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# A RELATION BETWEEN TRANSFINITE INDUCTION AND MATHEMATICAL INDUCTION IN ELEMENTARY NUMBER THEORY

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## Introduction.

The purpose of this paper is to clarify a relation which exists between transfinite induction deduced in elementary number theory and a kind of complexity of the induction formulas used in its derivation. Our result is stated as follows:

*Transfinite induction up to any ordinal number less than* $\omega_{\rho+2}$ *can be proved in elementary number theory only by the use of mathematical inductions whose induction formulas have at most* $\rho$ *quantifiers, but it is impossible to prove transfinite induction up to* $\omega_{\rho+2}$ *and higher ordinal numbers only by the use of those mathematical inductions, where the notation* $\omega_{\nu}$ *is defined as follows:* $\omega_0=0, \omega_1=1, \omega_2=\omega$ *etc.; in general,* $\omega_{\nu+1}=\omega^{\omega_{\nu}}$ (Main Theorem).

The impossibility of proving transfinite induction up to the ordinal number $\varepsilon_0$ in elementary number theory is inferred indirectly from Gödel's second incompleteness theorem ([5]) and Gentzen's consistency proof of elementary number theory ([2]). Gentzen [4] gave a direct proof for the nonprovability of transfinite induction up to $\varepsilon_0$ in elementary number theory, and further he remarked the fact that transfinite induction up to $\omega_{\nu+3}$ and higher ordinal numbers cannot be proved in elementary number theory only by the use of induction formulas having at most $\nu$ logical symbols.

Mints [7] gave an exact result by introducing the concept 'quantifier complexity' as a measure of the complexity of formulas, instead of the 'number of logical symbols'. His result is expressible as follows: Transfinite induction up to $\omega_{\nu+3}$ and higher ordinal numbers (or any ordinal number below $\omega_{\nu+3}$) is not (or is) provable in elementary number theory only by the use of induction formulas which are at most of quantifier complexity $\nu$. His proof for the nonprovability is based on the following facts:

1. The consistency of the system $R_{\nu}$, which is an extension of primitive re-

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cursive arithmetic by adding the schema of recursions on ordinal numbers less than $\omega_{\nu+3}$, is unprovable only by the use of induction formulas which are at most of quantifier complexity $\nu$.

2. The consistency of $R_\nu$ is provable in the system which arises from primitive recursive arithmetic by adding the principle of transfinite induction up to $\omega_{\nu+3}$ as axioms.

In the proof of the former he used Gödel's second incompleteness theorem for the system $R_\nu$. In this sense his proof is indirect.

Our proof presented below is direct one, because it is a modification of the procedure of proof in Gentzen [4]. Moreover, we can give a proof of Mints' result by means of a procedure similar to that used in our proof (cf. § 7). But, as a measure of the complexity of formulas, we adopt the number of quantifiers instead of Mints' quantifier complexity, because we want to make our proof of Main Theorem more simple.

The greater part of technical terms and conventions are adopted from English translation [8] of the brilliant works of G. Gentzen including [1], [2], [3] and [4].

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## § 1. Formal system and Main Theorem.

1.1. Formal system.

By Gentzen's result in [4] it is sufficient for our purpose to employ a formal system of elementary ordinal number theory whose individual domain is restricted within the set of all ordinal numbers below $\varepsilon_0$. In this paper, we shall deal with the following formal system for elementary theory of ordinal numbers, which is a slight modification of the system introduced by Gentzen in [4].

1.11. Terms and formulas. As *primitive symbols* we shall adopt the following:

The individual constant: 0.

The function constants: $\alpha+\beta$, $\alpha\cdot\beta$, $\omega^\alpha$, $fu_1(\alpha,\beta,\gamma)$, $fu_2(\alpha,\beta,\gamma)$, $\cdots$.

The predicate constants: $\alpha=\beta$, $\alpha<\beta$, $\alpha\leqq\beta$, $\cdots$.

The predicate variable: $\mathcal{E}(\alpha)$.

The letters $\alpha$, $\beta$ and $\gamma$ indicate the argument places.

Two functions $fu_1, fu_2$ are those defined in [4]. An arbitrary number of function constants and predicate constants other than those stated explicitly in the above may occur; but we require that those functions and predicates are not only defined in the ordinal numbers below $\varepsilon_0$ but also *decidably defined.*

By using those primitive symbols, *tems* and *formulas* are defined as usual.

All definite ordinal numbers below $\varepsilon_0$ can be represented uniquely as a kind of terms, called *numerical terms,* in the following way:

The individual constant 0 is a *numerical term.* If $\alpha$ and $\beta$ are *numerical terms* and $\beta$ is of the form $\omega^{\beta_1}$ or $\omega^{\beta_1}+\cdots$ and $\beta_1$ is not greater than $\alpha$, then $\omega^{\alpha}+\beta$ is a *numerical term.*

In the following, we simply call the numerical terms *ordinal numbers.*

1.12. The concept of '*sequent*' is defined as in [4].

A formula is said to be *prime* when it contains neither logical symbols nor the predicate variable $\mathcal{E}$. If a prime formula contains no free variables, then we can determine whether it represents a true or a false proposition on the basis of the decidable definition of functions and predicates concerned. Suppose that it is known of each formula in the antecedent and the succedent of a sequent without free variables whether it is true or false. Then the sequent is said to be '*true*' if there exists a true formula in the succedent or a false formula in the antecedent. In every other case the sequent is said to be '*false*'.

1.13. As '*basic sequents*' we shall admit 'basic logical sequents', 'basic equality sequents' and 'basic mathematical sequents'. A *basic logical sequent* is a sequent of the form

$$D \to D,$$

where $D$ is an arbitrary formula. A *basic equality sequent* is a sequent of the form

$$s=t, \mathcal{E}(s) \to \mathcal{E}(t)$$

where $s$ and $t$ are arbitrary terms. A *basic mathematical sequent* is a sequent consisting of prime formulas which becomes true sequent with every arbitrary substitution of ordinal numbers for possible occurrences of free variables.

1.14. As '*inference figures*' we shall use the same ones as in [4] except those of the implication '$\supset$' and *CJ*-inference figures. As schemata for introduction of $\supset$ in antecedent and introduction of $\supset$ in succedent we shall use $\supset$-*IA* and $\supset$-*IS* in Gentzen [1].

The schema for *CJ*-inference figures runs:

$$\frac{\Gamma \to \Theta, F(0) \quad a<\omega, F(a), \Gamma \to \Theta, F(a+1)}{t<\omega, \Gamma \to \Theta, F(t)},$$

where the eigenvariable $a$ must not occur in the lower sequent. The formula designated by $F(a)$ is called the *induction formula* of the *CJ*-inference figure concerned and the formula designated by $F(t)$ is called the *principal formula.*

1.15. By using the basic sequents and the inference figures the concept of the '*proof figure*' is defined as usual. It should be noted that we only refer to proof figures in tree form in the following. A sequent $S$ is said to be *provable* if there exists a proof figure whose lowermost sequent is $S$.

1.2. Main Theorem.

We use a notation $\omega_\nu$ of ordinal numbers, as in [4], in the following meaning: $\omega_0=0, \omega_1=1, \omega_2=\omega$ etc.; in general $\omega_{\nu+1}=\omega^{\omega_\nu}$.

By using this ordinal notation and the formal system, our result is expressible as the following

MAIN THEOREM. *Let $\alpha$ be an ordinal number, and $\rho$ a non-negative integer. Then the sequent, which expresses the validity of the transfinite induction up to $\alpha$,*

$$\forall y[\forall x(x<y \supset \mathcal{E}(x)) \supset \mathcal{E}(y)] \rightarrow \mathcal{E}(\alpha)$$

*is provable without using CJ-inference figures whose induction formulas have quantifiers greater than $\rho$ if and only if $\alpha$ is less than $\omega_{\rho+2}$.*

## § 2. *TJ*-derivations.

2.0. In this section we shall introduce the concept '*TJ-derivation*', which is a slight modification of '*TJ*-derivation' in [4].

2.1. We begin by formulating a schema for inference figures which mean the validity of the progressiveness of $\mathcal{E}$. The schema has the form

$$\frac{a<t, \Gamma \rightarrow \Theta, \mathcal{E}(a)}{\Gamma \rightarrow \Theta, \mathcal{E}(t)},$$

where $t$ is an arbitrary term and $a$ is a free variable not occurring in the lower sequent. The inference figures according to the schema are called '*TJ*-inference figures', though they differ from those used in [4]. The free variable $a$ is called the eigenvariable of the *TJ*-inference figure, and the formula designated by $\mathcal{E}(t)$ is called the principal formula.

Next we admit the *TJ*-inference figures into our formalism. The extended formal proofs obtained by adding those inference figures are called '*TJ*-derivations'. A sequent $S$ is said to be *TJ-derivable* if there exists a *TJ*-derivation whose lowermost sequent is $S$.

REMARK: In the following of this paper, we simply call the *TJ*-derivations *derivations* and we use a term '*derivable*' in the following meaning: '*TJ*-derivable'. Hence it should be noted that 'proof figure' and 'derivation' ('provable' and 'derivable') are used to have different meanings.

2.2. Suppose that we have proved the following

PROPOSITION 1. *A sequent $\Gamma\rightarrow\Theta$ is derivable only by the use of CJ-inference figures whose induction formulas have at most $\rho$ quantifiers if and only if the sequent*

$$\forall y[\forall x(x<y\supset \mathcal{E}(x))\supset \mathcal{E}(y)], \Gamma\rightarrow\Theta$$

*is provable only by the use of those CJ-inference figures.*

Then, by using Proposition 1, our main theorem is reformed in the following version:

*Let $\alpha$ be an ordinal number, and $\rho$ a non-negative integer. Then the sequent $\rightarrow\mathcal{E}(\alpha)$ is derivable only by the use of CJ-inference figures whose induction formulas have at most $\rho$ quantifiers if and only if $\alpha$ is less than $\omega_{\rho+2}$* (A reformed version of Main Theorem).

Hence in order to prove Main Theorem it suffices to show the reformed version.

2.3. PROOF OF PROPOSITION 1.

The 'if' part is obvious, because the sequent

$$\rightarrow\forall y[\forall x(x<y\supset \mathcal{E}(x))\supset \mathcal{E}(y)]$$

is derivable without *CJ* in the following way:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a<b\rightarrow a<b \quad \mathcal{E}(a)\rightarrow\mathcal{E}(a)}{a<b\supset \mathcal{E}(a), a<b\rightarrow\mathcal{E}(a)}}{\forall x(x<b\supset \mathcal{E}(x)), a<b\rightarrow\mathcal{E}(a)}}{a<b, \forall x(x<b\supset \mathcal{E}(x))\rightarrow\mathcal{E}(a)}}{\forall x(x<b\supset \mathcal{E}(x))\rightarrow\mathcal{E}(b)}}{\dfrac{\rightarrow\forall x(x<b\supset \mathcal{E}(x))\supset \mathcal{E}(b)}{\rightarrow\forall y[\forall x(x<y\supset \mathcal{E}(x))\supset \mathcal{E}(y)]}}$$

where $a$ and $b$ are distinct free variables.

