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A GENERALIZATION OF GROUPS WITH A ROOT DATA AND COVERINGS OF THE GROUPS

By

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0. Introduction

The groups of k-rational points of semi-simple algebraic groups defined over a field k or simple groups of Lie types have a structure of the groups with BN-pairs (Tits system, cf. [2] Chap. IV) or the groups with a root data (due to Bruhat-Tits [3]). On the other hand, Chevalley groups (normal or twisted) over a commutative ring with an identity have also root subgroups but in general, they are neither the groups with BN-pairs nor the groups with a root data. In this note, we treat these groups axiomatically. Namely, we generalize the axioms for the groups with a root data to be able to apply to these groups. Further, we can construct universal convering groups of these groups in the same way as those of R. Steinberg [7].

As for the central extensions of groups of Lie types, C.W. Curtis ([5]) has treated axiomatically and the universal extension of Chevalley groups over a commutative ring has been treated by M. Stein [6] and the result has been generalized to the twisted case by the author [1]. Some of these results can be generalized and simplified by our method.

I would like to express my warm thanks to Professor J. Tits who suggested me the axiomatic treatement of these groups in the same way as those of the groups with a root data and to Professor C.W. Curtis who has invited me to the University of Oregon at the fall term in 1976 and has given many kind discussions about the problem during my stay there.

1. Definition of a group with a root data.

Let E^n be a Euclidean space of dimension n. A subset Φ of E^n is called a *root system* if it satisfies the following properties:

(*SR* 1) *Φ is a finite subset of E^n such that $0 \notin \Phi$ and $\Phi = -\Phi$ and further Φ spans E^n.*

(*SR* 2) *For any $\alpha \in \Phi$, let σ_α be the orthogonal transformation of E^n defined by $\sigma_\alpha(x)$*

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$=x-\{2(x,\alpha)/(\alpha,\alpha)\}\alpha$ *for any* $x\in E^n$. *Then* $\sigma_\alpha(\Phi)=\Phi$.

(*SR* 3) *For any* $\alpha,\beta\in\Phi$, $2(\beta,\alpha)/(\alpha,\alpha)$ *is an integer.*

The subgroup W of the group of automorphisms of E^n generated by σ_α for all $\alpha\in\Phi$ is called the *Weyl group* of Φ.

A subset Π of Φ is called a *base* of Φ if it satisfies

(*B* 1) $\Pi=\{\alpha_1,\cdots,\alpha_n\}$ *is a base of* E^n.

(*B* 2) *For any root* $\alpha\in\Phi$, $\alpha=\sum_{i=1}^n c_i\alpha_i$ *where* c_i *are all non-negative integers or all non-positive integers.*

There exists always a base of Φ and with respect to a base Π of Φ, we set $\Phi^+=\left\{\alpha=\sum_{i=1}^n c_i\alpha_i\in\Phi;\ c_i\geqq 0\right\}$ and $\Phi^-=-\Phi^+$. Then $\Phi=\Phi^+\cup\Phi^-$ and $\Phi^+\cap\Phi^-=\phi$. An element of Φ^+ (resp. Φ^-) is called *positive* (resp. *negative*) root of Φ. $\Phi_{red}=\{\alpha\in\Phi;\ (1/2)\alpha\notin\Phi\}$ is also a root system and is called the reduced part of Φ. A root system Φ is called reduced if $\Phi=\Phi_{red}$. The Weyl group of Φ is generated by $\sigma_{\alpha_i}(1\leqq i\leqq n)$. As for the properties of root system, see [2], Chap. VI. We shall fix a base Π of Φ once for all.

A *root data* of type Φ in a group G due to Bruhat-Tits ([3], § 6.1) is a system $\{T, U_\alpha, M_\alpha\}_{\alpha\in\Phi}$ which satisfies the following properties:

(*DR* 1) *T is a subgroup of G and for any root $\alpha\in\Phi$, U_α is a subgroup of G which is not the identity group.*

(*DR* 2) *For any roots $\alpha,\beta\in\Phi$ such that $\beta\notin-\boldsymbol{Q}^+\alpha$, where $\boldsymbol{Q}^+$ is the set of positive ratinal numbers, the commutator subgroup $[U_\alpha, U_\beta]$ is contained in the group generated by $U_{p\alpha+q\beta}$ for $p,q\in\boldsymbol{N}$ such that $p\alpha+q\beta\in\Phi$, where we define $U_\phi=\{1\}$ for the empty set ϕ.*

(*DR* 3) *If α and 2α are roots in Φ, then $U_{2\alpha}\subsetneqq U_\alpha$.*

(*DR* 4) *For any root α, M_α is a left coset with respect to T and $U^*_{-\alpha}=U_{-\alpha}-\{1\}\subset U_\alpha M_\alpha U_\alpha$.*

(*DR* 5) *For any roots α, $\beta\in\Phi$ and $n\in M_\alpha$,*

$$nU_\beta n^{-1}=U_{\sigma_\alpha(\beta)}.$$

(*DR* 6) *If U^+ (resp. U^-) is the subgroup of G generated by U_α for all positive (resp. negative) roots $\alpha\in\Phi$, then $TU^+\cap U^-=\{1\}$.*

A root data is called *generative* if it satisfies further

(*DR* 7) *G is generated by T and U_α for all $\alpha\in\Phi$.*

Let $G=SL_{n+1}(k)$ be the special linear group over a field k. Let E^{n+1} be the Euclidean space of dimension $n+1$ and $\{e_1,\cdots,e_{n+1}\}$ be the set of unit vectors. Then $\Phi=\{\alpha_{ij}=e_i-e_j;\ i\neq j, 1\leqq i,j\leqq n+1\}$ is a root system in the subspace $E^n=\left\{\sum_{i=1}^{n+1} x_ie_i\in E^{n+1};\ x_1+\cdots+x_{n+1}=0\right\}$ of E^{n+1} of dimension n, and $\Pi=\{\alpha_i=e_i-e_{i+1};\ 1\leqq i\leqq n\}$ is a base of Φ. Let e_{ij} be the matrix unit, namely $(n+1)\times(n+1)$ matrix whose (i,j)-entry is

1 and all the other entries are 0. For any $t\in k$, we set $x_{ij}(t)=I+te_{ij}$, where I is the unit matrix of degree $n+1$. Now, let $U_{\alpha_{ij}}$ be the subgroup of G generated by $x_{ij}(t)$ for all $t\in k$. For any $t\in k^*=k-\{0\}$, set

$$w_{ij}(t)=x_{ij}(t)x_{ji}(-t^{-1})x_{ij}(t)$$

$$h_{ij}(t)=w_{ij}(t)w_{ij}(-1)$$

and let T be the subgroup of G generated by $h_{ij}(t)$ for all $t\in k^*$ and all (i, j) such that $i\neq j$, $1\leqq i, j\leqq n+1$ and let $M_{\alpha_{ij}}$ be the coset $Tw_{ij}(1)$. Then, the system $\{T, U_{\alpha_{ij}}, M_{\alpha_{ij}}\}_{\alpha_{ij}\in\Phi}$ is a generative root data for the group G.

Now, let R be a commutative ring with an identity and $G=SL_{n+1}(R)$. The subgroup $U_{\alpha_{ij}}$ of G generated by $x_{ij}(t)$ for all $t\in R$ is called a root subgroup of G. For any element t of the group R^* of the units of R, we can define $w_{ij}(t)$ and $h_{ij}(t)$ and also the subgroup T and the coset $M_{\alpha_{ij}}=Tw_{ij}(1)$. Then, the system $\{T, U_{\alpha_{ij}}, M_{\alpha_{ij}}\}_{\alpha_{ij}\in\Phi}$ satisfies the properties (DR 1) to (DR 6) except (DR 4).

