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A NOTE ON PROVABLE WELL-ORDERINGS IN FIRST ORDER SYSTEMS WITH INFINITARY INFERENCE RULES

By

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In [4], Shirai refined the well-known result on the provable well-orderings of the pure number theory in Gentzen [2] as follows:

Transfinite induction up to α is provable by only the induction rules (VJ-inferences) with induction formulae which have at most ρ quantifiers if and only if α represents an ordinal which is smaller than ω_{ρ}, where $\omega_0=\omega, \omega_{n+1}=\omega^{(\omega_n)}$.

In this paper we refine the result of Schütte [3] corresponding to the result of unprovability in Gentzen [2] according to the spirit of Shirai [4] as follows:

If P is an infinitary proof of the transfinite induction up to α, i.e., $\forall x(\forall y \prec xXy \supset Xx) \longrightarrow X\alpha$, and the ordinal number of P is smaller than $\omega \cdot \beta$, then α represents an ordinal which is smaller than $B(\beta, n)$, where n is the maximum length of the sequences of mutually regulating occurences of quantifiers in the cut formulae in P and B is the function defined at the beginning of §2 below.

In §1 we introduce our syntax and prove the reduction theorem which is a refined version of the Reduktionssatz of Schütte [3]. In §2 we prove the upper bound theorem which is the main result of the present note by the reduction theorem and two lemmata on the structure of the derivations of transfinite induction. As a corollary of the upper bound theorem we prove a part of the Shirai's result, i.e., if the transfinite induction up to α is provable by only the inductions whose induction formulae have at most ρ quantifiers and $\rho \geq 1$, then α is smaller than ω_{ρ}.

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§1. Syntax and the Reduction Theorem

We use only two logical symbols $\forall, \supset$. Let $\mathcal{L}$ be a first order language which has a binary predicate constant $\prec$ at least. Let $\mathfrak{A}$ be a structure of $\mathcal{L}$ in which $\prec$ is a well-ordering of $|\mathfrak{A}|$. We introduce a new constant c_a for each element $a \in |\mathfrak{A}|$. We understand that if $a \neq b$, then c_a and c_b are different symbols. We de-

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fine a system $\mathcal{L}(\mathfrak{A}, X)$ from $\mathcal{L}, \mathfrak{A}$ and a unary free predicate variable X.

DEFINITION OF $\mathcal{L}(\mathfrak{A}, X)$

(i) Formulae and terms are defined from $X, \mathcal{L}, \forall, \supset$ and the constants c_a $(a \in |\mathfrak{A}|)$ as usual.

(ii) Let Γ, Δ be finite sequences of formulae. Then $\Gamma \underset{\alpha}{\longrightarrow} \Delta$ is a sequent, where α is an ordinal number. The ordinal α is called the *ordinal of the sequent.*

(iii) $Xs \underset{\alpha}{\longrightarrow} Xt$ is an initial sequent, when $\mathfrak{A} \models \neg s < t$ and $\mathfrak{A} \models \neg s > t$. The notation $\neg A$ means the formula $A \supset c < c$, where c is a fixed constant.

(iv) $A_1, \cdots, A_m \underset{\alpha}{\longrightarrow} B_1, \cdots, B_n$ is an initial sequent if $\mathfrak{A} \models (A_1 \wedge \cdots \wedge A_m) \supset (B_1 \vee \cdots \vee B_n)$ and $A_1, \cdots, A_m, B_1, \cdots, B_n$ are formulae which do not contain the free variable X. The symbols $\wedge, \vee$ are defined from $\supset$ and $\neg$ as usual.

(v) Inference rules of $\mathcal{L}(\mathfrak{A}, X)$ are as follows:

1) Any structural inference of **LK** is adopted as an inference rule of $\mathcal{L}(\mathfrak{A}, X)$ provided that the ordinal of the upper sequent of the inference is equal to the ordinal of the lower sequent of the inference. We use the names: weakning left (right), exchange left (right), contraction left (right). (See § 2 of [5].)

2) The rules cut, $\supset$: left, $\supset$: right and $\forall$: left are the same as the ones of **LK** (cf. § 2 of [5]). The rule $\forall$: right of $\mathcal{L}(\mathfrak{A}, X)$ is the following inference rule:

$$\frac{\Gamma \underset{\alpha_a}{\longrightarrow} \Delta, A(c_a) \text{ for all } a \in |\mathfrak{A}|}{\Gamma \underset{\alpha}{\longrightarrow} \Delta, \forall x A x}.$$

The ordinals of the upper sequents of these four inferences must be smaller than the ordinal of the lower sequent.

