=\{x^sy^tz^u|0\leqq s, u\leqq p-1, 0\leqq t\leqq p^{r-2}-1, s+t+2u\geqq i\}$ is a K-basis of $J(KP)^i$, and so $t(P)=(p-1)+(p^{r-2}-1)+2(p-1)+1=p^{r-2}+3p-3$, a contradiction.

(iv) Assume $|a|=p^2$ and $b^{-f}a^{-1}ba\neq 1$. We may put $a^p=c$. Since $1\neq b^{-f}a^{-1}ba\in C$, $b^{-f}a^{-1}ba=c^h$ for some h with $1\leqq h\leqq p-1$. Thus $P=\langle a,b,c\,|\,a^p=c, b^{p^{r-2}}=c^p=1, a^{-1}ba=b^fc^h, a^{-1}ca=c, b^{-1}cb=c\rangle$. From $x^p=z$,

(10) $$z\in J(KP)^p\,.$$

Since $$f\equiv 1 \pmod{p},\ yx=\sum_{\substack{i+j+k\geqq 2\\ 0\leqq i\leqq 1}} a_{ijk}x^iy^jz^k+hz,\quad a_{ijk}\in K\,.$$

Hence

(11) $$yx\in \sum_{\substack{i+j+pk\geqq 2\\ 0\leqq i\leqq 1}} Kx^iy^jz^k\,.$$

Using this, as in the proof of Lemma 1.1, by induction we have

(12) $$y^tx\in \sum_{\substack{i+j+pk\geqq t+1\\ 0\leqq i\leqq 1}} Kx^iy^jz^k,\quad \text{for all } t\geqq 0\,,$$

(13) $$y^tx^s\in \sum_{\substack{i+j+pk\geqq s+t\\ 0\leqq i\leqq s}} Kx^iy^jz^k,\quad \text{for all } s,t\geqq 0\,.$$

Note $0\neq x^p=z$. It follows from (10) and (13) that $C_i=\{x^sy^tz^u\,|\,0\leqq s, u\leqq p-1, 0\leqq t\leqq p^{r-2}-1, s+t+pu\geqq i\}$ is a K-basis of $J(KP)^i$, and so $t(P)=(p-1)+(p^{r-2}-1)+p(p-1)+1=p^{r-2}+p^2-1$, a contradiction. This completes the proof of Lemma 1.5.

THEOREM 1.6. *Let P be a p-group of order p^r. If $r\geqq 2$, then the next four conditions* (i)-(iv) *are equivalent.*

(i) $t(P)=p^{r-1}+p-1$.

(ii) $p^{r-1}<t(P)<p^r$.

(iii) *P is not cyclic and has a cyclic subgroup of index p.*

(iv) *P is one of the following types;*

(a) *P is an abelian group of type (p^{r-1},p),*

(b) *$p=2, r=3$ and $P\cong D_3$ or Q_3,*

(c) *$p=2, r\geqq 4$ and $P\cong D_r, Q_r, S_r$ or $M_r(2)$,*

(d) *$p\geqq 3, r\geqq 3$ and $P\cong M_r(p)$.*

PROOF. (i)⇒(ii) is clear. (ii)⇒(iii) is obtained from [7, Theorem 1] and Lemma 1.5. (iii)⇒(iv) follows from [2, I 14.9 Satz]. (iv)⇒(i) is easy from Lemma 1.3.

COROLLARY 1.7. *Let G be a finite group with a p-Sylow subgroup P. If G is a p-solvable group of p-length 1 and P has order p^r with $r\geqq 2$, then the next four conditions are equivalent.*

(i) $t(G)=p^{r-1}+p-1$.

(ii) $p^{r-1}<t(G)<p^r$.

(iii) *Same as* (iii) *of Theorem* 1.6.

(iv) *Same as* (iv) *of Theorem* 1.6.

PROOF. It follows from [5, Theorems 2 and 7] (or [1, Theorem 2]) and [8, Lemma 2] that $t(G)=t(P)$. Thus this corollary is clear by Theorem 1.6.

REMARK 1. For a p-solvable group G of p-length $\geqq 2$, the same statement as Corollary 1.7 does not necessarily hold. Let G be the symmetric group of degree 4 and $p=2$. Then G is a 2-solvable group of 2-length 2 of order 24 with a dihedral 2-Sylow subgroup of order 8. On the other hand, by [7, Proposition], $t(G)=4\neq 2^2+2-1$.

2. p-Groups P with $t(P)=4$, 5 or 6

In this section, firstly, we shall compute $t(P)$ for all p-groups P of orders at most p^4. Using this we shall have all p-groups P such that $t(P)=4$, 5 or 6. All p-groups of order p^3 are found in [2, I 14.10 Satz] and all p-groups of order p^4 are found in [2, III 12.6 Satz] and [2, III § 12 Aufgaben (29), (30)].