Let $\mathfrak{m}$ be a maximal ideal of R and $k=R/\mathfrak{m}$. Then, we have a natural homomorphism

$$\phi_{\mathfrak{m}}:\ SL_{n+1}(R)\longrightarrow SL_{n+1}(R/\mathfrak{m})$$

where in the group $SL_{n+1}(R/\mathfrak{m})$, we can define a root data which satisfies (DR 1) $\sim$(DR 7). So that, corresponding to each maximal ideal of R, there is some system of subgroups in $SL_{n+1}(R)$ which induces the root data in $SL_{n+1}(R/\mathfrak{m})$. This leads to the following generalization of a group with a root data.

Let $\Lambda=\Lambda_0\cup\{1\}$ be a set, where $1\notin\Lambda_0$ and Φ be a root system. A system $\{T^\lambda, U_\alpha{}^\lambda, M_\alpha\}_{\alpha\in\Phi,\lambda\in\Lambda}$ which satisfies the following properties is called a (*generalized*) *root data* of type Φ in a group G.

(DRG 1) *T^λ, $U_\alpha{}^\lambda$ ($\lambda\in\Lambda$, $\alpha\in\Phi$) are subgroups of G. We denote by T, U_α the groups T^1, $U_\alpha{}^1$ respectively. Then $U_\alpha{}^\lambda$ is a proper normal subgroup of U_α for any $\lambda\in\Lambda_0$ and $\alpha\in\Phi$. T^λ contains T for any $\lambda\in\Lambda_0$.*

(DRG 2) *For any roots α, $\beta\in\Phi$ such that $\beta\notin-\boldsymbol{Q}^+\alpha$ and for any $\lambda\in\Lambda$, the commutator subgroup $[U_\alpha{}^\lambda, U_\beta]$ is contained in the group generated by $U_{p\alpha+q\beta}^{\lambda}$ for $p, q\in\boldsymbol{N}$ such that $p\alpha+q\beta\in\Phi$ where we define $U_\phi=\{1\}$ for the empty set ϕ.*

(DRG 3) *If α and 2α are roots in Φ, then $U_{2\alpha}{}^\lambda\subset U_\alpha{}^\lambda$ for all $\lambda\in\Lambda$ and $U_{2\alpha}\subsetneqq U_\alpha$.*

(DRG 4) *For any root $\alpha\in\Phi$, M_α is a left coset with respect to T such that $M_\alpha=M_{-\alpha}=M_\alpha{}^{-1}$. For any $\lambda\in\Lambda_0$, set $V_\alpha{}^\lambda=U_\alpha-U_\alpha{}^\lambda$ and let K^λ be the normal subgroup generated by $U_\alpha{}^\lambda$ for all $\alpha\in\Phi$. Then*

(i) *$U_\alpha\cap K^\lambda=U_\alpha{}^\lambda$ for any $\lambda\in\Lambda_0$ and $\alpha\in\Phi$.*

(ii) *$V_\alpha=\bigcap_{\lambda\in\Lambda_0}V_\alpha{}^\lambda\neq\phi$ and $V_{-\alpha}\subset V_\alpha M_\alpha V_\alpha$ for all $\alpha\in\Phi$.*

(iii) *Let $M_\alpha{}^\lambda$ be the coset $T^\lambda n$ for $n\in M_\alpha$. Then*

$$V_{-\alpha}^{\lambda} \subset V_\alpha{}^\lambda M_\alpha{}^\lambda V_\alpha{}^\lambda \text{ for any } \lambda \in \Lambda_0 \text{ and } \alpha \in \Phi.$$

(*DRG* 5) *For any roots* α, $\beta \in \Phi$ *and* $n \in M_\alpha$,

$$n U_\beta{}^\lambda n^{-1} = U_{\sigma_\alpha(\beta)}^\lambda \text{ for any } \lambda \in \Lambda .$$

(*DRG* 6) *If* U^+ *(resp.* U^-*) is the subgroup of* G *generated by* U_α *for all positive (resp. negative) roots* $\alpha \in \Phi$, *then*

(i) $TU^+ \cap U^- = \{1\}$.

(ii) $T^\lambda K^\lambda U^+ \cap K^\lambda U^- = K^\lambda$ *for any* $\lambda \in \Lambda_0$.

A root data is called to be *generative* if it satisfies (*DR* 7) and further called to be *strongly generative* if it satisfies

(*DR* 7′) *G is generated by* U_α *for all* $\alpha \in \Phi$.

Further, if a root data $\{T^\lambda, U_\alpha{}^\lambda, M_\alpha\}_{\alpha \in \Phi, \lambda \in \Lambda}$ satisfies the following properties, it is called a *strict root data* in G.

(*DRG* 8) *Let* $K = \bigcap_{\lambda \in \Lambda_0} K^\lambda$, *then* $K \subset U^+ T U^-$.

(*DRG* 9) $\bigcap_{\lambda \in \Lambda_0} T^\lambda K^\lambda = TK$.

If Λ_0 consists of a single element $\{\lambda\}$ and $U_\alpha{}^\lambda = \{1\}$ for any root $\alpha \in \Phi$ and $T^\lambda = T^1$, then the root data is called to be *simple*. A simple root data coincides with a root data defined by Bruhat-Tits. If Λ is a finite set, the root data is called to be *of finite type* and if $K = \{1\}$, the root data is called to be *semi-simple*.

Let $\{T^\lambda, U_\alpha{}^\lambda, M_\alpha\}_{\alpha \in \Phi, \lambda \in \Lambda}$ be a root data in a group G. For any $\lambda \in \Lambda_0$, set $G_\lambda = G/K^\lambda$ and define $U_{\lambda\alpha} = U_\alpha K^\lambda / K^\lambda$ which is isomorphic to $U_\alpha / U_\alpha \cap K^\lambda = U_\alpha / U_\alpha{}^\lambda$ (*DRG* 4–i), $M_{\lambda\alpha} = M_\alpha{}^\lambda K^\lambda / K^\lambda$ and $T_\lambda = T^\lambda K^\lambda / K^\lambda$. Then the system $\{T_\lambda, U_{\lambda\alpha}, M_{\lambda\alpha}\}_{\alpha \in \Phi}$ is a simple root data in the group G_λ. Also, set $\bar{G} = G/K$ and for any $\lambda \in \Lambda$, define $\bar{T}^\lambda = T^\lambda K/K$, $\bar{U}_\alpha{}^\lambda = U_\alpha{}^\lambda K/K$ and $\bar{M}_\alpha = M_\alpha K/K$. Then the system $\{\bar{T}^\lambda, \bar{U}_\alpha{}^\lambda, \bar{M}_\alpha\}_{\alpha \in \Phi, \lambda \in \Lambda}$ is a semi-simple root data in the group $\bar{G}$. In fact, by (*DRG* 4–ii), we have $\bar{U}_\alpha{}^\lambda \neq \bar{U}_\alpha$ for any $\lambda \in \Lambda_0$. It is clear that $U_\alpha K \cap K^\lambda \supset U_\alpha{}^\lambda K$ and conversely, if $k' = uk \in U_\alpha K \cap K^\lambda$, then $u = k'k^{-1} \in K^\lambda \cap U_\alpha = U_\alpha{}^\lambda$ by (*DRG* 4–i) and we have $k' \in U_\alpha{}^\lambda K$. Therefore, $U_\alpha K \cap K^\lambda \subset U_\alpha{}^\lambda K$. Further, $TKU^+ \cap KU^- \subset K^\lambda$ for all $\lambda \in \Lambda_0$ by (*DRG* 6–ii). Therefore, $TKU^+ \cap KU^- \subset K$. Since $TKU^+ \cap KU^- \supset K$, we have $TKU^+ \cap KU^- = K$. This gives (*DRG* 6–ii) for the system $\{\bar{T}^\lambda, \bar{U}_\alpha{}^\lambda, \bar{M}_\alpha\}_{\alpha \in \Phi, \lambda \in \Lambda}$. Other axioms follows from axioms for the group G.