(vi) *Degree and q-degree of a formula* is defined as follows and the degree and the q-degree of a formula A are denoted by $d(A)$ and $q(A)$ respectively.

1) If A is a prime formula, then $d(A) = q(A) = 0$.

2) If $d(At) = m$ and $q(At) = n$, then $d(\forall x A x) = m+1$ and $q(\forall x A x) = n+1$.

3) If $d(A_i) = m_i$ and $q(A_i) = n_i$ $(i=1,2)$, then $d(A_1 \supset A_2) = \max(m_1, m_2)+1$ and $q(A_1 \supset A_2) = \max(n_1, n_2)$.

(vii) *Degree and q-degree of a cut* is the degree and the q-degree of the cut formula of it.

(viii) We define *proofs and their degrees* inductively. We denote proofs by $P, P_0, \cdots, P_\lambda, \cdots$ and the degree of a proof P by $d(P)$.

1) An initial sequent is a proof and its degree is zero.

2) If $\Gamma \underset{\alpha}{\longrightarrow} \Delta$ is the lower sequent of an inference whose upper sequents are $\{\Gamma_\lambda \underset{\alpha_\lambda}{\longrightarrow} \Delta_\lambda\}_\lambda$, P_λ is a proof of $\Gamma_\lambda \underset{\alpha_\lambda}{\longrightarrow} \Delta_\lambda$ for each λ and $d(P_\lambda)$ is smaller than a natural

number n for any λ, then

$$\frac{\ldots;\ P_\lambda;\ \ldots}{\Gamma \underset{\alpha}{\longrightarrow} \Delta}$$

is a proof and its degree is the maximum of the degrees of the cuts in it.

(ix) If P is a proof, then its *q-degree* is the maximum of q-degrees of cuts which appear in the proof P. The ordinal of the end-sequent of a proof P is called the *ordinal of* P.

Now we prove the reduction theorem, which is a refinement of the Reduktionssatz of Schütte [3]. We use the following lemma without a proof. It is proved by the same method as in § 2 of [3].

LEMMA 1. *There are seven operations* $U_1, \cdots, U_7$ *which satisfy the following conditions respectively*:

1) *If* P *is a proof of* $\Gamma \underset{\alpha}{\longrightarrow} \Delta, A \supset B$, *then* $U_1(P)$ *is a proof of* $A, \Gamma \underset{\alpha}{\longrightarrow} \Delta, B$.

2) *If* P *is a proof of* $A \supset B, \Gamma \underset{\alpha}{\longrightarrow} \Delta$, *then* $U_2(P)$ *is a proof of* $B, \Gamma \underset{\alpha}{\longrightarrow} \Delta$ *and* $U_3(P)$ *is a proof of* $\Gamma \underset{\alpha}{\longrightarrow} \Delta, A$.

3) *If* P *is a proof of* $\Gamma \underset{\alpha}{\longrightarrow} \Delta, \forall x A x$ *and* t *is a term, then* $U_4(P, t)$ *is a proof of* $\Gamma \underset{\alpha}{\longrightarrow} \Delta, At$.

4) *If* P *is a proof of* $A, \Gamma \underset{\alpha}{\longrightarrow} \Delta$ *where* A *is a prime formula not containing the free variable* X *and* $\mathfrak{A} \models A$, *then* $U_5(P)$ *is a proof of* $\Gamma \underset{\alpha}{\longrightarrow} \Delta$.

5) *If* P *is a proof of* $\Gamma \underset{\alpha}{\longrightarrow} \Delta, A$ *where* A *is a prime formula not containing the free variable* X *and* $\mathfrak{A} \models \neg A$, *then* $U_6(P)$ *is a proof of* $\Gamma \underset{\alpha}{\longrightarrow} \Delta$.

6) *If* P *is a proof of* $\Gamma \underset{\alpha}{\longrightarrow} \Delta$ *and* $\alpha \leq \beta$, *then* $U_7(P, \beta)$ *is a proof of* $\Gamma \underset{\beta}{\longrightarrow} \Delta$.