Next we shall give a proof of the 'only if' part. Let be given a derivation of the sequent $\Gamma\rightarrow\Theta$ where all the induction formulas have at most $\rho$ quantifiers. Suppose that there occurs no *TJ*-inference figures in the derivation. Then the sequent $\Gamma\rightarrow\Theta$ is provable only by the use of *CJ*-inference figures whose induction formulas have at most $\rho$ quantifiers. Hence our assertion clearly holds. Thus we may assume that there occurs a *TJ*-inference figure in the given derivation. Then between the upper and the lower sequent of each *TJ*-inference figure——which has the form

$$\frac{a<t, \Delta\rightarrow\Lambda, \mathcal{E}(a)}{\Delta\rightarrow\Lambda, \mathcal{E}(t)}$$

——we insert the following derivational section:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Delta \rightarrow \Lambda,\ a<t \supset \mathcal{E}(a)}{\Delta \rightarrow \Lambda,\ \forall x(x<t \supset \mathcal{E}(x))} \quad \mathcal{E}(t) \rightarrow \mathcal{E}(t)}{\forall x(x<t \supset \mathcal{E}(x)) \supset \mathcal{E}(t),\ \Delta \rightarrow \Lambda,\ \mathcal{E}(t)}}{\forall y[\forall x(x<y \supset \mathcal{E}(x)) \supset \mathcal{E}(y)],\ \Delta \rightarrow \Lambda,\ \mathcal{E}(t)}}{\Delta,\ \forall y[\forall x(x<y \supset \mathcal{E}(x)) \supset \mathcal{E}(y)] \rightarrow \Lambda,\ \mathcal{E}(t)\,.} \quad \text{possibly several interchanges}$$

And the new formula $\forall y[\forall x(x<y \supset \mathcal{E}(x)) \supset \mathcal{E}(y)]$ is written down as the last antecedent formula of each individual sequent in the derivational *path* (cf. 1.5 of [3]) leading from the lower sequent of the *TJ*-inference figure to the endsequent. We may use some interchanges and contractions, if necessary, in order to turn the result of the above treatment into correct proof figure. Now the endsequent is

$$\Gamma,\ \forall y[\forall x(x<y \supset \mathcal{E}(x)) \supset \mathcal{E}(y)] \rightarrow \Theta\,.$$

Thus we may obtain a proof figure of the sequent

$$\forall y[\forall x(x<y \supset \mathcal{E}(x)) \supset \mathcal{E}(y)],\ \Gamma \rightarrow \Theta\,,$$

where all the induction formulas have at most $\rho$ quantifiers.

## § 3. Derivability of transfinite induction up to ordinal number below $\omega_{\rho+2}$.

3.0. In this section we shall prove the 'if' part of the reformed version of Main Theorem, which is stated as

THEOREM 1. *Let $\rho$ be a non-negative integer, and $\alpha$ any ordinal number less than $\omega_{\rho+2}$. Then the sequent $\rightarrow \mathcal{E}(\alpha)$ is derivable only by the use of CJ-inference figures whose induction formulas have at most $\rho$ quantifiers.*

3.1. For the sake of proof of Theorem 1, we shall introduce for every $\nu \geqq 0$ the relations: a formula is $\Pi_\nu$ (or $\Sigma_\nu$).

3.11. For every natural number $\nu$, a formula is said to be $\Pi_\nu$ (or $\Sigma_\nu$) if it is equivalent to a prenex formula $Q_1x_1 \cdots Q_\nu x_\nu F(x_1, \cdots, x_\nu)$ with $\nu$-quantifiers $Q_1x_1, \cdots, Q_\nu x_\nu$ where $F$ is quantifier-free and no two adjacent quantifiers are of the same kind and the first quantifier is universal (or existential). A formula is said to be $\Pi_0$ and $\Sigma_0$ if it is equivalent to a quantifier-free formula.

3.12. The following properties are proved:

(1) If a formula is $\Pi_\nu$ or $\Sigma_\nu$, then it is $\Pi_{\nu+1}$;

(2) If a formula $A$ is $\Pi_\nu$, then $\neg A$ is $\Sigma_\nu$;

(3) If a formula $A(a)$ is $\Pi_\nu$, so is $\forall x A(x)$ for $\nu \geqq 1$;

(4) If both formulas $A$ and $B$ are $\Sigma_\nu$, so is $A \vee B$;

(5) If both formulas $A$ and $B$ are $\Pi_\nu$, so is $A\vee B$.

The proof of the properties (4) and (5) is carried out by a simultaneous induction on $\nu$. In the proofs of (3) and (5), we shall use the fact: From a given formula containing successive quantifiers of the same kind (existential or universal), we can obtain the formula equivalent to the given one by condensing the successive quantifiers into one such quantifier.

This fact follows from the existence of a pair of calculable unary functions $g_1$ and $g_2$ such that any formula of the form $\forall x\forall yF(x, y)$ (or $\forall x\exists yF(x, y)$) is equivalent to $\forall zF(g_1(z), g_2(z))$ (or $\exists zF(g_1(z), g_2(z))$), where $g_1$ and $g_2$ mean unary function symbols which correspond to calculable functions $g_1$ and $g_2$ respectively. For example, the existence of such a pair of functions is proved as follows:

Let $g$ be the function obtained from Gödel's order isomorphism $P$ (cf. 7.9 of Gödel [6]) by the restriction of the domain over $\varepsilon_0\times\varepsilon_0$. Then $g$ is calculable isomorphism from $\varepsilon_0\times\varepsilon_0$ onto $\varepsilon_0$. Let $g_1$ and $g_2$ be the inverse functions of $g$ which satisfy the relations:

$$g_1(g(\alpha, \beta))=\alpha,\ g_2(g(\alpha, \beta))=\beta \ \text{ and } \ g(g_1(\alpha), g_2(\alpha))=\alpha\,.$$

Then $g_1$ and $g_2$ are calculable and the pair is one of the desired pairs.

3.13. We abbriviate $\forall x(x\leqq a\supset F(x))$ by $F^*(a)$ as in [4] and also abbriviate $\forall x(F^*(x)\supset F^*(x+\omega^a))$ by $F'(a)$. Then the following property is proved:

(6) For any natural number $\nu$ and for any term $t$, if $F(t)$ is $\Pi_\nu$, then $F'(t)$ is $\Pi_{\nu+1}$.

The property is proved by using (1)-(5) in the following way:

From the hypothesis it follows that $\nu\geqq 1$ and $F$ is $\Pi_\nu$. Hence by using (3) it follows that $F^*$ is $\Pi_\nu$. Hence from (2) it follows that $\neg F^*(a)$ is $\Sigma_\nu$. Hence it follows from (1) that $\neg F^*(a)$ is $\Pi_{\nu+1}$. On the other hand, it follows from the hypothesis that $F(a+\omega^t)$ is also $\Pi_\nu$. Hence it follows from (3) that $F^*(a+\omega^t)$ is $\Pi_{\nu+1}$. Hence it follows from (5) and above two facts that $\neg F^*(a)\vee F^*(a+\omega^t)$ is $\Pi_{\nu+1}$. Hence, by (3) we have that $\forall x(\neg F^*(x)\vee F^*(x+\omega^t))$ is $\Pi_{\nu+1}$. But, $F'(t)$ is equivalent to the formula. Hence we can conclude that $F'(t)$ is $\Pi_{\nu+1}$.

As a corollary of the property (6) we have the property (7).

$\mathcal{E}$ is $\Pi_1$, $\mathcal{E}'$ is $\Pi_2$, $\mathcal{E}''$ is $\Pi_3$ and in general $\mathcal{E}^{(\nu)}$ is $\Pi_{\nu+1}$.

3.2. For convenience's sake, we shall use a notation $\omega_\nu(\alpha)$ for every non-negative integer $\nu$ and ordinal number $\alpha$, which is defined recursively as follows: $\omega_0(\alpha)$ is $\alpha$; $\omega_{\nu+1}(\alpha)$ is $\omega^{\omega_\nu(\alpha)}$.

3.21. In order to prove Theorem 1, it suffices to prove the fact:

Let $m$ be a formal natural number. Then, the sequent $\rightarrow\mathcal{E}(\omega_\rho(m))$ is derivable only by the use of $CJ$-inference figures whose induction formulas have at most

$\rho$ quantifiers.

But, from the property (7), it follows that $\mathcal{E}^{(\mu)}$ is $\Pi_{\mu+1}$. Hence $\mathcal{E}^{(\mu)*}$ is also $\Pi_{\mu+1}$. Hence $\mathcal{E}^{(\mu)*}$ is equivalent to a formula with $\mu+1$ quantifiers. Thus in order to prove the above fact it suffices to show the fact:

Let $m$ be a formal natural number. Then, we can construct a derivation of the sequent $\rightarrow \mathcal{E}(\omega_\rho(m))$ where each induction formula is of the form $\mathcal{E}^{(\mu)*}$ for some $\mu<\rho$.

It remains for us to state a procedure for the construction of such a derivation.

3.22. The procedure is easily obtained by a slight modification from the Gentzen's procedure for the systematic construction of *TJ*-derivation up to $\omega_\nu$ i.e., *TJ*-derivation of the sequent $\mathcal{E}(0)\rightarrow\mathcal{E}(\omega_\nu)$. He gave it in § 2 of [4], by a mathematical induction on $\nu$.

3.221. Basis: In his construction a *TJ*-derivation up to 0 consists of the only one sequent $\mathcal{E}(0)\rightarrow\mathcal{E}(0)$.

On the contrary, we shall give a derivation of the sequent $\rightarrow\mathcal{E}(m)$, without *CJ* in the following way;

Let $\mu$ be a natural number, $m$ the formal one corresponding to $\mu$. And let $a_0, a_1, \cdots, a_\mu$ be distinct $\mu+1$ free variables. Then we see that the following diagram is a desired derivation

$$\begin{array}{r}
a_0<a_1, a_1<a_2, \cdots, a_{\mu-1}<a_\mu, a_\mu<m\rightarrow \\
\hline
a_0<a_1, a_1<a_2, \cdots, a_{\mu-1}<a_\mu, a_\mu<m\rightarrow\mathcal{E}(a_0) \\
\hline
a_1<a_2, \cdots, a_{\mu-1}<a_\mu, a_\mu<m\rightarrow\mathcal{E}(a_1) \\
\vdots \\
\hline
a_{\mu-1}<a_\mu, a_\mu<m\rightarrow\mathcal{E}(a_{\mu-1}) \\
\hline
a_\mu<m\rightarrow\mathcal{E}(a_\mu) \\
\hline
\rightarrow\mathcal{E}(m)
\end{array}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right\} \mu+1 \text{ } TJ\text{-inference figures.}$$

3.222. Induction step: Gentzen replaces, in a given *TJ*-derivation up to $\omega_\nu$, each occurrence of arbitrary expression of the form $\mathcal{E}(u)$ by $\mathcal{E}'(u)$. Next above the resulting sequent obtained from a *TJ*-upper sequent by the replacement he writes a derivation of the resulting sequent that contains only one *CJ* with induction formula of the form $\mathcal{E}^*$. Hence he has a derivation of the sequent $\mathcal{E}'(0)\rightarrow\mathcal{E}'(\omega_\nu)$. From this derivation he constructs a *TJ*-derivation up to $\omega_{\nu+1}$ by supplying some diagrams which contain no *CJ*.

Suppose that a derivation of the sequent $\rightarrow\mathcal{E}(\omega_\nu(m))$ is given such that each induction formula occurring in it is of the form $\mathcal{E}^{(\mu)*}$ for some $\mu<\nu$. Then by taking the similar procedure we may easily construct a derivation of the sequent $\rightarrow\mathcal{E}(\omega_{\nu+1}(m))$, in which each induction formula is of the form $\mathcal{E}^{(\mu)*}$ $(\mu<\nu+1)$, from the given derivation.