2. Examples of groups with a root data

As an example of a group with a root data, first we take up a Chevalley group over a commutative ring (cf. M. Stein [6]).

Let R be a commutative ring with an identity and G be a Chevalley-Demazure group Scheme of type Φ, $G(R)$ be the group of points of G in R. For each $\alpha \in \Phi$, there is a monomorphism

$$t \longmapsto e_\alpha(t)$$

of the additive group $G_a(R)$ of R onto a subgroup $U_\alpha = U_\alpha^1$ of $G(R)$. The subgroup $E(R)$ generated by U_α for all $\alpha \in \Phi$ is called the elementary subgroup of $G(R)$. Let $Spm\ R = \{\mathfrak{m}_\lambda;\ \lambda \in \Lambda_0\}$ be the set of all maximal ideals of R. Denote by U_α^λ the subgroup of U_α generated by $e_\alpha(t)$ for all $t \in \mathfrak{m}_\lambda$. Then, U_α^λ is a proper normal subgroup of U_α for any $\lambda \in \Lambda_0$ and $V_\alpha^\lambda = U_\alpha - U_\alpha^\lambda$ is the set of elements $e_\alpha(t)$ for all $t \in R - \mathfrak{m}_\lambda$. Then, we have

$$V_\alpha = \bigcap_{\lambda \in \Lambda_0} V_\alpha^\lambda = \{e_\alpha(t);\ t \in R^*\},$$

where R^* is the multiplicative group of the units of R.

For any $t \in R^*$, we set

$$w_\alpha(t) = e_\alpha(t) e_{-\alpha}(-t^{-1}) e_\alpha(t)$$
$$h_\alpha(t) = w_\alpha(t) w_\alpha(-1).$$

Further, for any pair of elements $t, u \in R - \mathfrak{m}_\lambda$ such that $tu \equiv 1 \pmod{\mathfrak{m}_\lambda}$, we set

$$w_\alpha^\lambda(t, u) = e_\alpha(t) e_{-\alpha}(-u) e_\alpha(t)$$
$$h_\alpha^\lambda(t, u) = w_\alpha^\lambda(t, u) w_\alpha(-1).$$

Now, let T^λ (resp. $T^1 = T$) be the subgroup of $E(R)$ generated by $h_\alpha^\lambda(t, u)$ for all $\alpha \in \Phi$ and all elements t, u of $R - \mathfrak{m}_\lambda$ such that $tu \equiv 1 \pmod{\mathfrak{m}_\lambda}$ (resp. by $h_\alpha(t)$ for all $\alpha \in \Phi$ and $t \in R^*$), and by M_α the left coset $Tw_\alpha(1)$. Then, we have

PROPOSITION 2.1 *Let $G(R)$ be a Chevalley group over a commutative ring R with an identity of type Φ. Then, the system $\{T^\lambda, U_\alpha^\lambda, M_\alpha\}_{\alpha \in \Phi, \lambda \in \Lambda}$ is a root data in the group $G(R)$ and it is a strongly generative, strict root data in the elementary subgroup $E(R)$ of $G(R)$.*

Note that if R is a field, the root data is simple and if R is semi-local (resp. without radical), then the root data is of finite type (resp. semi-simple).

PROOF (*DRG* 1, 2, 3) are clear from definition. Since K^λ is contained in the kernel of the natural homomorphism $\phi_\lambda : E(R) \to E(R/\mathfrak{m}_\lambda)$, we have $U_\alpha \cap K^\lambda = U_\alpha^\lambda$ (*DRG* 4-i). If we denote by J the Jacobson radical of R, then

$$K = \bigcap_{\lambda \in \Lambda_0} K^\lambda = \bigcap_{\lambda \in \Lambda_0} E(R, \mathfrak{m}_\lambda) \subset E(R, J).$$

Since $e_\alpha(1) \in V_\alpha$, $V_\alpha \neq \phi$. For any $u \in R^*$,

$$e_{-\alpha}(u) = e_\alpha(u^{-1}) w_\alpha(-u^{-1}) e_\alpha(u^{-1}) \in V_\alpha M_\alpha V_\alpha.$$

Thus, we see (*DRG* 4-ii). Also, for any pair of elements t, u of $R - \mathfrak{m}_\lambda$ such that

$tu \equiv 1 \pmod{\mathfrak{m}_\lambda}$,

$$e_{-\alpha}(u) = e_\alpha(t) w_\alpha{}^\lambda(-t, -u) e_\alpha(t) \in V_\alpha{}^\lambda M_\alpha{}^\lambda V_\alpha{}^\lambda .$$

Thus, we see (*DRG* 4-iii). (*DRG* 5) is trivial and we can see that $TU^+ \cap U^- = \{1\}$. Since $E(R/\mathfrak{m}_\lambda)$ is a group with the simple root data induced by the system $\{T^\lambda, U_\alpha{}^\lambda, M_\alpha\}_{\alpha \in \Phi, \lambda \in \Lambda}$, we have

$$T^\lambda K^\lambda U^+ \cap K^\lambda U^- = K^\lambda.$$

Thus, we have proved (*DRG* 6). Therefore, the system $\{T^\lambda, U_\alpha{}^\lambda, M_\alpha\}_{\alpha \in \Phi, \lambda \in \Lambda}$ is a root data in the group $G(R)$ and it is strongly generative in the group $E(R)$. Now, we shall show that the root data is strict in $E(R)$. Let $U_K{}^\pm = U^\pm \cap K$ and $T_K = T \cap K$. To claim (*DRG* 8), we shall show that $L = U_K{}^+ T_K U_K{}^-$ is a normal subgroup of $E(R)$ generated by $\bigcap_{\lambda \in \Lambda_0} U_\alpha{}^\lambda = U_\alpha \cap K$ for all $\alpha \in \Phi$, this shows that $K = L \subset U^+ T U^-$. To do this it is sufficient to show that $e_\alpha(t) L \subset L$ for any $e_\alpha(t) \in K, \alpha \in (-\Pi)$ and L is normalized by $e_\alpha(t) \in E(R)$ for any $\alpha \in (-\Pi)$. We claim that $e_\alpha(t) L \subset L$ for any $e_\alpha(t) \in K$, $\alpha \in (-\Pi)$. If $\alpha \in (-\Pi)$ then $e_\alpha(t)$ normalizes the subgroup $U_{K, \Phi^+ - \{-\alpha\}}$ generated by $e_\beta(t) \in U_K^+$ for all $\beta \in \Phi^+ - \{-\alpha\}$. Further, since $u = 1 + st$ is a unit in R if $s \in J$ or $t \in J$, we have

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ s & \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ su^{-1} & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & -t^2 su^{-1} \\ 0 & 1 \end{pmatrix} .$$

Therefore,

$$e_\alpha(t) e_{-\alpha}(s) = e_{-\alpha}(su^{-1}) h_\alpha(u) e_\alpha(-t^2 su^{-1}) e_\alpha(t)$$

for any $t \in J$ and $s \in R$. Thus, we have $e_\alpha(t) L \subset L$ for all $e_\alpha(t) \in K, \alpha \in (-\Pi)$. We claim that $e_\alpha(t)$ normalizes L for all $\alpha \in (-\Pi)$ and $t \in R$. We have

$$L = U^+_{K, \Phi^+ - \{-\alpha\}} U_{K, -\alpha} T_K U_K^-$$

where $U_{K,-\alpha} = U_{-\alpha} \cap K$. $e_\alpha(t)$ normalizes $U^+_{K, \Phi^+ - \{-\alpha\}}$ and U_K^- by the above equation, $e_\alpha(t) U_{K,-\alpha} e_\alpha(t)^{-1} \subset L$. Also, $e_\alpha(t) T_K e_\alpha(t)^{-1} \subset T_K U_K^- \subset L$. Therefore, $e_\alpha(t)$ normalizes L for all $t \in R$ and $\alpha \in (-\Pi)$. Finally, we shall show (*DRG* 9). It is trivial that $\bigcap_{\lambda \in \Lambda_0} T^\lambda K^\lambda \supset TK$. If $x \in \bigcap_{\lambda \in \Lambda_0} T^\lambda K^\lambda$, then by canonical representation of G, x is diagonal modulo K^λ for all $\lambda \in \Lambda_0$. Therefore, x is diagonal modulo K. So that $x \in TK$ and we see $\bigcap_{\lambda \in \Lambda_0} T^\lambda K^\lambda = TK$. q.e.d.