REDUCTION THEOREM. *There are three operations* R_1, R_2, R_3 *which satisfy the following conditions respectively*:

1) *If* P *is a proof of* $\Gamma \underset{\alpha}{\longrightarrow} \Delta$, *then* $R_1(P)$ *is a proof of* $\Gamma \underset{2\cdot\alpha}{\longrightarrow} \Delta$, $d(R_1(P)) \leq d(P)$, $q(R_1(P)) = q(P)$ *and* $R_1^{d(P)}(P)$ *has no cut formula whose outermost logical symbol is* $\supset$.[1)]

2) *If* P *is a proof of* $\Gamma \underset{\alpha}{\longrightarrow} \Delta$, *then* $R_2(P)$ *is a proof of* $\Gamma \underset{2^\alpha}{\longrightarrow} \Delta$, $d(R_2(P)) \leq d(P)$ *and* $q(R_2(P)) \leq q(P)$. *If* P *has no cut formula whose outermost logical symbol is* $\supset$ *and* $q(P) \neq 0$, *then* $q(R_2(P)) < q(P)$.

3) *If* P *is a proof of* $\Gamma \underset{\alpha}{\longrightarrow} \Delta$, *then* $R_3(P)$ *is a proof of* $\Gamma \underset{\alpha}{\longrightarrow} \Delta$, $d(R_3(P)) \leq d(P)$, $q(R_3(P)) \leq q(P)$ *and* $R_3(P)$ *has no cut formula which is a prime formula except Xs.*

1) If F is an operation, then F^n means the operation that is defined inductively by the equations: $F^1 = F$, $F^{n+1} = F \circ F^n$.

PROOF. We define R_1, R_2 and R_3 by the induction of the definition of the proofs. It is easy to see the conditions are satisfied by those. The proofs are left to readers.

1) R_1 is defined by the followings:

Case 1) P is an initial sequent $\Gamma \xrightarrow[\alpha]{} \Delta$. In this case $R_1(P)$ is the initial sequent $\Gamma \xrightarrow[2\cdot\alpha]{} \Delta$.

Case 2) P is the following form:

$$\frac{\Gamma \xrightarrow[\alpha_1]{P_1} \Delta, A \supset B \quad A \supset AB, \Gamma \xrightarrow[\alpha_2]{P_2} \Delta}{\Gamma \xrightarrow[\alpha_3]{} \Delta}.$$

Then $R_1(P)$ is the following proof:

$$\frac{\Gamma \xrightarrow[2\cdot\alpha_2]{U_3 \circ R_1(P_2)} \Delta, A \qquad \dfrac{A, \Gamma \xrightarrow[2\cdot\alpha_1]{U_1 \circ R_1 \circ (P_1)} \Delta, B \quad B, \Gamma \xrightarrow[2\cdot\alpha_2]{U_2 \circ R_1(P_2)} \Delta}{A, \Gamma \xrightarrow[2\cdot\max(\alpha_1, \alpha_2)+1]{} \Delta}}{\Gamma \xrightarrow[2\cdot\alpha_3]{} \Delta}$$

Case 3) P is not in the above two cases. Let P be the following:

$$\frac{\ldots;\ P_\lambda;\ \ldots}{\Gamma \xrightarrow[\alpha]{} \Delta}.$$

Then $R_1(P)$ is the following proof:

$$\frac{\ldots;\ R_1(P_\lambda);\ \ldots}{\Gamma \xrightarrow[2\cdot\alpha]{} \Delta}.$$

2) R_2 is defined by the followings:

Case 1) P is an initial sequent $\Gamma \xrightarrow[\alpha]{} \Delta$. In this case $R_2(P)$ is the initial sequent $\Gamma \xrightarrow[2^\alpha]{} \Delta$.

Case 2) P is the following form:

$$\frac{\Gamma \xrightarrow[\alpha_1]{P_1} \Delta, \forall x A x \quad \forall x A x, \Gamma \xrightarrow[\alpha_2]{P_2} \Delta}{\Gamma \xrightarrow[\alpha_3]{} \Delta}.$$

In this case $R_2(P)$ is constructed by the same methed as in '*IV. Fall*' of §3 of [3] from $R_2(P_2)$ and $U_4(R_2(P_1), t)$, where t is a term. The precise definition is left for readers.

Case 3) P is not in the above two cases. Similarly to Case 3 of the definition of R_1.