We replace in the given derivation each $\mathcal{E}(u)$ by $\mathcal{E}'(u)$. Then every $TJ$-inference figure has the form

$$\frac{a<t, \Gamma\to\Theta, \mathcal{E}'(a)}{\Gamma\to\Theta, \mathcal{E}'(t)}.$$

Then we replace it by the following

$$\cfrac{\cfrac{\cfrac{a<t, \Gamma\to\Theta, \mathcal{E}'(a)}{\Gamma\to\Theta, a<t\supset\mathcal{E}'(a)}}{\Gamma\to\Theta, \forall y[y<t\supset\mathcal{E}'(y)]} \quad \forall y[y<t\supset\mathcal{E}'(y)]\to\mathcal{E}'(t)}{\Gamma\to\Theta, \mathcal{E}'(t)}$$

and above the right upper sequent $\forall y[y<t\supset\mathcal{E}'(y)]\to\mathcal{E}'(t)$ of the new cut we write the derivation of the sequent which is obtained from the derivation of the corresponding sequent, stated in 2.22 of [4], by using $TJ$-inference figures instead of $TJ$-upper sequents. Thus we have a derivation of the sequent $\to\mathcal{E}'(\omega_\nu(m))$. From that derivation, as stated in [4], we can construct a derivation of the sequent $\to\mathcal{E}(\omega_{\nu+1}(m))$ without adding new $CJ$.

In order to see that the constructed derivation is a desired one, it suffices to show that in the derivation of the sequent $\to\mathcal{E}'(\omega_\nu(m))$ each induction formula is of the form $\mathcal{E}^{(\mu)*}$ for $\mu<\nu+1$. By the replacement of $\mathcal{E}$ by $\mathcal{E}'$, $\mathcal{E}^{(\mu)}$ changes into $\mathcal{E}^{(\mu+1)}$. Hence each induction formula of the form $\mathcal{E}^{(\mu)*}$ in the given derivation changes into an induction formula of the form $\mathcal{E}^{(\mu+1)*}$ $(\mu<\nu)$. All induction formulas that are newly added are of the form $\mathcal{E}^*$. Hence each induction formula in the derivation of the sequent $\to\mathcal{E}'(\omega_\nu(m))$ is of the form $\mathcal{E}^{(\mu)*}$ for $\mu<\nu+1$.

Thus we can construct a desired one. This concludes the proof of Theorem 1.

## § 4. Underivability of transfinite induction up to $\omega_{\rho+2}$ and higher ordinal number.

4.0. In this section we shall give a proof of the 'only if' part of the reformed version of Main Theorem (2.2):

Let $\alpha$ be an ordinal number, and $\rho$ a non-negative integer. If the sequent $\to\mathcal{E}(\alpha)$ is derivable only by the use of $CJ$-inference figures whose induction formulas have at most $\rho$ quantifiers, then $\alpha$ is less than $\omega_{\rho+2}$.

4.1. A derivation is called a *$\rho$-derivation* if it consists of prenex formulas with at most $\rho$ quantifiers ($\rho\geqq 0$). 0-derivation is a quantifier-free derivation.

Suppose that we have proved the following

PROPOSITION 2. *Let $\Pi\to\Sigma$ be a quantifier-free sequent. If the sequent is deriva-*

*ble only by the use of CJ-inference figures whose induction formulas have at most $\rho$ quantifiers, then there exists a $\rho$-derivation of the sequent $\Pi\rightarrow\Sigma$.*

Then, in order to prove the 'only if' part of the reformed version of Main Theorem, it suffices to show

THEOREM 2. *Let $\alpha$ be an ordinal number, and $\rho$ a non-negative integer. If there exists a $\rho$-derivation of the sequent $\rightarrow\mathcal{E}(\alpha)$, then $\alpha$ is less than $\omega_{\rho+2}$.*

4.2. PROOF OF PROPOSITION 2.

Proposition 2 follows from the following two lemmas:

LEMMA 1. *Let be given a quantifier-free sequent. If the sequent is derivable only by using CJ-inference figures whose induction formulas have at most $\rho$ quantifiers, then there exists a derivation of the sequent which consists of formulas with at most $\rho$ quantifiers.*

LEMMA 2. *If there exists a derivation of a quantifier-free sequent and it consists of formulas with at most $\rho$ quantifiers, then there exists a $\rho$-derivation of the sequent.*

4.21. PROOF OF LEMMA 1.

Gentzen's Hauptsatz (cf. [1]) plays an important role in this proof.

Let $\Pi\rightarrow\Sigma$ be a quantifier-free sequent and let be given a derivation of the sequent in which every induction formula has at most $\rho$ quantifiers.

4.211. Here we recall that a *LK*-derivation in [1] is a derivation which is defined only by using basic logical sequents, structural inference figures and operational (propositional and predicate) inference figures.

The following facts are easily seen:

(1) A sequent

$$t<\omega, \Gamma, t<\omega\supset[F(0)\supset\{\forall x(x<\omega\supset(F(x)\supset F(x+1)))\supset F(t)\}] \rightarrow \Theta, F(t)$$

is provable in the predicate calculus from both sequents

$$\Gamma\rightarrow\Theta, F(0) \quad \text{and} \quad a<\omega, F(a), \Gamma\rightarrow\Theta, F(a+1)$$

for each free variable $a$ not in $\Gamma, \Theta$ and $F(t)$.

(2) A sequent

$$\Gamma, \forall x(x<t\supset\mathcal{E}(x))\supset\mathcal{E}(t)\rightarrow\Theta, \mathcal{E}(t)$$

is provable in the predicate calculus from a sequent $a<t, \Gamma\rightarrow\Theta, \mathcal{E}(a)$ for each free variable $a$ not in $\Gamma, \Theta$ and $\mathcal{E}(t)$.

Contemporarily, a formula is called a *characteristic formula* if it is a closed formula of the form $\forall x_1 \cdots \forall x_\nu M(x_1, \cdots, x_\nu)$ such that with a substitution of some terms for distinct free variables $a_1, \cdots, a_\nu$ in $M(a_1, \cdots, a_\nu)$ the resulting formula is one of these: (i) a provable formula in the propositional calculus from basic mathematical sequents or basic equality sequents; (ii) a formula of the form

$$t < \omega \supset [F(0) \supset \{\forall x(x < \omega \supset (F(x) \supset F(x+1))) \supset F(t)\}]$$

where $F$ has at most $\rho$ quantifiers; (iii) a formula of the form

$$\forall x(x < t \supset \mathcal{E}(x)) \supset \mathcal{E}(t)\,.$$

Then, by (1) and (2), from the given derivation of the sequent $\Pi \rightarrow \Sigma$ we can construct a $LK$-derivation of a sequent $\Gamma_0, \Pi \rightarrow \Sigma$ for some sequence $\Gamma_0$ of characteristic formulas. Hence, by making use of Gentzen's Hauptsatz for $LK$-derivations, we shall have a $LK$-derivation of the same endsequent, in which no cuts occur.

4.212. Let be given a cut-free $LK$-derivation of $\Gamma_0, \Pi \rightarrow \Sigma$. Now from the cut-free $LK$-derivation we shall construct a derivation of $\Pi \rightarrow \Sigma$ which consists of formulas containing at most $\rho$ quantifiers.

A *CJ*-inference figure (or cut) is called a *CJ with rank* $\rho$ (or *cut with rank* $\rho$) if its induction formula (or cut formula) has at most $\rho$ quantifiers.

Suppose that we have proved the following

LEMMA 3. *Let $A$ be a characteristic formula. If a sequent $A, \Gamma \rightarrow \Theta$ is derivable only by using CJs and cuts with rank $\rho$, then so is $\Gamma \rightarrow \Theta$.*

Clearly the sequent $\Gamma_0, \Pi \rightarrow \Sigma$ is derivable only by using *CJ*s and cuts with rank $\rho$. Hence, by using Lemma 3 repeatedly, we shall obtain a derivation of the sequent $\Pi \rightarrow \Sigma$ which contains only *CJ*s and cuts with rank $\rho$.

We note that the following *subformula property* holds for derivations:

In a derivation, every derivational formula is a subformula of one of these; a formula in the endsequent, a cut formula, a formula in uppersequents of a *CJ* which is designated explicitly by $F(0), a < \omega, F(a)$ or $F(a+1)$ in the schema (1.14), or a formula in the upper sequent of a *TJ* which is designated explicitly by $a < t$ or $\mathcal{E}(a)$ in the schema (2.1).

Hence, by the subformula property we see that the derivation (obtained above) consists of formulas with at most $\rho$ quantifiers. Hence we have a derivation of $\Pi \rightarrow \Sigma$ which consists of formulas with at most $\rho$ quantifiers.

4.213. PROOF OF LEMMA 3. Let $\forall x_1 \cdots \forall x_\nu M(x_1, \cdots, x_\nu)$ be a characteristic formula, and $M$ a resulting formula of substitution of arbitrary terms in $M(a_1, \cdots, a_\nu)$. In order to prove the lemma it suffices to show the following assertion:

If $M, \Gamma \rightarrow \Theta$ is derivable only by using *CJ*s and cuts with rank $\rho$, so is $\Gamma \rightarrow \Theta$.

Suppose that $M$ has at most $\rho$ quantifiers. Then, since the sequent $\rightarrow M$ is derivable only by using *CJ*s with rank $\rho$ (but without use of any cut), we can easily show the assertion. Hence we may assume that $M$ has quantifiers greater than $\rho$. Then the assertion follows from the following three sublemmas:

SUBLEMMA 1. *Suppose that $A \supset B$ has quantifiers greater than $\rho$. If the sequent $\Gamma, A \supset B, \Delta \rightarrow \Theta$ is derivable only by using CJs and cuts with rank $\rho$, then so are both sequents $\Gamma, B, \Delta \rightarrow \Theta$ and $\Gamma, \Delta \rightarrow \Theta, A$.*

SUBLEMMA 2. *Suppose that $A \supset B$ has quantifiers greater than $\rho$. If the sequent $\Gamma \rightarrow \Theta, A \supset B, \Lambda$ is derivable only by using CJs and cuts with rank $\rho$, then so is the sequent $\Gamma, A \rightarrow \Theta, B, \Lambda$.*

SUBLEMMA 3. *Suppose that $\forall x F(x)$ has quantifiers greater than $\rho$. If the sequent $\Gamma \rightarrow \Theta, \forall x F(x), \Lambda$ is derivable only by using CJ*s *and cuts with rank $\rho$, then so is the sequent $\Gamma \rightarrow \Theta, F(t), \Lambda$ for any term $t$.*

4.213.1. In order to prove sublemmas, we shall introduce two auxiliary concepts:

A formula $A$ in an upper sequent of an arbitrary inference figure is called a *predecessor* of a formula $B$ in the lower sequent if and only if the sequent formulas $A$ and $B$ are identical and correspond to one another according to the inference figure schema (cf. 1.14 and 2.1).

The '*ancestors*' of a derivational formula $A$ are defined recursively as follows:

( i ) The formula $A$ itself is an *ancestor* of $A$;

(ii) If a derivational formula is an *ancestor* of $A$, then every predecessor of the derivational formula is an *ancestor* of $A$.