THEOREM 2.1. *Let P be a nonabelian p-group of order p^r. Then we have the followings.*

(I) $r=3, p\geqq 3$. *There are two nonisomorphic nonabelian groups of order p^3.*

(i) *If* $P=M_3(p), t(P)=p^2+p-1$.

(ii) *If* $P=M(p), t(P)=4p-3$.

(II) $r=3, p=2$. *There are two nonisomorphic nonabelian groups of order 2^3.*

(i)-(ii) *If* $P=D_8$ *or* Q_8, $t(P)=5$.

(III) $r=4, p\geqq 5$. *There are ten nonisomorphic nonabelian groups of order p^4.*

(i) *If* $P=M_4(p), t(P)=p^3+p-1$.

(ii) *If P is a direct product of* $M_3(p)$ *and a cyclic group of order* $p, t(P)=p^2+2p-2$.

(iii) *If P is a direct product of* $M(p)$ *and a cyclic group of order* p, $t(P)=5p-4$.

(iv) *If* $P=\langle a, b \mid a^{p^2}=b^{p^2}=1, a^{-1}ba=b^{p+1}\rangle$, $t(P)=2p^2-1$.

(v) *If* $P=\langle a, b, c \mid a^p=b^p=c^{p^2}=1, a^{-1}ba=bc^p, a^{-1}ca=c, b^{-1}cb=c\rangle, t(P)=p^2+2p-2$.

(vi) *If* $P=\langle a, b, c \mid a^p=b^p=c^{p^2}=1, a^{-1}ba=b, a^{-1}ca=bc, b^{-1}cb=c\rangle, t(P)=p^2+3p-3$.

(vii) *If* $P=\langle a, b, c \mid a^p=b^p=c^{p^2}=1, a^{-1}ba=bc^p, a^{-1}ca=bc, b^{-1}cb=c\rangle, t(P)=p^2+3p-3$.

(viii) *If* $P=\langle a, b, c \mid a^p=b^p=c^{p^2}=1, a^{-1}ba=bc^{fp}, a^{-1}ca=bc, b^{-1}cb=c\rangle$, *where f is a quadratic nonresidue modulo p*, $t(P)=p^2+3p-3$.

(ix) *If* $P=\langle a,b,c,d\,|\,a^p=b^p=c^p=d^p=1, b^{-1}cb=c, c^{-1}dc=d, b^{-1}db=d, a^{-1}ba=b, a^{-1}ca=bc, a^{-1}da=cd\rangle, t(P)=7p-6$.

(x) *If* $P=\langle a,b,c,d\,|\,a^p=b, b^p=c^p=d^p=1, b^{-1}cb=c, c^{-1}dc=d, b^{-1}db=d, a^{-1}ca=bc, a^{-1}da=cd\rangle, t(P)=p^2+3p-3$.

(IV) $r=4, p=3$. *There are ten nonisomorphic nonabelian groups of order* 3^4.

(i) *If* $P=\langle a,b,c\,|\,a^3=b^3, b^9=c^3=1, a^{-1}ba=bc, a^{-1}ca=b^3c, b^{-1}cb=c\rangle, t(P)=15$.

(ii)-(x) *For the other nine groups* P *of order* 3^4, *we can know* $t(P)$ *by putting* $p=3$ *in* (III), *where* (ix) *of* (III) *and* (x) *of* (III) *are isomorphic.*

(V) $r=4, p=2$. *There are nine nonisomorphic nonabelian groups of order* 2^4.

(i)-(iv) *If* $P=D_4, Q_4, S_4$ *or* $M_4(2)$, $t(P)=9$.

(v) *If* P *is a direct product of* D_3 *and a cyclic group of order* 2, $t(P)=6$.

(vi) *If* P *is a direct product of* Q_3 *and a cyclic group of order* 2, $t(P)=6$.

(vii) *If* $P=\langle a,b\,|\,a^4=b^4=1, a^{-1}ba=b^3\rangle, t(P)=7$.

(viii) *If* $P=\langle a,b,c\,|\,a^2=b^2=c^4=1, a^{-1}ba=bc^2, a^{-1}ca=c, b^{-1}cb=c\rangle, t(P)=6$.

(ix) *If* $P=\langle a,b,c\,|\,a^2=b^2=c^4=1, a^{-1}ba=b, a^{-1}ca=bc, b^{-1}cb=c\rangle, t(P)=7$.