3) R_3 is defined as follows:

Case 1) P is an initial sequent $\Gamma \xrightarrow[\alpha]{} \Delta$. Then $R_3(P)$ is the initial sequent $\Gamma \xrightarrow[\alpha]{} \Delta$ itself.

Case 2) P is the following form:

$$\frac{\overset{P_1}{\Gamma \xrightarrow[\alpha_1]{} \Delta, A} \quad \overset{P_2}{A, \Gamma \xrightarrow[\alpha_2]{} \Delta}}{\Gamma \xrightarrow[\alpha_3]{} \Delta},$$

where A is a prime formula except Xt. If $\mathfrak{A} \models A$, then $R_3(P)$ is the proof $U_7(U_5(R_3(P_2)), \alpha_3)$. If $\mathfrak{A} \models \neg A$, then $R_3(P)$ is the proof $U_7(U_6(R_3(P_1)), \alpha_3)$.

Case 3) P is not in the above two cases. Similarly to Case 3 of the definition of R_1.

§ 2. Upper bounds of order types of provable well-orderings

In this section we fix a system $\mathcal{L}(\mathfrak{A}, X)$ and its predicate symbol $\prec$ which represents a well-ordering of $|\mathfrak{A}|$. If t is a term of $\mathcal{L}(\mathfrak{A}, X)$, then $|t|$ means the ordinal α such that $\mathfrak{A}(t)$ is the α-th element of $|A|$ with respect to the well-ordering $\mathfrak{A}(\prec)$. Prog(X) means the formula $\forall x(\forall y(y \prec x \supset Xy) \supset Xx)$. If α is an ordinal and n is a natural number, then $B(\alpha, n)$ is defined as follows:

$$\begin{cases} B(\alpha, 0) = \omega \cdot \alpha , \\ B(\alpha, n+1) = 2^{B(\alpha, n)} . \end{cases}$$

Now we can state the upper bound theorem, which is the principal result of the present paper.

UPPER BOUND THEOREM. *If P is a proof of* Prog$(X) \xrightarrow[\alpha]{} Xt$, *$\alpha < \omega \cdot \beta$ and $q(P) = n$, then $|t| < B(\beta, n)$. We may replace $<$ by $\leq$.*

To prove this theorem we introduce a system **OC** (ordinal calculus) and prove two lemmata on it.[2]

DEFINETION OF **OC**.

(i) If Γ, Δ are finite sequences of ordinals, then $\Gamma \xrightarrow[\alpha]{} \Delta$ is a sequent of **OC**, where α is an ordinal.

2) The **OC** is introduced to make the proof of the theorem intelligible. We can prove the theorem without **OC**. Really, the present proof was inspired by a solution of an exercise on provable well-orderings in Feferman's lecture [1] where he seemed to claim to analyze the proofs of transfinite inductions directly. (See the Lemma 6.5 of [1].)

(ii) The sequent $\alpha \xrightarrow[\beta]{} \alpha$ is an initial sequent.

(iii) $\Gamma \xrightarrow[\beta]{} \Delta, 0$ is an initial sequent.

(iv) The inferences corresponding to the structural rules and cut of $\mathcal{L}(\mathfrak{A}, X)$ are introduced and called by the same names, e.g.

Weakning left)

$$\frac{\Gamma \xrightarrow[\beta]{} \Delta}{\alpha, \Gamma \xrightarrow[\gamma]{} \Delta},$$

where $\beta = \gamma$.

Cut)

$$\frac{\Gamma \xrightarrow[\beta]{} \Delta, \alpha \quad \alpha, \Gamma \xrightarrow[\gamma]{} \Delta}{\Gamma \xrightarrow[\delta]{} \Delta},$$

where $\max(\beta, \gamma) < \delta$.

(v) The following structural rule is adopted.

Repetition)

$$\frac{\Gamma \xrightarrow[\alpha]{} \Delta}{\Gamma \xrightarrow[\beta]{} \Delta} \quad (\alpha \leq \beta).$$

(vi) The following inference is adopted as the only non-structural inference rule. (Cf. Progressionsschluß of [3].)

PS)

$$\frac{\Gamma \xrightarrow[\alpha_\beta]{} \Delta, \beta \quad \text{for all } \beta < \gamma}{\Gamma \xrightarrow[\delta]{} \Delta, \gamma},$$

where $\gamma \neq 0$ and $\alpha_\beta < \delta$ for any $\beta < \gamma$.