As the second example, we take up twisted Chevalley groups over a commutative ring R with an identity and with an involution (cf. E. Abe [1]).

Let G be a Chevalley-Demazure group scheme of type $\Phi = A_l$, D_l or E_6 and of universal or adjoint type. Let $G_\sigma(R)$ be the twisted Chevalley group over R and $E(\Phi_\sigma, R)$ be its elementary subgroup. Let $Spm_\sigma R = \{\mathfrak{m}_\lambda ; \lambda \in \Lambda_0\}$ be the set of all σ-invariant maximal ideals of R. For convenience, we assume Φ is of type A_{2n} (The other case is simpler than the present case). We shall denote $(\Phi_\sigma)_{red}$ simply by Φ_σ. Let

$$\mathcal{R}=\{\xi=(a,b);\ a,b\in R, a\bar{a}=b+\bar{b}\}$$
$$\mathcal{R}^*=\{\xi=(a,b)\in\mathcal{R};\ b\in R^*\}.$$

For a long (resp. short) root $\alpha\in\Phi_\sigma$, there is a bijection $t\longmapsto x_\alpha(t)$ (resp. $\xi\longmapsto x_\alpha(\xi)$) of R (resp. $\mathcal{R}$) onto a subgroup $U_\alpha=U_\alpha{}^1$ of $E(\Phi_\sigma, R)$. Let $U_\alpha{}^\lambda$ be the subgroup of U_α consisting of the elements $x_\alpha(t)$ for all $t\in\mathfrak{m}_\lambda$ (resp. $x_\alpha(\xi)$ for all $\xi=(a,b)$ such that $a,b\in\mathfrak{m}_\lambda$ for a long (resp. short) root α. We set, for a long root $\alpha\in\Phi_\sigma$ and $t\in R^*$,

$$w_\alpha(t)=x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$$
$$h_\alpha(t)=w_\alpha(t)w_\alpha(-1)$$

and for a short root $\alpha\in\Phi_\sigma$ and $\xi=(a,b)$, $\eta=(c,d)\in\mathcal{R}^*$,

$$w_\alpha(\xi)=x_\alpha(\xi)x_{-\alpha}(-\bar{b}\longrightarrow\xi)x_\alpha(b\bar{b}^{-1}\longrightarrow\xi)$$
$$h_\alpha(\xi,\eta)=w_\alpha(\xi)w_\alpha(\eta).$$

Further, for a long root $\alpha\in\Phi_\sigma$ and a pair of elements t, u of $R-\mathfrak{m}_\lambda$ such that $tu\equiv1$ (mod $\mathfrak{m}_\lambda$), we define

$$w_\alpha{}^\lambda(t,u)=x_\alpha(t)x_{-\alpha}(-u)x_\alpha(t)$$
$$h_\alpha{}^\lambda(t,u)=w_\alpha{}^\lambda(t,u)w_\alpha(-1),$$

and also, for a short $\alpha\in\Phi_\sigma$ and for $\xi=(a,b)$, $\eta=(c,d)\in\mathcal{R}$ such that b, $d\in R-\mathfrak{m}_\lambda$, take a pair of elements $x,y\in R-\mathfrak{m}_\lambda$ such that $bx\equiv1$, $dy\equiv1$ (mod $\mathfrak{m}_\lambda$), we define

$$w_\alpha{}^\lambda(\xi,x)=x_\alpha(\xi)x_{-\alpha}(-\bar{b}\longrightarrow\xi)x_\alpha(b\bar{x}\longrightarrow\xi)$$
$$h_\alpha{}^\lambda(\xi,x;\ \eta,y)=w_\alpha{}^\lambda(\xi,x)w_\alpha{}^\lambda(\eta,y).$$

Let T^1 be the subgroup generated by $h_\alpha(t)$ for all long roots α and all $t\in R^*$ and $h_\alpha(\xi,\eta)$ for all short roots α and all ξ, $\eta\in\mathcal{R}^*$, and let M_α be the coset $Tw_\alpha(t)$ for some $t\in R^*$ if α is long and the coset $Tw_\alpha(\xi)$ for some $\xi\in\mathcal{R}^*$ if α is short. Let T^λ be the subgroup generated by $h_\alpha{}^\lambda(t,u)$ for all t, $u\in R-\mathfrak{m}_\lambda$ such that $tu\equiv1$ (mod $\mathfrak{m}_\lambda$) for all long roots α, and $h_\alpha{}^\lambda(\xi,x;\ \eta,y)$ for all $\xi=(a,b)$, $\eta=(c,d)\in\mathcal{R}^*$ and $x,y\in R-\mathfrak{m}_\lambda$ such thnt $bx\equiv1$, $dy\equiv1$ (mod $\mathfrak{m}_\lambda$), for all short roots α.

If the intersection of all σ-invariant maximal ideals of R coincides with the Jacobson radical of R (for example if R is a local ring) and $\mathcal{R}^*\neq\phi$, we can show that the system $\{T^\lambda, U_\alpha{}^\lambda, M_\alpha\}_{\alpha\in\Phi_\sigma,\lambda\in\Lambda}$ is a strongly generative strict root data in $E(\Phi_\sigma, R)$. We shall omit the proof.

3. Properties of a group with a root data

We shall give here some fundamental properties of a group G with a root data

$\{T^\lambda, U_\alpha^\lambda, M_\alpha\}_{\alpha\in\Phi,\lambda\in\Lambda}$. We shall omit the proof of statements which can be seen in the same way as [3], §6.1.

(3.1) *$U_\alpha \neq U_{-\alpha}$ for each $\alpha\in\Phi$. $U_\alpha M_\alpha U_\alpha \cap N(U_\alpha)=\phi$ and in particular $V_{-\alpha}\cap N(U_\alpha)=\phi$.*

(3.2) *For each root $\alpha\in\Phi$ and an element $u\in V_{-\alpha}^\lambda$ for $\lambda\in\Lambda_0$, there exists an element $m(u)$ of M_α^λ such that $u\in V_\alpha^\lambda m(u) V_\alpha^\lambda$. If $m_1(u)$ is an another such element, then $m_1(u)\in U_\alpha^\lambda m(u) U_\alpha^\lambda$.*

PROOF By (DRG-ii), we see $u\in V_\alpha^\lambda M_\alpha^\lambda V_\alpha^\lambda$. Thus, $u=u'mu''$ for some $u', u''\in V_\alpha^\lambda$ and $m\in M_\alpha^\lambda$. If $u=u'mu''=u_1'm_1u_1''$ for u', u'', u_1, $u_1''\in V_\alpha^\lambda$ and m, $m_1\in M_\alpha^\lambda$, then $u_1'^{-1}u'=m_1m^{-1}(mu_1''^{-1}m^{-1})$, where $u_1'^{-1}u'=u_2'\in U_{-\alpha}$, $mu_1''u''^{-1}m^{-1}\in U_{-\alpha}$ and $m_1m^{-1}\in T^\lambda$. Therefore, by (DRG 6-ii), $u_2'=m_1m^{-1}u_2''\in T^\lambda U_{-\alpha}\cap U_\alpha\subset K^\lambda\cap U_\alpha=U_\alpha^\lambda$. So we have $u_1'\equiv u' \bmod U_\alpha^\lambda$. Further, $mm_1^{-1}=mu_1''u''^{-1}m^{-1}\in T^\lambda\cap U_{-\alpha}\subset K^\lambda\cap U_{-\alpha}=U_{-\alpha}^\lambda$. Thus, we have $u_1''\equiv u'' \bmod U_\alpha^\lambda$. q.e.d.