4.213.2. The proofs of three sublemmas are easy. For example, Sublemma 3 is proved as follows:

Let be given a derivation of the sequent $\Gamma \rightarrow \Theta, \forall x F(x), \Lambda$ which contains only *CJ*s and cuts with rank $\rho$. Then, since $\forall x F(x)$ has quantifiers greater than $\rho$, each ancestor of the derivational formula $\forall x F(x)$ in the endsequent is neither a principal formula of a *CJ* nor a principal formula of a *TJ*-inference figure. We replace by $F(t)$ all ancestors of the derivational formula $\forall x F(x)$. And, if there occurs an introduction of $\forall$ in succedent whose principal formula is replaced by $F(t)$, then we substitute (in the derivational section standing above the lower sequent) the free variable used as its eigenvariable for $t$. By these treatment we obtain a derivation of the sequent $\Gamma \rightarrow \Theta, F(t), \Lambda$ which contains *CJ*s and cuts with rank $\rho$ only.

This completes the proof of Lemma 3, and consequently the proof of Lemma 1.

4.22. PROOF OF LEMMA 2. Let be given a derivation of a quantifier-free sequent which consists of formulas with at most $\rho$ quantifiers. Now we recall that every formula $A$ with at most $\rho$ quantifiers has a prenex normal form $A^\circ$ which contains at most $\rho$ quantifiers. Then from the given derivation we can obtain a $\rho$-derivation of the sequent in the following way:

We first replace in the entire given derivation each formula $A$ by its prenex normal form $A^\circ$ stated in the above. Then all inference figures except operational ones remain correct. The correctness of operational inference figures is destroied in general, but it is recovered by supplying some cuts and some derivations consisting of prenex formulas with at most $\rho$ quantifiers. For example, suppose that there occurs an introduction of $\wedge$ in antecedent of the form

$$\frac{A, \Gamma \to \Theta}{A \wedge B, \Gamma \to \Theta}$$

in the given derivation. After the replacement it runs:

$$\frac{A^\circ, \Gamma^\circ \to \Theta^\circ}{(A \wedge B)^\circ, \Gamma^\circ \to \Theta^\circ},$$

where $\Gamma^\circ$ and $\Theta^\circ$ are the sequences of formulas obtained from $\Gamma$ and $\Theta$ by the replacement. Then we replace it by the following cut

$$\frac{(A \wedge B)^\circ \to A^\circ \quad A^\circ, \Gamma^\circ \to \Theta^\circ}{(A \wedge B)^\circ, \Gamma^\circ \to \Theta^\circ}$$

and above the sequent $(A \wedge B)^\circ \to A^\circ$ we write a cut-free derivation of the sequent without *CJ* or *TJ*.

In this way we may obtain a $\rho$-derivation of the given quantifier-free sequent.

4.3. Proof of Theorem 2. Let $\alpha$ be an ordinal number, and $\rho$ a non-negative integer. Let be given a $\rho$-derivation of the sequent $\to \mathcal{E}(\alpha)$.

Suppose that we have proved the following

Fundamental Lemma. *Let $\Pi \to \Sigma$ be a quantifier-free and closed sequent, and $\rho$ a non-negative integer. If there exists a $\rho$-derivation of the sequent $\Pi \to \Sigma$, then the sequent*

$$\forall x(x < \gamma \supset \mathcal{E}(x)), \Pi \to \Sigma$$

*is provable for some ordinal number $\gamma$ less than $\omega_{\rho+2}$.*

Then, by applying Fundamental Lemma to the given $\rho$-derivation we know that there exists a proof figure of the sequent

$$\forall x(x < \gamma \supset \mathcal{E}(x)) \to \mathcal{E}(\alpha)$$

for some $\gamma$ less than $\omega_{\rho+2}$. In the proof figure we replace $\mathcal{E}$ as follows: $\mathcal{E}(v)$ is in each case turned into $v<\omega_{\rho+2}$, where $v$ in each case designates the expression standing in the argument place of $\mathcal{E}$. Then all inference figures and all basic sequents in the proof figure remain correct. Hence we have a proof figure of the sequent

$$\forall x(x<\gamma\supset x<\omega_{\rho+2})\rightarrow\alpha<\omega_{\rho+2}.$$

On the other hand, a sequent $a<\gamma\rightarrow a<\omega_{\rho+2}$ is a basic mathematical sequent. Hence the sequent $\rightarrow\forall x(x<\gamma\supset x<\omega_{\rho+2})$ is provable. Hence we have that the sequent $\rightarrow\alpha<\omega_{\rho+2}$ is provable.

From the provability of the sequent $\rightarrow\alpha<\omega_{\rho+2}$ we can conclude the fact that $\alpha$ is less than $\omega_{\rho+2}$ by a reduction of absurdity:

Suppose that $\alpha$ were not less than $\omega_{\rho+2}$. Then the sequent $\alpha<\omega_{\rho+2}\rightarrow$ were a basic mathematical sequent. From the above argument it follows that the sequent $\rightarrow\alpha<\omega_{\rho+2}$ is provable. Hence we could obtain that the empty sequent $\rightarrow$ were provable. This is a contradiction. Hence $\alpha$ is less than $\omega_{\rho+2}$.

It remains for us to prove Fundamental Lemma. We shall carry out the proof by dividing it into two cases according as $\rho=0$ or $\rho\geqq 1$. The proof for the case of $\rho=0$ is given in § 5, and that for the case of $\rho\geqq 1$ in § 6.

## § 5. Proof of Fundamental Lemma for the case of $\rho=0$.

5.0. In order to prove Fundamental Lemma in the case of $\rho=0$, it suffices to prove Prof. Maehara's unpublished result:

Let $\Pi\rightarrow\Sigma$ be a quantifier-free sequent. If there exists a quantifier-free derivation of $\Pi\rightarrow\Sigma$, then the sequent

$$\forall x(x<k\supset\mathcal{E}(x)),\Pi\rightarrow\Sigma$$

is provable for some formal natural number $k$.

5.1. A formula in a sequent is called a *negative subformula of the sequent* if it is a negative subformula of some formula in the succedent or a positive subformula of some formula in the antecedent.

PROPOSITION 3. *Let $\Pi\rightarrow\Sigma$ be a quantifier-free sequent and let $\mathcal{E}(t_1),\cdots,\mathcal{E}(t_\nu)$ be the negative subformulas of the form $\mathcal{E}$ occurring in it. If a sequent*

$$\forall x(x<t\supset\mathcal{E}(x)),\Pi\rightarrow\Sigma,\mathcal{E}(s)$$

*is provable, then so is the sequent*

$$\forall x(x<t\supset\mathcal{E}(x)),\Pi\rightarrow\Sigma,s<t\vee s=t_1\vee\cdots\vee s=t_\nu.$$

5.11. PROOF OF PROPOSITION 3. Let be given a proof figure of the sequent $\forall x(x<t \supset \mathcal{E}(x)), \Pi \rightarrow \Sigma, \mathcal{E}(s)$. We replace, in the entire proof figure, each occurrence of arbitrary expression of the form $\mathcal{E}(v)$ by $\mathcal{E}(v) \wedge (v<t \vee v=t_1 \vee \cdots \vee v=t_\nu)$. It is easily seen that all inference figures, all basic logical sequents and all basic mathematical sequents remain correct. The resulting sequents obtained from basic equality sequents by the replacement are provable. Hence the sequent

$$\forall x[x<t \supset \{\mathcal{E}(x) \wedge (x<t \vee x=t_1 \vee \cdots \vee x=t_\nu)\}], \tilde{\Pi}$$
$$\rightarrow \tilde{\Sigma}, \mathcal{E}(s) \wedge (s<t \vee s=t_1 \vee \cdots \vee s=t_\nu)$$

is provable, where $\tilde{\Pi}$ and $\tilde{\Sigma}$ are sequences of formulas obtained from $\Pi$ and $\Sigma$ respectively by the replacement. The sequent

$$\forall x(x<t \supset \mathcal{E}(x)) \rightarrow \forall x[x<t \supset \{\mathcal{E}(x) \wedge (x<t \vee x=t_1 \vee \cdots \vee x=t_\nu)\}]$$

and

$$\mathcal{E}(s) \wedge (s<t \vee s=t_1 \vee \cdots \vee s=t_\nu) \rightarrow s<t \vee s=t_1 \vee \cdots \vee s=t_\nu$$

are also provable. Hence it follows that the sequent

$$\forall x(x<t \supset \mathcal{E}(x)), \tilde{\Pi} \rightarrow \tilde{\Sigma}, s<t \vee s=t_1 \vee \cdots \vee s=t_\nu$$

is provable.

On the other hand, from the assumptions it follows that the sequent $\Pi \rightarrow \Sigma$ is quantifier-free and the formulas $\mathcal{E}(t_1), \cdots, \mathcal{E}(t_\nu)$ are the negative subformulas of the sequent. Hence, from the definition of the replacement, it follows that for each formula $A$ in $\Pi$ the sequent $A \rightarrow \tilde{A}$ is provable and for each formula $A$ in $\Sigma$ the sequent $\tilde{A} \rightarrow A$ is provable.

Hence the sequent

$$\forall x(x<t \supset \mathcal{E}(x)), \Pi \rightarrow \Sigma, s<t \vee s=t_1 \vee \cdots \vee s=t_\nu$$

is provable.

5.2. PROOF OF MAEHARA'S RESULT. Let be given a quantifier-free derivation of the sequent $\Pi \rightarrow \Sigma$ and let $\mu$ be the total number of the derivational sequents in it. Then by mathematical induction on $\mu$ we shall prove the existence of a formal natural number $\boldsymbol{k}$ such that the sequent

$$\forall x(x<\boldsymbol{k} \supset \mathcal{E}(x)), \Pi \rightarrow \Sigma$$

is provable.

5.21. Suppose that $\mu$ is 1. Then our assertion is clear, because the sequent $\Pi \rightarrow \Sigma$ itself is a basic sequent.

5.22. Suppose that $\mu$ is greater than 1. Let $I$ be the lowermost inference figure in the given derivation. By the hypothesis of the induction, for each upper sequent of $I$, the existence of a desired formal natural number has already been proved. We shall show the existence of such a formal natural number for the lower sequent of $I$.

5.221. Suppose that $I$ is an inference figure with one upper sequent other than $TJ$-inference figures. Then as the desired formal natural number we may take the same as that for the upper sequent.

5.222. Suppose that $I$ is an inference figure with two upper sequents. Then as the desired formal natural number we may take the maximum of the two ones for the upper sequents.

5.223. Suppose that $I$ is a $TJ$-inference figure. We may assume that the upper sequent of $I$ runs $a<t, \Gamma\to\Theta, \mathcal{E}(a)$. Let $k_1$ be the formal natural number for the upper sequent. Then by the hypothesis of the induction the sequent

$$\forall x(x<k_1 \supset \mathcal{E}(x)), a<t, \Gamma \to \Theta, \mathcal{E}(a)$$

is provable.

Let $\mathcal{E}(t_1), \cdots, \mathcal{E}(t_\nu)$ be the negative subformulas of the form $\mathcal{E}$ occurring in the sequent $a<t, \Gamma\to\Theta$. We know that the sequent $a<t, \Gamma\to\Theta$ is quantifier-free. Hence by applying Proposition 3 to the provable sequent

$$\forall x(x<k_1 \supset \mathcal{E}(x)), a<t, \Gamma \to \Theta, \mathcal{E}(a)$$

we can see that the sequent

$$\forall x(x<k_1 \supset \mathcal{E}(x)), a<t, \Gamma \to \Theta, a<k_1 \vee a=t_1 \vee \cdots \vee a=t_\nu$$

is provable. Here we recall that the free variable $a$ is the eigenvariable of $I$ and doesn't occur in the lower sequent of $I$. Hence the free variable $a$ does not occur in the sequent

$$\forall x(x<k_1 \supset \mathcal{E}(x)), \Gamma \to \Theta, \forall x[x<t \supset (x<k_1 \vee x=t_1 \vee \cdots \vee x=t_\nu)].$$

Hence the sequent is provable.