LEMMA 2. *If* $\mathrm{Prog}(X) \xrightarrow[\alpha]{} Xt$ *is provable in* $\mathcal{L}(\mathfrak{A}, X)$ *without a cut whose cut formula is not the form* Xs, *then* $\xrightarrow[\alpha]{} |t|$ *is provable in* **OC**.

LEMMA 3. *If* $\alpha_1, \cdots, \alpha_m \xrightarrow[\gamma]{} \beta_1, \cdots, \beta_n$ *(possibly* $m=0$*) is provable in* **OC** *and* $\{\alpha_1, \cdots, \alpha_m\} \cap \{\beta_1, \cdots, \beta_n\} = \emptyset$, *then*

$$\gamma + \mathrm{Card}(\{\alpha_i \colon \alpha_i < \min(\beta_1, \cdots, \beta_n)\}) \geq \min(\beta_1, \cdots, \beta_n),$$

where for any set S $\mathrm{Card}(S)$ *means the cardinal of* S.

Before proofs of these lemmata we prove the upper bound theorem by them.

PROOF OF THE UPPER BOUND THEOREM. Let P be a proof of $\mathrm{Prog}(X) \xrightarrow[\alpha]{} Xt$, $d(P)=m$ and $q(P)=n$. If $\alpha < \omega \cdot \beta$, then by the reduction theorem we see $R_3 \circ (R_1{}^m \circ R_2)^n \circ R_1{}^m(P)$ is a proof of $\mathrm{Prog}(X) \xrightarrow[\gamma]{} Xt$, where $\gamma < B(\beta, n)$. It is easy to see that any cut of the resulting proof has the form Xs. By Lemma 2 we see there is a proof of $\xrightarrow[\gamma]{} |t|$ in **OC**. By Lemma 3 we see $t \leq \gamma < B(\beta, n)$. The present proof holds even if $<$ is replaced by $\leq$. Hence the upper bound theorem is proved by Lemma 2 and 3.

Now we prove Lemma 2 and 3.

PROOF OF LEMMA 2. Assume P is a proof of $\mathrm{Prog}(X) \xrightarrow[\alpha]{} Xt$ and P has no cut whose cut formula is not Xs. If we ignore the order of the occurences of formulae, the possible form of sequents in P is the following:

$$\Gamma \xrightarrow[\beta]{} \Delta ,$$

where

$\Gamma = \mathrm{Prog}(X)$; $\forall y(y \prec s_1 \supset Xy) \supset Xs_1, \cdots, \forall y(y \prec s_l \supset Xy) \supset Xs_l$; $Xt_1, \cdots, Xt_m$; $u_1 \prec v_1, \cdots, u_n \prec v_n$,

$\Delta = \forall y(y \prec w_1 \supset Xy), \cdots, \forall y(y \prec w_p \supset Xy)$; $y_1 \prec x_1 \supset Xy_1, \cdots, y_q \prec x_q \supset Xy_q$; $Xz_1, \cdots, Xz_r$.

For each $\Gamma \xrightarrow[\beta]{} \Delta$ of P we assign a sequent $\Gamma^* \xrightarrow[\beta]{} \Delta^*$ of OC such that

$\Gamma^* = |t_1|, \cdots, |t_m|$,

$\Delta^* = |w_1|, \cdots, |w_p|, |\tilde{y}_1|, \cdots, |\tilde{y}_s|, |z_1|, \cdots, |z_r|$,

where $\tilde{y}_1, \cdots, \tilde{y}_s$ is a subsequence of $y_1, \cdots, y_q$ and y_i belongs to the subsequence if and only if $\mathfrak{A} \models y_i \prec x_i$.

We define a subtree of P, say P', as follows:

1) The end sequent of P belongs to P'.

2) Assume a sequent $\Gamma \xrightarrow[\beta]{} \Delta$ of P belongs to P' and $\Gamma \xrightarrow[\beta]{} \Delta$ is not an initial sequent of P.