(3.3) *For each root $\alpha\in\Phi$ and an element $u\in V_{-\alpha}$, there exsists a unique element $m(u)$ of M_α such that $u\in V_\alpha m(u) V_\alpha$.*

(3.4) *T normalizes M_α for all $\alpha\in\Phi$.*

(3.5) *If $\alpha/2\in\Phi$, then $M_\alpha\subset M_{\alpha/2}$.*

(3.6) *$T\cup M_\alpha$ is a subgroup of G.*

(3.7) *$V_\alpha M_\alpha U_\alpha = V_{-\alpha}TU_\alpha$ for each $\alpha\in\Phi$.*

(3.8) *Let N be the subgroup of G generated by M_α for all $\alpha\in\Phi(\neq\phi)$. Then there exists an onto homorphism ν: $N\to W$ such that $\nu(m)=\sigma_\alpha$ for any $m\in M_\alpha$. N normalizes T.*

(3.9) *For each root α, let $M_\alpha^0=m(V_\alpha)=\{m(u);\ u\in V_\alpha\}$ and $T^0=T\cap\langle M_\alpha^0;\ \alpha\in\Phi\rangle$. Then T^0 is normal in T and also in N.*

(3.10) *Let G^0 be the subgroup of G generated by U_α for all root $\alpha\in\Phi$ and $T^{0\lambda}=T^\lambda\cap G^0$ for each $\lambda\in\Lambda$. Then $\{T^{0\lambda}, U_\alpha^\lambda, T^0M_\alpha^0\}_{\alpha\in\Phi,\lambda\in\Lambda}$ is a root data in G^0. In general, let $X\subset T$ be a subgroup nomalized by M_α^0 for all $\alpha\in\Phi$ and G_X be the group generated by X and U_α for all $\alpha\in\Phi$. Then the system $\{XT^{0\lambda}, U^\lambda, XT^0M_\alpha^0\}_{\alpha\in\Phi,\lambda\in\Lambda}$ is a root data in G_X.*

(3.11) *Let $\{T, U_\alpha, M_\alpha\}_{\alpha\in\Phi}$ be a simple root data in a group G. Let L_α be the subgroup generated by T, U_α and $U_{-\alpha}$. Then*

(i) $L_\alpha=TU_\alpha\cup U_\alpha M_\alpha U_\alpha=TU_\alpha\cup U_{-\alpha}TU_\alpha\neq U_{-\alpha}TU_\alpha$

(ii) $N(U_\alpha)\cap L_\alpha=TU_\alpha$, $N(U_\alpha)\cap N(U_{-\alpha})\cap L_\alpha=T$

(iii) $M_\alpha=\{x\in L_\alpha;\ xU_\alpha x^{-1}=U_{-\alpha}, xU_{-\alpha}x^{-1}=U_\alpha\}$.

In particular, M_α is completely determined by T, U_α and $U_{-\alpha}$.

(3.12) *Let $\Phi=\bigcup_i \Phi_i$ be the decomposition of Φ into irreducible components, and G_i^0 be the subgroup of G generated by U_α for all $\alpha\in\Phi_i$. Then $\{T^\lambda\cap G_i^0, U_\alpha^\lambda, M_\alpha\cap G_i^0\}_{\alpha\in\Phi_i,\lambda\in\Lambda}$ is a root data in G^0, G_i^0 is a normal subgroup of G^0 and $G_i^0\cap G_j^0$ is contained in*

the center of G^0 *if* $i\neq j$. G^0 *is a central product of* G_i^0*'s.*
(3.13) U^+ *is a nilpotent subgroup of* G. *Let* Ψ *be a subset of* Φ^+ *closed under addition of roots and let* $\Psi_{red}=\{\alpha\in\Psi;\ \alpha/2\notin\Psi\}$. *For each* $\alpha\in\Psi$, *let* Y_α *be a subgroup of* U_α *and we set* $X_\alpha=Y_\alpha$ *if* $2\alpha\in\Psi$ *and* $X_\alpha=Y_\alpha Y_{2\alpha}$ *if* $2\alpha\notin\Psi$. *Let* X_Ψ *be the subgroup of* G *generated by* X_α *for all* $\alpha\in\Psi_{red}$. *Assume,*

(i) *For each linearly independent roots* α, β *in* Ψ, *the commutator subgroup* $[Y_\alpha, Y_\beta]$ *contained in the subgroup generated by* $Y_{p\alpha+q\beta}$, $p, q>0$ *and* $p\alpha+q\beta\in\Psi$.

Then, there exists a bijection from $\prod_{\alpha\in\Psi_{red}} X_\alpha$ *onto* X_Ψ *under some fixed order of* Ψ_{red}.

(ii) *For each* $v\in E^n$, $\langle Y_\alpha\rangle_{\alpha\in\Psi,\alpha(v)\geq 0}\cap\langle Y_\alpha\rangle_{\alpha\in\Psi,\alpha(v)<0}=\{1\}$.

Then, X_Ψ *is nilpotent. In particular, if we apply the result for* $Y_\alpha=U_\alpha^\lambda$, *we denote* X_Ψ *by* U_Ψ^λ.
(3.14) *Let* Ψ *be a subsystem of* Φ, Π_1 *a base of* Ψ. *For each* $\alpha\in\Pi_1$, *let* N_α *be a non-empty subset of* M_α, *let* N_1 *be the subgroup generated by* N_α *for all* $\alpha\in\Pi_1$ *and let* X *be a subgroup of* T *normalized by* N_1 *such that* $N_\alpha^2\subset X$ *for each* $\alpha\in\Pi_1$. *Then, there exists a homomorphism* μ: $N_1X\rightarrow W$ *such that* $\mu(n)=\sigma_\alpha$ *for all* $n\in XN_\alpha$ *and* $\mu(N_1X)$ *is isomorphic to the Weyl group* W_1 *of* Ψ *and* $Ker\,\mu=X$.
(3.15) *Let* $N=\langle M_\alpha;\ \alpha\in\Pi\rangle$, *then* $\nu^{-1}(1)=T=N\cap TU^+$.
(3.16) *Let* $\{T, U_\alpha, M_\alpha\}_{\alpha\in\Phi}$ *be a simple root data in a group* G *and* $S=\{\nu(M_\alpha);\ \alpha\in\Pi\}$ *be the subset of* $W\cong N/T$. *Then,* $\{G, TU^+, M, S\}$ *is a Tits system with the Weyl group* W.

4. Coverings of a group with a root data

We shall construct a covering of a group with a strongly generative, strict root data assuming further some conditions. (As for coverings of a group, see M. Stein [6].)

Let Φ be an irreducible root system of rank>1 and let $\Pi=\{\alpha_1, \cdots, \alpha_n\}$ be a base of Φ. For any integers $i, j (1\leqq i, j\leqq n)$, set

$$\Psi_{ij}=\{p\alpha_i+q\alpha_j\in\Phi;\ p, q\in\boldsymbol{Z}\}$$
$$\Psi=\bigcup_{i,j}\Psi_{ij}.$$

Then the following lemma is well known.