PROOF. Put $x=a-1, y=b-1, z=c-1$ and $w=d-1$ in KP if they exist.

(I) (i) and (ii) are verified by Theorem 1.6 and Lemma 1.4, respectively.

(II) Clear from Theorem 1.6.

(III) (i) Trivial by Theorem 1.6.

(ii)-(iii) These follow from [6, Theorem] and (I).

(iv) Easy from Theorem 1.2.

(v) Since $yx=xyz^p+xz^p+yz^p+z^p+xy$, we have

$$yx\in\sum_{\substack{i+j+k\geqq 2\\ 0\leqq i\leqq 1}} Kx^iy^jz^k. \tag{14}$$

Using this, as in the proof of Lemma 1.1, we know

$$y^tx\in\sum_{\substack{i+j+k\geqq t+1\\ 0\leqq i\leqq 1}} Kx^iy^jz^k,\quad \text{for all } t\geqq 0, \tag{15}$$

$$y^tx^s\in\sum_{\substack{i+j+k\geqq s+t\\ 0\leqq i\leqq s}} Kx^iy^jz^k,\quad \text{for all } s,t\geqq 0. \tag{16}$$

By (16), it is seen that $C_i=\{x^sy^tz^u\,|\,0\leqq s,t\leqq p-1, 0\leqq u\leqq p^2-1, s+t+u\geqq i\}$ is a K-basis of $J(KP)^i$. Hence $t(P)=p^2+2p-2$.

(vi) As in Lemma 1.4, $t(P)=(p-1)+2(p-1)+(p^2-1)+1=p^2+3p-3$.

(vii) Since $yx=xyz^p+xz^p+yz^p+z^p+xy$,

$$yx\in\sum_{\substack{i+2j+k\geqq 3\\ 0\leqq i\leqq 1}} Kx^iy^jz^k. \tag{17}$$

By induction it follows from (17) that

(18) $$y^t x \in \sum_{\substack{i+2j+k\geqq 2t+1\\ 0\leqq i\leqq 1}} Kx^i y^j z^k, \quad \text{for all } t\geqq 0.$$

On the other hand, since $zx=xyz+xz+yz+xy+y$, we have

(19) $$y\in J(KP)^2,$$

(20) $$zx\in \sum_{\substack{i+2j+k\geqq 2\\ 0\leqq i\leqq 1}} Kx^i y^j z^k.$$

Using (20), as (18), it is seen that

(21) $$z^u x\in \sum_{\substack{i+2j+k\geqq u+1\\ 0\leqq i\leqq 1}} Kx^i y^j z^k, \quad \text{for all } u\geqq 0.$$

From (21) and (18), we can show

(22) $$y^t x^s\in \sum_{\substack{i+2j+k\geqq s+2t\\ 0\leqq i\leqq s}} Kx^i y^j z^k, \quad \text{for all } s, t\geqq 0$$

by induction on s. Similarly, from (21) and (18),

(23) $$z^u x^s\in \sum_{\substack{i+2j+k\geqq s+u\\ 0\leqq i\leqq s}} Kx^i y^j z^k, \quad \text{for all } s, u\geqq 0.$$

Now, we shall prove that $C_i=\{x^s y^t z^u | 0\leqq s, t\leqq p-1, 0\leqq u\leqq p^2-1, s+2t+u\geqq i\}$ is a K-basis of $J(KP)^i$ by induction on i. Put $i\geqq 2$. By (19), $C_i\subseteqq J(KP)^i$. As usual it is sufficient to show that $(x^s y^t z^u)(x^{s'} y^{t'} z^{u'})$ can be written as a K-linear combination of C_i if $0\leqq s, s', t, t'\leqq p-1$, $0\leqq u, u'\leqq p^2-1$, $s+2t+u\geqq 1$ and $s'+2t'+u'\geqq i-1$. Using (23) and (22) we can show this. Hence $t(P)=(p-1)+2(p-1)+(p^2-1)+1=p^2+3p-3$.

(viii) We can put $2\leqq f\leqq p-1$. Hence we have (17). Thus, just as in (vii), we obtain $t(P)=p^2+3p-3$.