On the other hand, it follows that for any terms $t_1, \cdots, t_\nu$, $t$ and $s$ the sequent

$$\forall x[x<t \supset (x<s \vee x=t_1 \vee \cdots \vee x=t_\nu)] \to t<s+n+1$$

is provable, where $n$ is the formal natural number corresponding to $\nu$.

Hence from the provability of these two sequents it follows that the sequent

$$\forall x(x<k_1 \supset \mathcal{E}(x)), \Gamma \to \Theta, t<k_1+n+1$$

is provable. Clearly both sequents

$$\forall x(x<k_1+n+1 \supset \mathcal{E}(x)) \to \forall x(x<k_1 \supset \mathcal{E}(x))$$

and

$$\forall x(x<k_1+n+1\supset \mathcal{E}(x)), t<k_1+n+1 \rightarrow \mathcal{E}(t)$$

are provable. Hence we can conclude that the sequent

$$\forall x(x<k_1+n+1\supset \mathcal{E}(x)), \Gamma \rightarrow \Theta, \mathcal{E}(t)$$

is provable. Thus as the desired formal natural number for the lower sequent of $I$ we may take one which is equal to $k_1+n+1$.

## § 6. Proof of Fundamental Lemma for the case of $\rho\geqq 1$.

6.0. In this section, we shall give a proof of Fundamental Lemma for the case of $\rho\geqq 1$ along similar lines of the proof of Gentzen's fundamental lemma ('grundlegende Satz') stated in 3.1 of [4]. Fundamental Lemma in the case of $\rho\geqq 1$ is stated as follows:

*Let $\Pi\rightarrow\Sigma$ be a quantifier-free and closed sequent. Let $\rho$ be a natural number (i.e., $\rho\geqq 1$). If there exists a $\rho$-derivation of the sequent $\Pi\rightarrow\Sigma$, then the sequent*

$$\forall x(x<\gamma\supset \mathcal{E}(x)), \Pi \rightarrow \Sigma$$

*is provable for some ordinal number $\gamma$ less than $\omega_{\rho+2}$.*

REMARK: The only derivations that we shall consider in the following are *TJ*-derivations consisting of prenex formulas. Hence by '*derivations*' we shall mean those derivations.

6.1. Our proof of Fundamental Lemma in the case of $\rho\geqq 1$ consists of four stages: 6.11; 6.12; 6.13 and 6.14. From an exact version of Fundamental Lemma in the case of $\rho\geqq 1$ (6.13) and the estimate of the values of $\rho$-derivations (6.14), the proof of Fundamental Lemma in the case of $\rho\geqq 1$ can be deduced immediately.

6.11. We define '*reduction steps*' for arbitrary derivations which satisfy the following conditions:

(i) The endsequent is quantifier-free and closed;

(ii) A *TJ*-inference figure occurs in it.

The precise definition of the reduction steps will be given in 6.3. Each reduction concerns to a specified inference figure in such a derivation. According to the kinds of the specified inference figures, the reductions are divided into those: *CJ*-reductions; quantifier-free cut reductions; propositional reductions; predicate reductions; the first critical reductions and the second critical reductions.

When a reduction concerns to an inference figure with one upper sequent (i.e., it is one of the reductions: *CJ*-reductions, propositional reductions for propositional inference figures with one upper sequent, predicate reductions or the second critical reductions), its reduction step transforms every derivation satisfying the conditions

(i) and (ii) into another derivation with a quantifier-free and closed endsequent. Especially in the case of a *CJ*-reduction or a predicate reduction the endsequent of the derivation is preserved in the process. And in every other case the endsequent is not preserved in the process, but the endsequent of old derivation is derivable from the new one without use of *TJ*-inference figures.

When a reduction concerns to an inference figure with two upper sequents (i.e., it is one of the propositional reductions for propositional inference figures with two upper sequents, quantifier-free cut reductions or the first critical reductions), its reduction step transforms a derivation into two derivations with quantifier-free and closed endsequents. From two newly resulting endsequents, the old one is derivable without use of *TJ*-inference figures.

6.12. We furthermore correlate an *ordinal number* with each derivation, together with a rule for its calculation; we call it the '*value*' of the derivation. As values only transfinite ordinal numbers below $\varepsilon_0$ (but not equal to 0) will be used. It is then proved that *with each reduction step the value diminishes.* Especially *with the second critical reduction step the value diminishes at least* $\omega$ *in the sense of natural sum*, i.e., the natural sum of $\omega$ and the value of the reduced derivation is not over the value of the given derivation.

The correlation of ordinal numbers with derivations and the proof of the above facts will be given in 6.4.

6.13. Once these concepts in 6.11 and 6.12 have been defined and the stated assertions have been proved, then the following important lemma follows:

*Let be given a derivation of a quantifier-free and closed sequent and let* $\gamma$ *be the value of the derivation. Suppose that the endsequent of the derivation is* $\Pi\rightarrow\Sigma$. *Then the sequent*

$$\forall x(x<\gamma\supset\mathcal{E}(x)),\ \Pi\rightarrow\Sigma$$

*is provable* (An exact version of Fundamental Lemma in the case of $\rho\geqq 1$).

The lemma will be proved by a transfinite induction on the value $\gamma$ of the derivation.

6.131. Suppose that the value $\gamma$ is 0. Then the assertion is vacuously true, because there is no derivation with value 0.

6.132. Suppose that $\gamma$ is greater than 0. We may assume that the assertion has been recognized as true for all derivations whose value is smaller than $\gamma$ and whose endsequent is quantifier-free and closed. Now suppose that an arbitrary derivation with the value $\gamma$ is given and its endsequent is $\Pi\rightarrow\Sigma$.

6.132.1. Suppose that the given derivation contains no *TJ*-inference figure. Then the endsequent itself is provable. Hence our assertion holds clearly.

6.132.2. Suppose that the given derivation contains a *TJ*-inference figure.

The concept 'critical' will be explained later (6.37). We distinguish two cases according as the given derivation is critical or not.

6.132.21. Suppose that the derivation is not critical. Then a reduction step may be carried out which reduces the derivation to one or two derivations with a quantifier-free closed endsequent and a smaller value. And from the one or two endsequents the old endsequent is derivable without use of any *TJ*-inference figures.

6.132.211. First we suppose that the derivation is reduced to one derivation with the endsequent $\Pi_1 \rightarrow \Sigma_1$ and the value $\gamma_1$. Since $\gamma_1$ is smaller than $\gamma$, by the hypothesis of the transfinite induction the assertion holds for the reduced derivation. Hence the sequent $\forall x(x<\gamma_1 \supset \mathcal{E}(x)), \Pi_1 \rightarrow \Sigma_1$ is provable. Hence the sequent $\forall x(x<\gamma \supset \mathcal{E}(x)), \Pi_1 \rightarrow \Sigma_1$ is provable. But from the sequent $\Pi_1 \rightarrow \Sigma_1$ the sequent $\Pi \rightarrow \Sigma$ is derivable without use of any *TJ*-inference figures. Hence the sequent $\forall x(x<\gamma \supset \mathcal{E}(x)), \Pi \rightarrow \Sigma$ is provable. Thus the assertion holds for the given one.

6.132.212. Next we suppose that the derivation is reduced to two derivations, one with the endsequent $\Pi_1 \rightarrow \Sigma_1$ and the value $\gamma_1$ and the other with the endsequent $\Pi_2 \rightarrow \Sigma_2$ and the value $\gamma_2$. Since $\gamma_1$ and $\gamma_2$ are smaller than $\gamma$, by the hypothesis of the transfinite induction the assertion holds for the reduced derivations. Hence both sequents

$$\forall x(x<\gamma_1 \supset \mathcal{E}(x)), \Pi_1 \rightarrow \Sigma_1 \quad \text{and} \quad \forall x(x<\gamma_2 \supset \mathcal{E}(x)), \Pi_2 \rightarrow \Sigma_2$$

are provable. Hence the sequents

$$\forall x(x<\gamma \supset \mathcal{E}(x)), \Pi_1 \rightarrow \Sigma_1 \quad \text{and} \quad \forall x(x<\gamma \supset \mathcal{E}(x)), \Pi_2 \rightarrow \Sigma_2$$

are provable. But from the sequents $\Pi_1 \rightarrow \Sigma_1$ and $\Pi_2 \rightarrow \Sigma_2$ the sequent $\Pi \rightarrow \Sigma$ is derivable without use of any *TJ*-inference figures. Hence $\forall x(x<\gamma \supset \mathcal{E}(x)), \Pi \rightarrow \Sigma$ is provable. Thus the assertion holds for the given derivation.

6.132.22. Suppose that the given derivation is critical. Then the first critical reduction step may be carried out. And, if it is not in that case, there occurs in the succedent of the endsequent a formula $\mathcal{E}(s)$ such that $s$ is less than $\gamma$, or the second critical reduction step may be carried out.

6.132.221. Suppose that the first critical reduction step is carried out. Then it reduces the given derivation to two derivations with a quantifier-free closed endsequent and a smaller value. And from two newly resulting endsequents the old one is derivable without use of any *TJ*-inference figures. Hence, as in 6.132.212, it is proved that the assertion holds for the given derivation.

6.132.222. Suppose that we cannot carry out the first critical reduction step. Then, in the succedent of the endsequent of the given derivation, there occurs a formula $\mathcal{E}(s)$ such that the second critical reduction step may be carried out if $s$

is not less than $\gamma$.

6.132.222.1. Suppose that $s$ is less than $\gamma$. Then the sequent $\rightarrow s<\gamma$ is provable. Hence the sequent $\forall x(x<\gamma\supset\mathcal{E}(x))\rightarrow\mathcal{E}(s)$ is provable. Hence in this case we shall obtain the conclusion without use of the hypothesis of the transfinite induction.

6.132.222.2. Suppose that $s$ is not less than $\gamma$. Then the second critical reduction step is carried out. It reduces the given derivation to a derivation with an endsequent $\Pi, \beta<t\rightarrow\mathcal{E}(\beta), \Sigma$ and a smaller value $\gamma_1$, where $s=t$ is true, $\beta$ is an arbitrary ordinal number and $\gamma_1\#\omega\leqq\gamma$. Let $\mathcal{E}(t_1), \cdots, \mathcal{E}(t_\nu)$ be the negative subformulas of the form $\mathcal{E}$ in $\Pi\rightarrow\Sigma$. By using the facts that $\gamma_1\#\omega\leqq\gamma$ and that $s=t$ is true, we may choose as $\beta$ an ordinal number that satisfies the conditions: $\beta<t$ is true; all the formulas $\beta=t_1, \beta=t_2, \cdots, \beta=t_\nu$ are false; $\beta<\gamma_1$ is false. By the hypothesis of the transfinite induction the assertion holds for the reduced derivation. Hence the sequent

$$\forall x(x<\gamma_1\supset\mathcal{E}(x)), \Pi, \beta<t\rightarrow\mathcal{E}(\beta), \Sigma$$

is provable.