LEMMA 4,1 *For any root* $\alpha\in\Phi$, *there exists a closed subset* Ψ_α *of* Φ *such that* $\alpha\in\Psi_\alpha$ *and* $w(\Psi_\alpha)\subset\Phi^+\cap\Psi_{ij}$ *for some* $w\in W$ *and for some* Ψ_{ij} *and* $w(\alpha)\notin\Pi$.

PROPOSITION 4.2 *Let* $\{T^\lambda, U_\alpha^\lambda, M_\alpha\}_{\alpha\in\Phi,\lambda\in\Lambda}$ *be a generative root data in a group* G. *Assume*

(*DRG* 10) *For any root* $\alpha\in\Phi$ *and* $\lambda\in\Lambda$, U_α^λ *is contained in the subgroup of G generated by* U_β^λ *for any* $\beta\in\Psi_\alpha$ *linearly independent to* α.

Then, K^λ *is the subgroup generated by* U_α^λ *for all* $\alpha\in\Phi$.

PROOF Since the root data is generative, it is sufficient to show that T and U_α normalize the subgroup L^λ generated by U_β^λ for all $\beta\in\Phi$. If β is linearly independent to α and $u\in U_\alpha$, then by (*DRG* 2), $uU_\beta^\lambda u^{-1}\subset L^\lambda$. Now, assume α,β are linearly dependent. Let L_β^λ be the subgroup of L^λ generated by U_γ^λ for all $\gamma\in\Psi_\beta$ which is linearly independent to β. Then, by (*DRG* 10), $U_\beta^\lambda\subset L_\beta^\lambda$ and we have that $uL_\beta^\lambda u^{-1}\subset L^\lambda$ for any $u\in U_\alpha$. Therefore, $uU_\beta^\lambda u^{-1}\subset L^\lambda$ for any $u\in U_\alpha$. q.e.d.

Note that (*DRG* 10) is true for Chevalley groups over a commutative ring of type Φ which is irreducible of rank$>$1.

THEOREM 4.3 *Let G be a group with a strongly generative root data* $\{T^\lambda, U_\alpha^\lambda, M_\alpha\}_{\alpha\in\Phi,\lambda\in\Lambda}$ *where* Φ *is irreducible of rank*$>$1. *Assume that the root data satisfies* (*DRG* 10). *Let* $\hat{G}$ *be the group generated by* $\hat{U}_\alpha$ *for all* $\alpha\in\Psi$ *which is isomorphic to* U_α *together with relations* (*DRG* 2) *for all roots* $\beta,\gamma\in\Psi_{ij}$ *for all* i, j $(1\leqq i,j\leqq n)$ *such that* $\beta\in-\boldsymbol{Q}^+\gamma$. *Then,* $\hat{G}$ *is a group with a root data of type* Φ.

(4.3.1) *From* (3.3), *for each root* $\alpha\in\Psi$ *and an element* $u\in V_{-\alpha}$, *there exists a unique element* $m(u)$ *of* M_α *such that* $u\in V_\alpha m(u)V_\alpha$. *In* (3.8), *we denote by* $M_\alpha=M_\alpha^0=m(V_\alpha)=\{m(u)\,;\ u\in V_\alpha\}$, *where* $m(u)=u'uu''$ *for some* $u', u''\in V_\alpha$. *We define* $\hat{m}(u)$ *the element of* $\hat{G}$ *expressed by* $\hat{u}'\hat{u}\hat{u}''$, *and let* $\hat{M}_\alpha$ *be the set of all elements* $\hat{m}(u)$ *for all* $\hat{u}\in\hat{V}_{-\alpha}$. *If* $\hat{m}_\alpha$ *is an element of* $\hat{M}_\alpha$, *then* $\nu(\pi(\hat{m}_\alpha))=\sigma_\alpha$ *and for any elements* $\alpha,\beta\in\Psi_{ij}$, *we have*

$$\hat{m}_\alpha\hat{U}_\beta\hat{m}_\alpha^{-1}=\hat{U}_{\sigma_\alpha(\beta)}$$

where π *is the natural homomorphism of* $\hat{G}$ *onto G.*

PROOF First, let α,β be linearly independent roots of $\Psi_{ij}\cap\Phi_{red}$. We set $\Phi'=\{p\alpha+q\beta\in\Phi\,;\ p,q>0\}$. Then,

$$\hat{m}_\alpha\hat{U}_\beta\hat{m}_\alpha^{-1}\subset\hat{U}_{\Phi'} \quad\text{and}\quad \hat{U}_{\sigma_\alpha(\beta)}\subset\hat{U}_{\Phi'}\,.$$

Since $\hat{U}_{\Phi'}\cong U_{\Phi'}$ and the expression of an element of $U_{\Phi'}$ as a product of elements of X_α $(\alpha\in\Phi'\cap\Phi_{red})$ is unique (cf. (3. 13)), we have

$$\hat{m}_\alpha\hat{U}_\beta\hat{m}_\alpha^{-1}=\hat{U}_{\sigma_\alpha(\beta)}\,.$$

Next, let α, β be linearly dependent. From the assumption (*DRG* 10), $\hat{U}_\beta$ is contained in the subgroup $\hat{U}_{\Psi'}$, where Ψ' is the set of roots $\gamma\in\Psi_\beta$ linearly independent to β and we have

$$\hat{m}_\beta\hat{U}_\beta\hat{m}_\beta^{-1}\subset\hat{U}_{\sigma_\beta(\Psi')}\,.$$

By the uniqueness of the expression, we see

$$\hat{m}_\beta \hat{U}_\beta \hat{m}_\beta^{-1} = \hat{U}_{-\beta}\,. \qquad \text{q.e.d.}$$

(4.3.2) *Let* $\alpha \in \Psi_{red} = \Psi \cap \Phi_{red}$. *Define*

$$\hat{T}_\alpha = \langle \hat{m}_\alpha^{(1)} \cdots \hat{m}_\alpha^{(2k)}\,;\ k=1,2,\cdots,\ \hat{m}_\alpha^{(i)} \in \hat{M}_\alpha \rangle$$
$$\hat{T}_{ij} = \langle \hat{T}_\alpha\,;\ \alpha \in \Psi_{ij} \rangle\,,$$
$$\hat{N}_{ij} = \langle \hat{T}_{ij}, \hat{M}_\alpha\,;\ \alpha \in \Psi_{ij} \rangle\,.$$

From (3.2), for each root $\alpha \in \Psi$ and an element u of $V_{-\alpha}^\lambda$, there exists an element $m_\alpha^\lambda(u)$ of M_α^λ such that $u \in V_\alpha^\lambda m_\alpha^\lambda(u) V_\alpha^\lambda$. Therefore, $m_\alpha^\lambda(u) = u'uu''$ for some u', $u'' \in V_\alpha^\lambda$. For any $\alpha \in \Psi$, we define

$$\hat{M}_\alpha^\lambda = \{\hat{m}_\alpha^\lambda(u) = \hat{u}'\hat{u}\hat{u}''\,;\ \hat{u}', \hat{u}'' \in \hat{V}_\alpha^\lambda, \hat{u} \in \hat{V}_{-\alpha}^\lambda\}$$
$$\hat{T}_\alpha^\lambda = \langle \hat{m}_\alpha^{\lambda(1)} \cdots \hat{m}_\alpha^{\lambda(2k)}\,;\ k=1,2,\cdots,\ \hat{m}_\alpha^{\lambda(i)} \in \hat{M}_\alpha^\lambda \rangle$$
$$\hat{T}^\lambda = \langle \hat{T}_\alpha^\lambda\,;\ \alpha \in \Psi \rangle$$

We shall show that for each $\alpha \in \Psi$ there exists a subgroup $\hat{U}_\alpha$ of $\hat{G}$ isomorphic to U_α and $\hat{M}_\alpha$ can be defined as above and that the system $\{\hat{T}^\lambda, \hat{U}_\alpha^\lambda, \hat{M}_\alpha\}_{\alpha \in \Phi, \lambda \in \Lambda}$ is a root data in the group $\hat{G}$.