(ix) It is clear that

(24) $$zy=yz, wz=zw, wy=yw \quad \text{and} \quad yx=xy.$$

Since

(25) $$zx=xyz+xz+yz+xy+y,$$

$y\in J(KP)^2$. Similarly, since

(26) $$wx=xzw+xw+zw+xz+z,$$

(27) $$z\in J(KP)^2.$$

From (25), (27) and $y\in J(KP)^2$, we have

(28) $$y\in J(KP)^3.$$

It follows from (25) and (26) that

(29)
$$zx \in \sum_{\substack{i+3j+2k\geqq 3\\ 0\leqq i\leqq 1}} Kx^iy^jz^k\ ,$$

and

(30)
$$wx \in \sum_{\substack{i+2k+h\geqq 2\\ 0\leqq i\leqq 1}} Kx^iz^kw^h\ ,$$

respectively. From (24) and (29),

(31)
$$z^ux \in \sum_{\substack{i+3j+2k\geqq 2u+1\\ 0\leqq i\leqq 1}} Kx^iy^jz^k,\quad \text{for all } u\geqq 0\ .$$

Similarly, from (24) and (30), we have

(32)
$$w^vx \in \sum_{\substack{i+2k+h\geqq v+1\\ 0\leqq i\leqq 1}} Kx^iz^kw^h,\quad \text{for all } v\geqq 0\ .$$

By (31) and (24),

(33)
$$z^ux^s \in \sum_{\substack{i+3j+2k\geqq s+2u\\ 0\leqq i\leqq s}} Kx^iy^jz^k,\quad \text{for all } s, u\geqq 0\ .$$

By (32), (31) and (24), we also have

(34)
$$w^vx^s \in \sum_{\substack{i+3j+2k+h\geqq s+v\\ 0\leqq i\leqq s}} Kx^iy^jz^kw^h,\quad \text{for all } s, v\geqq 0\ .$$

As usual, by (24), (27), (28), (33) and (34), we can show that $C_i=\{x^sy^tz^uw^v|0\leqq s, t, u, v\leqq p-1, s+3t+2u+v\geqq i\}$ is a K-basis of $J(KP)^i$. So $t(P)=(p-1)+3(p-1)+2(p-1)+(p-1)+1=7p-6$.

(x) Since $x^p=y$, it follows

(28′)
$$y\in J(KP)^p\ .$$

Using (28′) instead of (28), as in (ix), we can show that $C_i=\{x^sy^tz^uw^v|0\leqq s, t, u, v\leqq p-1, s+pt+2u+v\geqq i\}$ is a K-basis of $J(KP)^i$. Thus $t(P)=(p-1)+p(p-1)+2(p-1)+(p-1)+1=p^2+3p-3$.

(IV) (i) $C_i=\{x^sy^tz^u|0\leqq s, u\leqq 2,\ 0\leqq t\leqq 8,\ s+t+2u\geqq i\}$ is a K-basis of $J(KP)^i$. Hence $t(P)=15$.

(V) (i)-(iv) are easy by Theorem 1.6. (v) and (vi) are obtained from [6, Theorem] and (II). (vii), (viii) and (ix) follow from (iv) of (III), (v) of (III) and (vi) of (III), respectively.

COROLLARY 2.2. *For a p-group P, we have the followings.*

(I) *$t(P)=4$ if and only if P is one of the following types;*

(i) *$p=2$ and P is a cyclic group of order 2^2,*

(ii) *$p=2$ and P is an elementary abelian group of order 2^3.*

(II) $t(P)=5$ *if and only if* P *is one of the following types*;
 (i) $p=2$ *and* P *is an abelian group of type* $(2^2, 2)$,
 (ii) $p=2$ *and* $P\cong D_3$,
 (iii) $p=2$ *and* $P\cong Q_3$,
 (iv) $p=2$ *and* P *is an elementary abelian group of order* 2^4,
 (v) $p=3$ *and* P *is an elementary abelian group of order* 3^2,
 (vi) $p=5$ *and* P *is a cyclic group of order* 5.

(III) $t(P)=6$ *if and only if* P *is one of the following types*;
 (i) $p=2$ *and* P *is an abelian group of type* $(2^2, 2, 2)$,
 (ii) $p=2$ *and* P *is a direct product of* D_3 *and a cyclic group of order* 2,
 (iii) $p=2$ *and* P *is a direct product of* Q_3 *and a cyclic group of order* 2,
 (iv) $p=2$ *and* $P\cong\langle a, b, c \mid a^2=b^2=c^4=1, a^{-1}ba=bc^2,\ a^{-1}ca=c, b^{-1}cb=c\rangle$,
 (v) $p=2$ *and* P *is an elementary abelian group of order* 2^5.

PROOF. The assertions are proved by [3, Theorem 3.7] (cf. [10, Lemma 2.3]), [7, Theorem 1], [6, Theorem] and Theorem 2.1.

REMARK 2. As noting in the proof of Corollary 1.7 it is seen that $t(G)=t(P)$ for a p-solvable group G of p-length 1 with a p-Sylow subgroup P. Thus, by Corollary 2.2, we can have all p-solvable groups G of p-length 1 with $t(G)=4, 5$ or 6.

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