We recall that the sequent $\Pi, \beta<t\rightarrow\Sigma$ is quantifier-free and that $\mathcal{E}(t_1), \cdots, \mathcal{E}(t_\nu)$ are the negative subformulas of the form $\mathcal{E}$ in the sequent. Hence by applying Proposition 3 (5.1) to the provable sequent

$$\forall x(x<\gamma_1\supset\mathcal{E}(x)), \Pi, \beta<t\rightarrow\Sigma, \mathcal{E}(\beta)$$

we shall obtain the fact that the sequent

$$\forall x(x<\gamma_1\supset\mathcal{E}(x)), \Pi, \beta<t\rightarrow\Sigma, \beta<\gamma_1\vee\beta=t_1\vee\cdots\vee\beta=t_\nu$$

is provable. On the other hand, from the definition of $\beta$, it follows that the following sequents

$$\rightarrow\beta<t \quad \text{and} \quad \beta<\gamma_1\vee\beta=t_1\vee\cdots\vee\beta=t_\nu\rightarrow$$

are provable. Hence the sequent

$$\forall x(x<\gamma\supset\mathcal{E}(x)), \Pi\rightarrow\Sigma$$

is provable. Thus the assertion holds for the given derivation.

This completes the proof of the exact version of Fundamental Lemma in the case of $\rho\geqq 1$.

6.14. Between $\rho$-derivations and their values we have the following relationship: *The value of $\rho$-derivation is less than $\omega_{\rho+2}$* ($\rho\geqq 1$). The proof is given in 6.5.

6.2. For convenience of the discussion that follows, the formalism defined in 1.1 in §1 will be modified: We now admit further inference figures '*substitutions of terms*', as in 3.2 of Gentzen [4], according to the following schemata:

$$\frac{\Gamma_1, F(s), \Gamma_2 \rightarrow \Theta}{\Gamma_1, F(t), \Gamma_2 \rightarrow \Theta} \quad \text{or} \quad \frac{\Gamma \rightarrow \Theta_1, F(s), \Theta_2}{\Gamma \rightarrow \Theta_1, F(t), \Theta_2},$$

where $s$ and $t$ are terms without free variables and designate the same ordinal number.

6.3. *Definition of the reduction steps* for derivations.

6.30. Suppose that a derivation is given which contains at *TJ*-inference figure and that its endsequent is a quantifier-free and closed sequent $\Pi \rightarrow \Sigma$.

The definition of a reduction step will, in essence, be taken over from Gentzen [4], § 3. Let us therefore first examine the essential differences between our present concept of a derivation and its counterpart in [4]:

The endsequent now may be an arbitrary quantifier-free closed sequent; in [4] it has the special form. The basic equality sequent is restricted for formula of the form $\mathcal{E}$; in [4] it is admitted for arbitrary formula. The inference figures for introduction of $\supset$ in antecedent or in succedent are now admitted. The *CJ*-inference figures are now modified to have two upper sequents. Finally the *TJ*-inference figures are introduced but the *TJ*-upper sequents are omitted.

6.31. The reduction step begins with the *first preparatory step* as in 3.32 of [4]: the replacement of 'redundant' free variables by 1.

The '*ending*' of the given derivation is defined somewhat differently from 3.32 of [4], as follows:

The ending includes all those derivational sequents which are encountered if we trace each individual path from the endsequent upwards and stop as soon as we reach the line of inference of a *predicate, CJ* or *TJ-inference figure.*

The ending can therefore contain not only structural inference figures and substitutions of terms but also propositional inference figures and, after the preparatory step, no free variables. The uppermost sequents of the ending may be lower sequents of predicate, *CJ* or *TJ*-inference figures as well as basic sequents of any one of these three kinds.

As in [4], we add to the preparatory step an additional step: the replacement of *all terms occurring in the ending* by the ordinal numbers of their values. After the additional step, *basic equality sequents* no longer occur in the ending.

6.32. If an uppermost sequent of the ending is the lower sequent of a *CJ*-inference figure, then a *CJ-reduction* is carried out. We may adapt the definition of *CJ*-reduction in [4] with minor change which is caused by the difference in forms between the schema for *CJ*-inference figures in 1.14 and that in [4].

Let us consider such a *CJ*-inference figure. It has a form

$$\frac{\Gamma \rightarrow \Theta, F(0) \quad a<\omega, F(a), \Gamma \rightarrow \Theta, F(a+1)}{t<\omega, \Gamma \rightarrow \Theta, F(t)},$$

where $t$ is an *ordinal number*. It can be decided whether $t<\omega$ is true or not. If $t<\omega$ is not true, we simply from

$$\frac{t<\omega \rightarrow}{t<\omega, \Gamma \rightarrow \Theta, F(t)} \quad \text{thinnings and possibly several interchanges}$$

and the rest of the given derivation is then added unchanged below the sequent $t<\omega, \Gamma\rightarrow\Theta, F(t)$.

If $t<\omega$ is true, $t$ is 0 or a natural number $n$. If $t$ is 0, we replace the *CJ*-inference figure by a thinning

$$\frac{\Gamma \rightarrow \Theta, F(0)}{t<\omega, \Gamma \rightarrow \Theta, F(t)}$$

and omit the derivational part standing above the right upper sequent of the *CJ*-inference figure. If $t$ is a natural number $n$, we replace the *CJ*-inference figure by the following diagram

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \rightarrow \Theta, F(0)\end{array} \quad \begin{array}{c}\vdots\\ F(0), \Gamma \rightarrow \Theta, F(n)\end{array}}{\Gamma, \Gamma \rightarrow \Theta, \Theta, F(n)}}{\Gamma \rightarrow \Theta, F(n)}}{t<\omega, \Gamma \rightarrow \Theta, F(t)},$$

where we write the derivational section standing above the left upper sequent of the *CJ*-inference figure above the sequent $\Gamma\rightarrow\Theta, F(0)$ and we write the diagram stated in 3.33 of [4], above the sequent $F(0), \Gamma\rightarrow\Theta, F(n)$. From the sequent $t<\omega$, $\Gamma\rightarrow\Theta, F(t)$ downwards, the rest is finally continued by adjoining the unchanged remainder of the old derivation.

6.33. If the ending includes a quantifier-free cut, then a *quantifier-free cut reduction* is carried out.

Let us therefore consider such an inference figure. We may consider that the derivation looks like this:

$$\frac{\begin{array}{c}\vdots\\ \Gamma \rightarrow \Theta, D\end{array} \quad \begin{array}{c}\vdots\\ D, \Delta \rightarrow \Lambda\end{array}}{\Gamma, \Delta \rightarrow \Theta, \Lambda} \quad \text{The quantifier-free cut}$$

$$\vdots$$

$$\Pi \rightarrow \Sigma \quad \text{The endsequent.}$$

The reduction step consists of the transformation of the derivation into the two derivations:

One of them is obtained from the following diagram:

$$\begin{array}{cl} \vdots & \\ \underline{\underline{\Gamma \to \Theta, D}} & \text{possibly several thinnings} \\ \Gamma, \Delta \to D, \Theta, \Lambda & \text{and interchanges} \\ \cdot \quad \cdot \quad \cdot & \\ \cdot \;\; \cdot \;\; \cdot & \\ \cdots & \\ \cdot & \\ \Pi \to D, \Sigma & \text{The endsequent.} \end{array}$$

It is obtained from the given derivation first by omitting the section standing above the right upper sequent of the quantifier-free cut, next by replacing the cut by possibly several thinnings and interchanges and finally by inserting the formula $D$ as the first succedent formula into every sequent below the sequent

$$\Gamma, \Delta \to D, \Theta, \Lambda .$$

The other of them is obtained from the diagram:

$$\begin{array}{cl} \vdots & \\ \underline{\underline{D, \Delta \to \Lambda}} & \text{possibly several thinnings} \\ \Gamma, \Delta, D \to \Theta, \Lambda & \text{and interchanges} \\ \cdot \quad \cdot \quad \cdot & \\ \cdot \;\; \cdot \;\; \cdot & \\ \cdots & \\ \cdot & \\ \Pi, D \to \Sigma & \text{The endsequent.} \end{array}$$

Since the cut formula $D$ is quantifier-free and closed, both new endsequents $\Pi \to D, \Sigma$ and $\Pi, D \to \Sigma$ are quantifier-free and closed. From these two, the original endsequent $\Pi \to \Sigma$ is derivable only by using structural inference figures.

6.34. If the ending includes a propositional inference figure and does not include a quantifier-free cut, then a *propositional reduction* is carried out.

Let us consider one of the lovermost those propositional inference figures.

When the inference figure has two upper sequents, the reduction step consists of the transformation of the derivation into the two derivations, which are obtained by the same way as in 6.33. Otherwise, the reduction step consists of the transformation of the derivation into one derivation. Since the reductions are similar we shall only deal with the case when the propositional inference figure is an introduction of $\wedge$ in antecedent. Suppose that our derivation looks like this:

$$\begin{array}{cl} \vdots & \\ \underline{\underline{A, \Gamma \to \Theta}} & \text{The propositional inference figure} \\ A \wedge B, \Gamma \to \Theta & \\ \cdot \quad \cdot \quad \cdot & \\ \cdot \;\; \cdot \;\; \cdot & \\ \cdots & \\ \cdot & \\ \Pi \to \Sigma & \text{The endsequent.} \end{array}$$

Then the transformation is to construct a derivation of the form:

$$\begin{array}{ll} \vdots & \\ \dfrac{A, \Gamma \to \Theta}{A\wedge B, \Gamma, A \to \Theta} & \text{possibly several interchanges and thinnings} \\ \vdots & \\ \Pi, A \to \Sigma & \text{The endsequent.} \end{array}$$

It is obtained from the derivation first by replacing the propositional inference figure by a thinning and possibly several interchanges and next by inserting the formula $A$ as the last antecedent formula in each sequent below the sequent $A\wedge B, \Gamma, A\to\Theta$. Since the given derivation consists of prenex formulas, the subformula $A$ of $A\wedge B$ is quantifier-free. Hence the sequent $\Pi, A\to\Sigma$ is quantifier-free and closed. In the ending of the given derivation, the propositional inference figure concerned is one of the lowermost ones and no quantifier-free cuts occur. Hence there occurs a formula $A\wedge B$ in the antecedent of the endsequent, one of whose ancestors is the principal formula of the inference figures. Therefore from the new endsequent the original endsequent is derivable only by using an introduction of $\wedge$ in antecedent, a contraction and possibly several interchanges.

6.35. If the ending is nowhere bounded above by a *CJ*-inference figure and it contains neither quantifier-free cuts nor propositional inference figures, then the actual reduction step is preceded by a *second preparatory step* as in 3.34 of [4].

Now the ending can contain only structural inference figures (except quantifier-free cuts) and substitutions of terms. Hence the concepts '*clustered*' and '*cluster of formulas*' are defined in the same way as in [4].

The second preparatory step is carried out precisely as in 3.34 of [4]; for the *elimination of thinnings* there is nothing to do except those stated in [4]. For the *elimination of the basic logical sequents*, as stated in [4], it is important that the derivation concerned does not consist of basic logical sequents and thinnings only; this requirement is also met, since the derivation contains at least one *TJ*-inference figure (cf. 6.11 and 6.30).

For the ending the following therefore holds: the inference figures which it contains are 'substitutions of terms' and structural inference figures except quantifier free cuts, with thinnings at most immediately above the endsequent. It contains no free variables. Its uppermost sequents are basic mathematical sequents or lower sequents of predicate inference figures or *TJ*-inference figures.