(4.3.3) $\hat{m}_\alpha \hat{T}_\beta \hat{m}_\alpha^{-1} = \hat{T}_{\sigma_\alpha(\beta)}$ *for any* $\alpha, \beta \in \Psi_{ij} \cap \Phi_{red}$.

Proof We have $\hat{m}_\alpha \hat{m}_\beta \hat{m}_\alpha^{-1} \in \hat{m}_\alpha(\hat{V}_\beta \hat{V}_{-\beta} \hat{V}_\beta) m_\alpha^{-1} = \hat{V}_{\sigma_\alpha(\beta)} \hat{V}_{-\sigma_\alpha(\beta)} \hat{V}_{\sigma_\alpha(\beta)}$. Since $\pi(\hat{m}_\alpha \hat{m}_\beta \hat{m}_\alpha^{-1}) \in V_{\sigma_\alpha(\beta)} V_{-\sigma_\alpha(\beta)} V_{\sigma_\alpha(\beta)} \cap N$ and $\nu(\pi(\hat{m}_\alpha \hat{m}_\beta \hat{m}_\alpha^{-1})) = \sigma_{\sigma_\alpha(\beta)}$, we have $\hat{m}_\alpha \hat{m}_\beta \hat{m}_\alpha^{-1} \in \hat{M}_{\sigma_\alpha(\beta)}$. Therefore, $\hat{m}_\alpha \hat{T}_\beta \hat{m}_\alpha^{-1} \subset \hat{T}_{\sigma_\alpha(\beta)}$. By symmetry,

$$\hat{m}_\alpha \hat{T}_\beta \hat{m}_\alpha^{-1} = \hat{T}_{\sigma_\alpha(\beta)}\,.$$

(4.3.4) $\hat{T}_{ij} \triangleleft \hat{N}_{ij}$ *and the natural homomorphism*

$$\phi\colon \hat{N}_{ij}/\hat{T}_{ij} \longrightarrow N_{ij}/T_{ij}$$

is an isomorphism.

Proof Set $\bar{m}_\alpha = \hat{T}_{ij} \hat{m}_\alpha$ and $\bar{m}_\alpha^2 = 1$, $\bar{m}_\alpha \bar{m}_\beta \bar{m}_\alpha = \bar{m}_{\sigma_\alpha(\beta)}$. So we have that ϕ is an isomorphism by (3.14).

(4.3.5) *Let* $\hat{N} = \langle \hat{T}_{\alpha_i}, \hat{m}_{\alpha_i} (1 \leqq i \leqq n) \rangle$ *and* $\hat{T} = \langle \hat{T}_{\alpha_i}\,; 1 \leqq i \leqq n \rangle$. *Then* $\hat{T} \triangleleft \hat{N}$ *and* $\hat{N}/\hat{T} \cong W$.

Proof By definition, we see $\hat{T} \triangleleft \hat{N}$. For any $\alpha, \beta \in \Psi_{ij} \cap \Phi_{red}$, $\hat{h}_\alpha \hat{T}_\beta \hat{h}_\alpha^{-1} = \hat{T}_\beta$ for $\hat{h}_\alpha \in \hat{T}_\alpha$. Thus, $\hat{T}_\alpha \triangleleft \hat{T}_{ij}$. Now, we shall show that $\hat{T}_{ij} = \langle \hat{T}_{\alpha_i}, \hat{T}_{\alpha_j} \rangle$ which shows that $\hat{T} = \langle \hat{T}_{\alpha_i}, 1 \leqq i \leqq n \rangle$. Let $\beta \in \Psi_{ij}$ be linearly independent from α_i, α_j and assume that $\hat{h}_{\beta'} \in \langle \hat{T}_{\alpha_i}, \hat{T}_{\alpha_j} \rangle$ for any $\beta' < \beta$. Then, since $\alpha_{\alpha_i}(\beta) = \beta' < \beta$, we have

$$\begin{aligned}\hat{h}_\beta \hat{m}_{\alpha_i}^{(1)} \hat{m}_{\alpha_i}^{(2)} &= \hat{h}_\beta \hat{m}_{\alpha_i}^{(1)} \hat{h}_\beta^{-1} (\hat{h}_\beta \hat{m}_{\alpha_i}^{(2)}) = \hat{m}'_{\alpha_i} \hat{h}_\beta \hat{m}_{\alpha_i}^{(2)} \\ &= (\hat{m}'_{\alpha_i} \hat{m}_{\alpha_i}^{(2)}) \hat{m}_{\alpha_i}^{(2)-1} \hat{h}_\beta \hat{m}_{\alpha_i}^{(2)} \in \hat{T}_{\alpha_i} \hat{T}_{\beta'} \subset \hat{T}_{\alpha_i} \hat{T}_{\alpha_j}\end{aligned}$$

Therefore, $\hat{h}_\beta\in\langle\hat{T}_{\alpha_i}, \hat{T}_{\alpha_j}\rangle$. $\hat{N}/\hat{T}\cong N/T$ follows from (4.3.4).

(4.3.6) *For any* $\alpha\in\Phi^+$, *there exists subgroup* $\hat{U}_\alpha$ *of* $\hat{G}$ *satisfying*

(i) $\pi(\hat{U}_\alpha)=U_\alpha$.

(ii) $\hat{T}$ *normalizes* $\hat{U}_\alpha$.

(iii) $\hat{U}_\alpha=\hat{U}_{\alpha_i}$ *if* $\alpha=\alpha_i$.

(iv) $\hat{m}_{\alpha_i}\hat{U}_\alpha\hat{m}_{\alpha_i}^{-1}=\hat{U}_{\sigma_i(\alpha)}$ *if* α, α_i *are linearly independent.*

PROOF The proof is the same as the proof of step (c), 14–08 [4]. We shall give here the proof for convenience of the reader. If $\alpha\notin\Phi^+_{red}$, then $\alpha/2\in\Phi_{red}$ and $U_\alpha\subsetneqq U_{\alpha/2}$. Therefore, we may set $\hat{U}_\alpha\cong U_\alpha\subsetneqq U_{\alpha/2}$. Therefore, we assume that $\alpha\in\Phi^+_{red}$. We shall construct subgroups $\hat{U}_\alpha$ by induction of the lexicographic order of the roots. Assume that we have already constructed the subgroups $\hat{U}_\beta$ for each β such that $0<\beta<\alpha$ which satisfy (i)~(iii) and (iv) for any root α_i such that $0<\sigma_i(\beta)<\alpha, 0<\alpha_i<\alpha$. We shall construct the subgroup $\hat{U}_\alpha$. If $\alpha\in\Pi$ and $\alpha=\alpha_i$, we may take $\hat{U}_\alpha=\hat{U}_{\alpha_i}$. Now, assume $\alpha\notin\Pi$ and $0<\sigma_i(\alpha)<\alpha$. We take $\hat{U}_\alpha=\hat{m}_{\alpha_i}\hat{U}_{\sigma_i(\alpha)}\hat{m}_{\alpha_i}^{-1}$, then $\hat{U}_\alpha$ satisfies (i)~(iii). We shall claim that if $0<\sigma_j(\alpha)\leqq\alpha$,

$$\hat{m}_{\alpha_j}\hat{U}_\alpha\hat{m}_{\alpha_j}^{-1}=\hat{U}_{\sigma_j(\alpha)}\,.$$

If $\alpha\in\Psi_{ij}$, then this follows from (4.3.1). Assume $\alpha\notin\Psi_{ij}$, then for any $\sigma\in\langle\sigma_i, \sigma_j\rangle$, $\sigma(\alpha)>0$ and $\sigma(\alpha)\leqq\alpha$. If $i=j$, then $\hat{m}_{\alpha_i}\hat{U}_\alpha\hat{m}_{\alpha_i}^{-1}=\hat{m}_{\alpha_i}^2\hat{U}_{\sigma_i(\alpha)}\hat{m}_{\alpha_i}^{-2}=\hat{U}_{\sigma_i(\alpha)}$. If $i\neq j$, we shall divide the proof into the following two cases.