Case 1) $\Gamma \xrightarrow[\beta]{} \Delta$ is the lower sequent of a $\forall$: right. Then the inference has the following form:

$$\frac{\Gamma \xrightarrow[\beta_a]{} \Delta', c_a \prec t \supset X(c_a) \text{ for all } a \in |\mathfrak{A}|}{\Gamma \xrightarrow[\beta]{} \Delta', \forall y(y \prec t \supset Xy)} .$$

If $\mathfrak{A} \models c_a \prec t$, then $\Gamma \xrightarrow[\beta_a]{} \Delta', c_a \prec t \supset X(c_a)$ belongs to P'.

Case 2) $\Gamma \xrightarrow[\beta]{} \Delta$ is not the lower sequent of a $\forall$: right. In this case all upper sequents of the inference whose lower sequent is $\Gamma \xrightarrow[\beta]{} \Delta$ belong to P'.

Replace each sequent $\Gamma \xrightarrow[\beta]{} \Delta$ of P' by the sequent $\Gamma^* \xrightarrow[\beta]{} \Delta^*$. We denote the resulting tree of sequents of **OC** by P^*. We show P^* is a proof of **OC**.

Case 1) $\Gamma \xrightarrow[\beta]{} \Delta$ is an uppermost sequent of P'. In this case the sequent is uppermost in P or the lower sequent of a $\forall$: right whose principal formula (Hauptformel) is the form $\forall y(y \prec t \supset Xy)$ and $\mathfrak{A} \models \forall y(\neg y \prec t)$. In the latter case $\forall y(y \prec t \supset Xy)$ is replaced by $|t|$. Hence $\Gamma^* \xrightarrow[\beta]{} \Delta^*$ is an initial sequent of **OC**. By the definition of

P' we can easily see that if $s \prec t$ appears in the antecedent of a sequent in P', then $\mathfrak{A} \models s \prec t$. Hence in the former case $\Gamma \xrightarrow[\beta]{} \Delta$ must be the initial sequent obtained by the clause (iii) of definition of $\mathcal{L}(\mathfrak{A}, X)$. Hence $\Gamma^* \xrightarrow[\beta]{} \Delta^*$ is an initial sequent of **OC**.

Case 2) $\Gamma \xrightarrow[\beta]{} \Delta$ is not uppermost in P', and $\Gamma \xrightarrow[\beta]{} \Delta$ is the lower sequent of a cut or a structural rule in P. Then the inference turns a cut, a structural rule or a repetition in P^*.

Case 3) $\Gamma \xrightarrow[\beta]{} \Delta$ is the lower sequent of a $\forall$: right in P. Then the $\forall$: right is the form of the figure of 2) of the definition of P'. Hence we see the inference turns to a PS in P^*.

Case 4) $\Gamma \xrightarrow[\beta]{} \Delta$ is the lower sequent of a $\forall$: left in P. Then the inference turns to a repetition.

Case 5) $\Gamma \xrightarrow[\beta]{} \Delta$ is the lower sequent of a $\supset$: right in P. Then the $\supset$: right is the following form:

$$\frac{s \prec t, \Gamma \xrightarrow[\alpha]{} \Delta', Xs}{\Gamma \xrightarrow[\beta]{} \Delta', s \prec t \supset Xs}.$$

As mentioned above $\mathfrak{A} \models s \prec t$. Hence this inference turns to a repetition in P^*.

Case 6) $\Gamma \xrightarrow[\beta]{} \Delta$ is the lower sequent of a $\supset$: left in P. Then the $\supset$: left is the following form:

$$\frac{\Gamma' \xrightarrow[\beta_1]{} \Delta, \forall y(y \prec t \supset Xt) \quad Xt, \Gamma' \xrightarrow[\beta_2]{} \Delta}{\forall y(y \prec t \supset Xt) \supset Xt, \Gamma' \xrightarrow[\beta]{} \Delta}.$$

This turns to a cut in P^*.

Hence we have proved P^* is a proof of $\xrightarrow[\alpha]{} |t|$ in **OC**. This completes the proof of Lemma 2.

PROOF OF LEMMA 3. Let P be a proof of the sequent $\alpha_1, \cdots, \alpha_m \xrightarrow[\gamma]{} \beta_1, \cdots, \beta_n$. We say a sequent of **OC** is tautological, when the antecednt and the succedent have an ordinal in common. We say a sequent in P belongs to $\hat{P}^*$ if and only if there is no tautological sequent between the sequent and the end-sequent of P, including the sequent and the end-sequent. Note that $\hat{P}^*$ is well-founded. We prove the inequality of Lemma 3 for any sequent $\Gamma \xrightarrow[\alpha]{} \Delta$ in $\hat{P}^*$ by the induction on $\hat{P}^*$.