6.36. If the second preparatory step produces in the ending a '*cluster of formulas suitable for the application of a predicate reduction*'—i.e., a cluster of formulas which possesses an associated cut and whose right and left sides contain at least one formula which is the principal formula of a predicate inference figure and contains at least two quantifiers—then a *predicate reduction* is carried out as in 3.35 of [4]. Our concept 'cluster of formulas suitable for the application of a predicate reduction' is different from that in [4]. The difference is caused by the difference of the definition of the 'level' of a derivational sequent. The definition of '*level*' is now given as follows:

By the *level* of a derivational sequent we mean the maximum of two numbers $\nu-1$ and 0 where $\nu$ is the greatest number of quantifiers of any formula occurring in or below the sequent concerned.

6.37. We shall now examine the question of what a derivation looks like to which none of the described reductions is applicable. In this case, we shall call the derivation '*critical*' and shall state '*critical reduction steps*' for it.

We recall that our derivation contains at least one *TJ*-inference figure. We now adapt the proof from 3.36 of [4], in such a way that the *TJ*-inference figures (instead of *TJ*-upper sequents) are given equal status with the predicate inference figures. (The formula designated by $\mathcal{E}(t)$ in the schema of *TJ*-inference figure is called the 'principal formula'.) Then, from the adapted considerations, we can see that one of the following cases arises for the derivation: There exists a *cluster of formulas* which possesses an associated cut and whose cut formula contains only one quantifier and whose both right and left sides contain at least one formula which is the principal formula of a predicate inference figure; or there exists a *succedent formula* of the endsequent which belongs to the same cluster of formulas as the principal formula of one of the *TJ*-inference figures.

6.38. If there exists in the ending a cluster of formulas which possesses an associated cut and whose cut formula contains only one quantifier and whose right and left sides contain at least one principal formula of a predicate inference figure, then the '*first critical reduction step*' is carried out.

Let us therefore select such a cluster of formulas and from each of its sides one uppermost formula of the kind mentioned. We shall deal with the case in which the outermost logical symbol of the cluster formulas in a $\forall$, since the reduction for the case in which the outermost logical symbol is a $\exists$ proceeds completely symmetrically to the case $\forall$. The derivation therefore looks like this:

$$\begin{array}{ccl}
\vdots\,(a) & \vdots & \\
\dfrac{\Gamma_1 \to \Theta_1, F_1(a)}{\Gamma_1 \to \Theta_1, \forall x F(x)} & \dfrac{F_2(t), \Gamma_2 \to \Theta_2}{\forall x F_2(x), \Gamma_2 \to \Theta_2} & \text{The two predicate inference figures} \\
\vdots & \vdots & \\
\multicolumn{2}{c}{\dfrac{\Gamma \to \Theta, \forall x F_1(x) \qquad \forall x F(x), \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda}} & \text{The cut associated with the cluster} \\
\multicolumn{2}{c}{\vdots} & \\
\multicolumn{2}{c}{\Pi \to \Sigma \quad \text{The endsequent.}} &
\end{array}$$

In the above diagram the formulas $\forall x F_1(x), \forall x F_2(x)$ and $\forall x F(x)$ all belong to the cluster. The reduction step consists now of the transformation of the derivation into two derivations. One of them is of the form indicated by the following diagram:

$$\begin{array}{rcc}
 & \vdots\,(t) & \vdots \\
\text{thinning and possibly several interchanges and substitutions of terms} & \dfrac{\Gamma_1 \to \Theta_1, F_1(t)}{\Gamma_1 \to F(t), \Theta_1, \forall x F_1(x)} & \dfrac{F_2(t), \Gamma_2 \to \Theta_2}{\forall x F_2(x), \Gamma_2 \to \Theta_2} \\
 & \vdots & \vdots \\
 & \multicolumn{2}{c}{\dfrac{\Gamma \to F(t), \Theta, \forall x F(x) \qquad \forall x F(x), \Delta \to \Lambda}{\Gamma, \Delta \to F(t), \Theta, \Lambda}\ \text{cut}} \\
 & \multicolumn{2}{c}{\vdots} \\
 & \multicolumn{2}{c}{\Pi \to F(t), \Sigma \quad \text{The endsequent.}}
\end{array}$$

The other is of the form indicated by the following diagram:

$$\begin{array}{ccl}
\vdots\,(a) & \vdots & \\
\dfrac{\Gamma_1 \to \Theta_1, F_1(a)}{\Gamma_1 \to \Theta_1, \forall x F_1(x)} & \dfrac{F_2(t), \Gamma_2 \to \Theta_2}{\forall x F_2(x), \Gamma_2, F(t) \to \Theta_2} & \text{thinning and possibly several interchanges and substitutions of terms} \\
\vdots & \vdots & \\
\multicolumn{2}{c}{\dfrac{\Gamma \to \Theta, \forall x F(x) \qquad \forall x F(x), \Delta, F(t) \to \Lambda}{\Gamma, \Delta, F(t) \to \Theta, \Lambda}\ \text{cut}} & \\
\multicolumn{2}{c}{\vdots} & \\
\multicolumn{2}{c}{\Pi, F(t) \to \Sigma \quad \text{The endsequent.}} &
\end{array}$$

The former is obtained from the given derivation first by substituting the term $t$ for each occurrence of the free variable $a$ in the section of the derivation standing above the lower sequent of the introduction of $\forall$ in succedent, next by replacing the predicate inference figure by a thinning and possibly several interchanges and substitutions of terms, and finally inserting the formula $F(t)$ as the first succedent formula into every sequent below the sequent $\Gamma_1 \to F(t), \Theta_1, \forall x F_1(x)$. Since $t$ is a closed term and the formula $F(t)$ is quantifier-free, the new endsequent is also quantifier-free and closed.

The latter is obtained by the similar treatment.

6.39. If there exists a formula in the succedent of the endsequent which belongs to the same cluster of formulas as the principal formula of one of the *TJ*-inference figures, then the second critical reduction step may be carried out.

If there are several those formulas, we shall agree to choose that *TJ*-inference figure with the above property which stands furthest to the right. Suppose that its principal formula is $\mathcal{E}(t)$ and the derivation looks like this:

$$\begin{array}{cl} \vdots\ (a) & \\ \dfrac{a<t, \Gamma \to \Theta, \mathcal{E}(a)}{\Gamma \to \Theta, \mathcal{E}(t)} & \text{The } TJ\text{-inference figure} \\ \vdots & \\ \Pi \to \Sigma & \text{The endsequent.} \end{array}$$

We now define the '*second critical reduction step*':

We first specify an arbitrary ordinal number and designate it by $\beta$. The reduction step consists of the transformation of the derivation into the form indicated by the following diagram:

$$\begin{array}{cl} \vdots\ (\beta) & \\ \dfrac{\beta<t, \Gamma \to \Theta, \mathcal{E}(\beta)}{\Gamma, \beta<t \to \mathcal{E}(\beta), \Theta, \mathcal{E}(t)} & \begin{array}{l}\text{thinnings and possibly} \\ \text{several interchanges}\end{array} \\ \vdots & \\ \Pi, \beta<t \to \mathcal{E}(\beta), \Sigma & \text{The endsequent.} \end{array}$$

It is obtained from the given derivation first by substituting the ordinal number $\beta$ for each occurrence of the free variable $a$ in the section standing above the lower sequent of the *TJ*-inference figure, next by replacing the *TJ*-inference figure by a thinning and possibly several interchanges, and finally by inserting the formula $\beta<t$ as the last antecedent formula and $\mathcal{E}(\beta)$ as the first succedent formula into every sequent below the sequent

$$\Gamma, \beta<t \to \mathcal{E}(\beta), \Theta, \mathcal{E}(t).$$

This completes the definition of the reduction steps.

6.4. The *correlation of ordinal numbers with derivations* and the proofs of the two facts that with each reduction step the value of the derivation diminishes and that with the second critical reduction step the value diminishes at least by $\omega$ in the sense of natural sum.

6.41. We can adapt the correlation of ordinal numbers with derivations with minor three modifications:

If the inference figure has two upper sequents and is not $CJ$, then the ordinal number of the line of inference is defined to be the natural sum of the ordinal numbers of the two upper sequents.

If the inference figure is a $CJ$, then the ordinal number of the line of inference is defined to be the natural sum of the ordinal number of the left upper sequent and $\omega^{\alpha_1+1}$, where $\omega^{\alpha_1}+\cdots$ is the ordinal number of the right upper sequent.

If the inference figure is a $TJ$, then the ordinal number of the line of inference is defined to be the natural sum of $\omega$ and the ordinal number of the upper sequent.

6.42. Then we may also verify that in each reduction step the value of the derivation diminishes.

In the 'first preparatory step' the value of the derivation remains unchanged.

For the $CJ$-reduction, the value of the derivation diminishes as in [4].

For the quantifier-free cut reduction, the value of the derivation diminishes. Since the cut formula $D$ is quantifier-free, by the replacement of the cut by thinnings and interchanges the level of a whole collection of sequents does not change. Moreover the derivational sections standing above the sequent $\Gamma\rightarrow\Theta, D$ (or $D, \Delta\rightarrow\Lambda$) in the first (or second) reduced derivation have the correspondent in the old derivation. Hence the sequent $\Gamma\rightarrow\Theta, D$ (or $D, \Delta\rightarrow\Lambda$) in the reduced derivation and the corresponding upper sequent of the cut receives the same ordinal number. And a decrease in the ordinal number results from the replacement of the natural sum of two numbers by only one of these two numbers. But, since no $CJ$-inference figures occurs below the sequent, the decrease is preserved in the calculation of the ordinal number further down to the endsequent. Thus the value of each reduced derivation is smaller than that of the given derivation.

For the propositional reduction, the value of the derivation diminishes. This case is dealt with as in the case of quantifier-free cut reduction. By the replacement of a propositional inference figure by a thinning and possibly several interchanges the level of a whole collection of sequents does not change. A decrease in the ordinal number results from the replacement, and the decrease is preserved in the calculation of the ordinal number further down to the endsequent. Thus the value of the derivation diminishes.

For the second preparatory step, as proved in [4], we can verify that this preparatory step cannot cause an increase in the ordinal number.

For the predicate reduction we must demonstrate a decrease of the value. Our definition of 'level' is different from that in [4]. But in this case the principal formula of the selected predicate inference figure contains at least two quantifiers and we can still prove the existence of the *level line*, and the proof of a decrease of the ordinal number is carried out in the same way as in [4].

We now come to the first critical reduction. In the derivation the only predicate inference figure has disappeared and been replaced by structural inference figures. This replacement has no influence on the level of a whole derivational sequents and on the ordinal number of the sequent corresponding to the upper sequent of the predicate inference figure. At this point a decrease in the ordinal number has therefore taken place which is preserved down to the endsequent. Hence a decrease of the value is proved.

6.43. Finally we shall examine the second critical reduction. In this case we must prove not only the fact that the value $\gamma_1$ of the reduced derivation is smaller than the value $\gamma$ of the derivation but also the fact that $\gamma_1 \# \omega \leqq \gamma$. In the derivation the *TJ*-inference figureh as replaced by some interchanges. Clearly this replacement has no influence on the level of a whole derivational sequents. Hence the sequent $\beta < t,\ \Gamma \rightarrow \Theta, \mathcal{E}(\beta)$ receives the same ordinal number (designated by $\alpha$) as the sequent $a < t,\ \Gamma \rightarrow \Theta,\ \mathcal{E}(a)$. Then the place of the lower sequent of *TJ*-inference figure with the ordinal number $\alpha \# \omega$ is taken by a derivational section whose lowest sequent is the sequent $\Gamma, \beta < t \rightarrow \mathcal{E}(\beta), \Theta,\ \mathcal{E}(t)$ correlated with the ordinal number $\alpha$. The decrease is preserved or is progressed as we pass down to the endsequent. (If there occurs a cut below the *TJ*-inference figure and the difference in levels between the lower and the upper sequent is greater than 0, then and only then the decrease of the ordinal number is progressed.) Thus we can conclude that $\gamma_1 \# \omega \leqq \gamma$. Hence we have proved that with the second critical reduction step the value diminishes at least by $\omega$ in the sense of natural sum.

6.5. *Conclusion.* Now the exact version of Fundamental Lemma in the case of $\rho \geqq 1$, stated in 6.13, follows:

Suppose that a $\rho$-derivation is given and that its endsequent is a quantifier-free and closed sequent $\Pi \rightarrow \Sigma$. Then the sequent $\forall x(x < \gamma \supset \mathcal{E}(x)), \Pi \rightarrow \Sigma$ is provable for the value $\gamma$ of the $\rho$-derivation.

Between $\rho$-derivations and their values we have the following relationship:

The value of $\rho$-derivation is less than $\omega_{\rho+2}$ for $\rho \geqq 1$.

This is easily seen from the definition of the correlation of ordinal numbers and the fact that the level of every sequent in a $\rho$-derivation is at most $\rho - 1$.

From these two results we shall obtain Fundamental Lemma in the case of $\rho \geqq 1$:

*Let $\Pi \to \Sigma$ be a quantifier-free and closed sequent, and $\rho$ a natural number. If there exists a $\rho$-derivation of $\Pi \to \Sigma$, then the sequent*

$$\forall x(x<\gamma \supset \mathcal{E}(x)), \Pi \to \Sigma$$

*is provable for some ordinal number $\gamma$ less than $\omega_{\rho+2}$.*

## § 7. A direct proof of Mints' result.

7.0. In this section we shall state a direct proof of Mints' result by using our method.

The *quantifier complexity* of a formula, defined by Mints [7], is the maximum number of quantifier alternations in the chains of mutually regulating occurrences of quantifiers in that formula. Then Mints' result can be stated as follows:

*Transfinite induction up to $\omega_{\nu+3}$ and higher ordinal numbers cannot be proved in elementary number theory only by using mathematical inductions whose induction formulas are at most of quantifier complexity $\nu$.*

7.01. We shall introduce an auxiliary concept.

A formula $A$ is called a *normal formula* if it satisfies the following conditions (i) and (ii):

( i ) If $\forall xF(x)$ or $\exists xF(x)$ is a subformula (of $A$) with quantifier complexity 0, then the formula $F(a)$ is quantifier-free.

( ii ) If $\forall xF(x)$ or $\exists xF(x)$ is a subformula (of $A$) with quantifier complexity $\nu+1$, then the formula $F(a)$ is of quantifier complexity $\nu$.

Then the following properties are easily proved:

(1) Every formula has an equivalent normal formula with the same quantifier complexity.

(2) Every subformula of a normal formula is normal.

The proof of property (1) is carried out by a mathematical induction on the number of quantifiers in the given formula.

7.1. In the arguments stated in § 2 and § 4, we read '*quantifier complexity at most $\nu$*' and '*normal formula*' for 'at most $\rho$ quantifiers' and 'prenex formula' respectively, and we can easily see that both Proposition 1 and Proposition 2 hold. Hence, in order to prove Mints' result, it suffices to prove

THEOREM 2. *Let $\alpha$ be an ordinal number. If there exists a $\nu$-derivation of the sequent $\to \mathcal{E}(\alpha)$, then $\alpha$ is less than $\omega_{\nu+3}$.*

In the above, *$\nu$-derivation* is, in this case, a derivation consisting of normal formulas with quantifier complexity at most $\nu$.

On the other hand, Theorem 2 can be inferred (as in 4.3) from

FUNDAMENTAL LEMMA. *Let $\Pi \rightarrow \Sigma$ be a quantifier-free and closed sequent. If there exists a $\nu$-derivation of the sequent $\Pi \rightarrow \Sigma$, then the sequent*

$$\forall x(x < \gamma \supset \mathcal{E}(x)), \Pi \rightarrow \Sigma$$

*is provable for some ordinal number $\gamma$ less than $\omega_{\nu+3}$.*

Hence, for our purpose, it suffices to prove the lemma. In the case of the proof we need not consider the classification of cases such that $\nu=0$ or $\nu \geqq 1$. The proof of the lemma is carried out by similar procedure as that of Fundamental Lemma in the case of $\rho \geqq 1$ (cf. § 6).

7.2. The proof of the lemma consists of four stages. The only *derivations* considered in the following are those which consists of normal formulas and in which all basic logical sequents are quantifier-free.

7.21. We define '*reduction steps*' for arbitrary derivations of quantifier-free closed sequent which contains at least one *TJ*-inference figure. The reduction steps are those defined in 6.3 and a new kind of reduction steps.

The *CJ-reduction step* (6.32) is now carried out when there is a *CJ*-inference figure with *closed principal formula* (1.14) in the entire derivation. (It is not necessary that the lower sequent of the *CJ*-inference figure belongs to the ending.)

The *propositional reduction* (6.34) is carried out only when the ending includes a propositional inference figure with *quantifier-free principal formula* but does not include a cut with *quatifier-free cut formula*.

7.210. In order to define the new kind of reduction steps, we begin by modifying the concepts 'predecessor' and 'ancestor' defined in 4.213.1. In the 'substitutions of terms' (6.2), the formula $F(s)$ in the upper sequent is a predecessor of the formula $F(t)$ in the lower sequent, and every formula of $\Gamma_1, \Gamma_2$, etc., occurring in the upper sequent is a predecessor of one occurring in the same place in the lower sequent (as usual).

Now the ancestors of a formula therefore need not be entirely identical, but may differ formally in the values of their terms.

7.211. The *new kind of reduction steps* is necessary, because the formulas in the given derivation are not always in prenex forms in this case. It is carried out when the given derivation satisfies the following conditions (i) and (ii):

(i) There occurs no *CJ*-inference figure with closed principal formula.

(ii) There occurs a cut in the ending whose cut formula has a propositional connective as its outermost logical symbol and contains at least one quantifier.

Let us consider such a cut in (ii). Since the other cases are similar we shall only deal with the case when the outermost logical symbol is $\wedge$. Suppose that our derivation looks like this:

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \rightarrow \Theta, A\vee B\end{matrix} \qquad \begin{matrix}\vdots \\ A\vee B, \Delta \rightarrow \Lambda\end{matrix}}{\Gamma, \Delta \rightarrow \Theta, \Lambda} \quad \text{The cut concerned}$$

$$\begin{matrix}\cdot \quad \cdot \quad \cdot \\ \cdot \ \ \cdot \ \ \cdot \\ \cdots \\ \cdot \\ \Pi \rightarrow \Sigma .\end{matrix}$$

Then the transformation is to construct a derivation of the form:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \rightarrow \Theta, B\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots \\ \Gamma \rightarrow \Theta, A\end{matrix} \quad \begin{matrix}\vdots \\ A, B, \Delta \rightarrow \Lambda\end{matrix}}{\Gamma, B, \Delta \rightarrow \Theta, \Lambda}\ \text{The first new cut}}{B, \Gamma, \Delta \rightarrow \Theta, \Lambda}}{\Gamma, \Gamma, \Delta \rightarrow \Theta, \Theta, \Lambda}\ \text{The second new cut}}{\Gamma, \Delta \rightarrow \Theta, \Lambda}$$

$$\begin{matrix}\cdot \quad \cdot \quad \cdot \\ \cdot \ \ \cdot \ \ \cdot \\ \cdots \\ \cdot \\ \Pi \rightarrow \Sigma .\end{matrix}$$

In the above diagram, above the left upper sequent $\Gamma \rightarrow \Theta, A$ of the first new cut and above the left upper sequent $\Gamma \rightarrow \Theta, B$ of the second new cut we write those obtained with slight modifications from the derivational section (of the given derivation) standing above the left upper sequent $\Gamma \rightarrow \Theta, A\wedge B$ of the cut concerned; and above the right upper sequent $A, B, \Delta \rightarrow \Lambda$ of the first new cut we write a modification of that standing above the right upper sequent $A\wedge B, \Delta \rightarrow \Lambda$ of the cut concerned. Except the derivational section standing above the sequent $\Gamma, \Delta \rightarrow \Theta, \Lambda$, everything else is left exactly as it was in the given derivation.

For example, the derivational section standing above the sequent $\Gamma \rightarrow \Theta, A$ is defined as follows:

In the derivational section standing above the sequent $\Gamma \rightarrow \Theta$, $A\wedge B$, we first replace all ancestors of the left cut formula $A\wedge B$ by their first conjuncts respectively. Next, if an uppermost ancestor of the cut formula is a principal formula of an introduction of $\wedge$ in succedent, then we omit the section standing above the right upper sequent of the inference figure.

In the given derivation, the ending contains no free variable (by the first preparatory step), and every ancestor of the left cut formula is closed. Hence, by (i), each ancestor of the cut formula is not a principal formula of *CJ*. Moreover each ancestor contains a quantifier. Hence each uppermost ancestor is either a thinning formula or a principal formula of an introduction of $\wedge$ in succedent. This ensures that by the treatments stated before the correctness of all basic sequents and inference figures is preserved.

7.22. We define the '*value*' of the derivation (6.12, 6.4). The correlation of ordinal numbers with derivations are defined in the same way as in 6.4 with the following modifications (i), (ii) and (iii):

( i ) If the inference figure is a *CJ*, then the ordinal number of the line of the inference is $\omega^{\alpha_1+1}+\cdots+\omega^{\alpha_\nu+1}$, where $\omega^{\alpha_1}+\cdots+\omega^{\alpha_\nu}$ is the natural sum of the ordinal number of the right upper sequent and the left upper sequent.

( ii ) If the inference figure is a cut, then the ordinal number of the line of the inference is the natural sum of $2^\nu$ $\alpha$'s and $2^\nu$ $\beta$'s, where $\nu$ is the number of occurrences of propositional connectives not included in the scope of quantifiers in the cut formula, and $\alpha$ and $\beta$ are the ordinal numbers of the upper sequents.

(iii) The *level* of a derivational sequent is defined now to be the maximum of quantifier complexities of all formulas occurring in or below the sequent concerned.

The modifications of definition on the correlation of ordinal numbers are caused by the following facts: The *CJ*-reduction is carried out whenever there occurs a *CJ* with closed principal formula in the entire derivation; a new kind of reduction steps is introduced.

Then it is proved that with each reduction step the value diminishes. Especially with the second critical reduction step the value diminishes at least $\omega$ in the sense of natural sum.

7.23. As in 6.13 by using a transfinite induction on the value of the derivation, we can prove an exact version of Fundamental Lemma:

*Let be given a derivation of a quantifier-free and closed sequent. Let $\gamma$ be the value of the derivation. Suppose that its endsequent is $\Pi\rightarrow\Sigma$. Then the sequent*

$$\forall x(x<\gamma\supset\mathcal{E}(x)), \Pi\rightarrow\Sigma$$

*is provable.*

7.24. The estimate of the value of $\nu$-derivation is easily obtained:

*The value of $\nu$-derivation is less than $\omega_{\nu+3}$ for all $\nu\geqq 0$.*

7.25. *Conclusion.* From the estimate (7.24) and the exact version of Fundamental Lemma (7.23), we can obtain Fundamental Lemma in 7.1, and consequently Mints' result.

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