Case 1) $\Gamma \xrightarrow[\alpha]{} \Delta$ is an initial sequent in P. Since $\Gamma \xrightarrow[\alpha]{} \Delta$ is not tautological, we see 0 belongs to Δ. Hence the inequality holds for this sequent.

Case 2) $\Gamma \xrightarrow[\alpha]{} \Delta$ is the lower sequent of one of structural inferences except the rule

of weakning right in P. Obviously by the induction hypothesis.
Case 3) $\Gamma \xrightarrow[\alpha]{} \Delta$ is the lower sequent of a weakning right in P. Then the upper sequent belongs to $\hat{P}^*$. Let the inference be the following form:

$$\frac{\Gamma \xrightarrow[\alpha]{} \Delta'}{\Gamma \xrightarrow[\alpha]{} \Delta', \beta}.$$

Then by the induction hypothesis we see

$$\alpha + \mathrm{Card}(\Gamma \cap \{\delta :\ \delta < \min(\Delta')\}) \geq \min(\Delta').$$

Hence if $\beta \geq \min(\Delta')$, then the inequality holds for $\Gamma \longrightarrow \Delta', \beta$. Assume $\beta < \min(\Delta')$. Set $\beta + \gamma = \min(\Delta')$. Evidently there is a natural number n such that

$$\mathrm{Card}(\Gamma \cap \{\delta :\ \delta < \beta\}) + n = \mathrm{Card}(\Gamma \cap \{\delta :\ \delta < \beta + \gamma\}) \quad \text{and} \quad n \leq \gamma.$$

Hence $\alpha + \mathrm{Card}(\Gamma \cap \{\delta :\ \delta < \beta\}) + n \geq \beta + \gamma \geq n$ holds. Since n is a natural number, the desired inequality $\alpha + \mathrm{Card}(\Gamma \cap \{\delta :\ \delta < \beta\}) \geq \beta$ holds.
Case 4) $\Gamma \xrightarrow[\alpha]{} \Delta$ is the lower sequent of a cut in P. Assume the cut is the following form:

$$\frac{\Gamma \xrightarrow[\alpha_1]{} \Delta, \beta \qquad \beta, \Gamma \xrightarrow[\alpha_2]{} \Delta}{\Gamma \xrightarrow[\alpha]{} \Delta}.$$

Subcase 1) The β belongs to Δ. Since $\min(\Delta, \beta) = \min(\Delta)$ and $\alpha_1 \leq \alpha$, we see the inequality holds for $\Gamma \xrightarrow[\alpha]{} \Delta$ by the induction hypothesis on $\Gamma \xrightarrow[\alpha_1]{} \Delta, \beta$.
Subcase 2) The β does not belong to Δ. In the case $\beta \geq \min(\Delta)$, evidently by the induction hypothesis on $\beta, \Gamma \xrightarrow[\alpha_2]{} \Delta$. Assume $\beta < \min(\Delta)$. By the induction hypothesis we see $\alpha_2 + 1 + \mathrm{Card}(\Gamma \cap \{\gamma :\ \gamma < \min(\Delta)\}) \geq \min(\Delta)$. Since $\alpha_2 + 1 \leq \alpha$, the inequality holds for $\Gamma \xrightarrow[\alpha]{} \Delta$.
Case 5) $\Gamma \xrightarrow[\alpha]{} \Delta$ is the lower sequent of an application of PS in P. Assume the PS is the following form:

$$\frac{\Gamma \xrightarrow[\alpha_\beta]{} \Delta', \beta \quad \text{for all } \beta < \gamma}{\Gamma \xrightarrow[\alpha]{} \Delta', \gamma}.$$

Subcase 1) In the case $\min(\Delta', \gamma) < \gamma$. Set $\delta = \min(\Delta', \gamma)$. Since $\delta \in \Delta'$, the sequent $\Gamma \xrightarrow[\alpha_\delta]{} \Delta', \delta$ belongs to $\hat{P}^*$. Hence by the induction hypothesis we see $\alpha_\delta + \mathrm{Card}\ (\Gamma \cap \{\varepsilon :\ \varepsilon < \delta\}) \geq \delta$. Since $\alpha > \alpha_\delta$, the inequality holds for $\Gamma \xrightarrow[\alpha]{} \Delta$.
Subcase 2) In the case $\min(\Delta', \gamma) = \gamma$. Let $\delta_1, \cdots, \delta_p$ (possibly $p = 0$) be the sequence

$\Gamma\cap\{\varepsilon:\ \varepsilon<\gamma\}$. Set $\delta_{p+1}=\gamma$. We define a natural number q $(1\leq q\leq p+1)$ such that

$$\delta_q=\min\{\zeta:\ \forall\eta(\zeta\leq\eta\leq\gamma\Rightarrow\eta\in\{\delta_1,\cdots,\delta_{p+1}\})\}.$$

If $\beta<\delta_q$ and $\beta\notin\Gamma$, then by the induction hypothesis on $\Gamma\xrightarrow[\alpha_\beta]{}\Delta',\beta$ we see $\alpha_\beta+\mathrm{Card}(\Gamma\cap\{\varepsilon:\ \varepsilon<\delta_q\})+1\geq\beta+1$. By the definition of δ_q, we see there is no element whose successor is δ_q in $\Gamma\cap\{\varepsilon:\ \varepsilon<\gamma\}$. Hence $\sup\{\beta+1:\ \beta<\delta_q$ and $\beta\notin\Gamma\}=\sup\{\beta+1:\ \beta<\delta_q\}$. Since $\alpha_\beta<\alpha$ for all $\beta<\delta_q$, we see $\alpha+\mathrm{Card}(\Gamma\cap\{\varepsilon:\ \varepsilon<\delta_q\})\geq\sup\{\beta+1:\ \beta<\delta_q\}=\delta_q$. By the definition of δ_q, we finally see $\alpha+\mathrm{Card}(\Gamma\cap\{\varepsilon:\ \varepsilon<\gamma\})\geq\gamma$. This is the desired inequality.

We have just completed the proof of the upper bound theorem. As a consequence of the theorem we prove the following corollary.

COROLLARY (Shirai [4]). *If t is a closed term for which* $\mathrm{Prog}(X)\longrightarrow Xt$ *is ρ-derivable* ($\rho\geq1$) *in the pure number theory of* [4], *then* $|t|<\omega_\rho$, *where* $\omega_0=\omega$, $\omega_{n+1}=\omega^{(\omega_n)}$ and $\prec$ is the cannonical ordering up to ε_0.

Let $\mathfrak{N}$ be the standard model of the pure number theory. A sequent S_1 is called a *closed instance* of S_2, when S_1 is obtained by substitutions of numerals for all free individual variables in S_2. A *closed instance of a formula* is defined by the same manner. To prove the corollary we use the following lemma.

LEMMA 4. *If P is a proof of a sequent S in the pure number theory of* [4], *then for each closed instance of S we can find a proof $\tilde{P}$ of $\mathcal{L}(\mathfrak{N},X)$ such that* (i) *the end-sequent of $\tilde{P}$ is the given closed instance of S,* (ii) *the ordinal of $\tilde{P}$ is at most $\omega\cdot n$, where n is the number of the inferences of P,* (iii) *each cut formula in $\tilde{P}$ is a closed instance of one of VJ-formulae or cut formulae in P.*

PROOF. By the induction on P. Details are left for readers.

PROOF OF THE COROLLARY. Assume $\mathrm{Prog}(X)\longrightarrow Xt$ is ρ-derivable. As in [4] we may assume the sequent has a proof P in the pure number theory such that any formula which is a cut formula or a VJ-formula in P has at most ρ quantifiers. Hence by Lemma 4 we can find a proof $\tilde{P}$ in $\mathcal{L}(\mathfrak{N},X)$ such that the q-degree of $\tilde{P}$ does not exceed ρ and the ordinal of $\tilde{P}$ is smaller than $\omega\cdot\omega$. Hence by the upper bound theorem we see $|t|<B(\omega,\rho)$. In the case $\rho\geq1$ we see $B(\omega,\rho)=\omega_\rho$. Hence we have reached the desired conclusion.

REMARK. The results of the present note hold even if $\mathrm{Prog}(X)\xrightarrow[\alpha]{}Xt$ is replaced by $\mathrm{Prog}(X)\xrightarrow[\alpha]{}\forall x\prec tXx$ (cf. [3]).

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