Case 1 $\alpha>\sigma_j(\alpha)$: Since $\sigma(\alpha)<\alpha$ for each $\sigma\in\langle\sigma_i, \sigma_j\rangle$ and $\sigma_j(\alpha)=(\sigma_i\sigma_j)^{n_{ij}-1}\sigma_i(\alpha)$, we have $0<(\sigma_i\sigma_j)^l\sigma_i(\alpha)<\alpha$ and $0<\sigma_j(\sigma_i\sigma_j)^l\sigma_i(\alpha)<\alpha$. Therefore,

$$\begin{aligned}\hat{U}_{\sigma_j(\alpha)}&=(\hat{m}_{\alpha_i}\hat{m}_{\alpha_j})^{n_{ij}-1}\hat{U}_{\sigma_i(\alpha)}(\hat{m}_{\alpha_i}\hat{m}_{\alpha_j})^{-n_{ij}+1}\\&=(\hat{m}_{\alpha_i}\hat{m}_{\alpha_j})^{n_{ij}-1}\hat{m}_{\alpha_i}^{-1}(\hat{m}_{\alpha_i}\hat{U}_{\sigma_i(\alpha)}\hat{m}_{\alpha_i}^{-1})\hat{m}_{\alpha_i}(\hat{m}_{\alpha_i}\hat{m}_{\alpha_j})^{-n_{ij}+1}\\&=\hat{m}_{\alpha_j}\hat{U}_\alpha\hat{m}_{\alpha_j}^{-1}\,.\end{aligned}$$

Case 2 $\alpha=\sigma_j(\alpha)$: Let $P=\boldsymbol{R}\alpha_i+\boldsymbol{R}\alpha_j$ and $Q=P^\perp$. Since $\alpha=\sigma_j(\alpha)$, for each $\sigma\in\langle\sigma_i, \sigma_j\rangle$, The restriction of σ to Q is the identity and $(\alpha, \alpha_j)=0$. Therefore, the orthogonal projection α' of α on P is orthogonal to α_j and for any $\sigma\in\langle\sigma_i, \sigma_j\rangle$, $\sigma(\alpha)=\alpha$ if and only if $\sigma(\alpha')=\alpha'$; if $\sigma(\alpha)=\alpha$, then the restrictions of σ and σ_j on P are coincide. Thus, we have $\sigma(\alpha)<\alpha$ for each $\sigma\in\langle\sigma_i, \sigma_j\rangle$ such that $\alpha\neq1$, $\sigma\neq\sigma_j$. Now, we set

$$\tau_l=\sigma_j(\sigma_i\sigma_j)^{l-1}\sigma_i,\ \ \rho_l=(\sigma_i\sigma_j)^l\sigma_i\,.$$

Then, τ_l and ρ_l are $\neq1$ and also $\neq\sigma_j$ for all l $(1\leqq l\leqq n_{ij})$. So we have $\tau_l(\alpha)<\alpha$, $\rho_l(\alpha)<\alpha(1\leqq l\leqq n_{ij})$. By induction assumption,

$$\begin{aligned}\hat{U}_{\sigma_j(\alpha)}&=(\hat{m}_{\alpha_i}\hat{m}_{\alpha_j})^{n_{ij}-1}\hat{U}_{\sigma_i(\alpha)}(\hat{m}_{\alpha_i}\hat{m}_{\alpha_j})^{-n_{ij}+1}\\&=(\hat{m}_{\alpha_i}\hat{m}_{\alpha_j})^{n_{ij}-1}\hat{m}_{\alpha_i}^{-1}(\hat{m}_{\alpha_i}\hat{U}_{\sigma_i(\alpha)}\hat{m}_{\alpha_i}^{-1})\hat{m}_{\alpha_i}(\hat{m}_{\alpha_i}\hat{m}_{\alpha_i})^{-n_{ij}+1}\\&=\hat{m}_{\alpha_j}\hat{U}_\alpha\hat{m}_{\alpha_j}^{-1}.\end{aligned}$$

q.e.d.

(4.3.7) *For any* $\alpha\in\Phi^-$, *there exists a subgroup* $\hat{U}_\alpha$ *of* $\hat{G}$ *such that* (*i*), (*ii*) *and* (*iv*) *hold.*

PROOF For a given root $\alpha\in\Phi$, let σ be an element of W such that $\sigma(\alpha)\in\Phi^+$, and define $\hat{U}_\alpha$ the subgroup $\hat{m}(\sigma)^{-1}\hat{U}_{\sigma(\alpha)}\hat{m}(\sigma)$. Now, we shall prove that the group $\hat{U}_\alpha$ is independent of the choice of σ. It is sufficient to show that if $\sigma(\alpha)>0$ and $\sigma'(\alpha)>0$, then $\hat{m}(\sigma)^{-1}\hat{U}_{\sigma(\alpha)}\hat{m}(\sigma)=\hat{m}(\sigma')^{-1}\hat{U}_{\sigma'(\alpha)}\hat{m}(\sigma')$. Let $\sigma'\sigma^{-1}=\sigma_{j_1}\sigma_{j_2}\cdots\sigma_{j_s}$ be a reduced expression, then $\sigma_{j_i}\sigma_{j_{i+1}}\cdots,\sigma_{j_s}(\sigma(\alpha))>0$ $(1\leqq i\leqq s)$. Therefore, by (4.3.6), we have

$$\hat{m}_{\alpha_{j_1}}\cdots\hat{m}_{\alpha_{j_s}}\hat{U}_{\sigma(\alpha)}\hat{m}_{\alpha_{j_s}}^{-1}\cdots\hat{m}_{\alpha_{j_1}}^{-1}=\hat{U}_{\sigma'(\alpha)}. \qquad \text{q.e.d.}$$

(4.3.8) $\hat{m}_{\alpha_j}\hat{U}_\alpha\hat{m}_{\alpha_j}^{-1}=\hat{U}_{\sigma_j(\alpha)}$ *for all* $\alpha_j\in\Pi$ *and all* $\alpha\in\Phi$.

PROOF If $\alpha\in\Phi^+$ and $\alpha_j\neq\alpha$, then this is (4.3.6). If $\alpha=\alpha_j$ then, $\hat{U}_{-\alpha_j}=\hat{m}_{\alpha_j}^{-1}\hat{U}_{\alpha_j}\hat{m}_{\alpha_j}$. If $\alpha\in\Phi^-$, then there exists $\sigma\in W$ such that $\sigma(\alpha)>0$. Therefore,

$$\hat{m}_{\alpha_j}^{-1}\hat{U}_\alpha\hat{m}_{\alpha_j}=\hat{m}_{\alpha_j}^{-1}\hat{m}(\sigma)^{-1}\hat{U}_{\sigma(\alpha)}\hat{m}(\sigma)\hat{m}_{\alpha_j}.$$

Since $\sigma(\sigma_j(\sigma_j(\alpha)))>0$, we have

$$\hat{U}_{\sigma_j(\alpha)}=\hat{m}_{\alpha_j}^{-1}\hat{m}(\sigma)^{-1}\hat{U}_{\sigma(\sigma_j(\sigma_j(\alpha)))}\hat{m}(\sigma)\hat{m}_{\alpha_j}=\hat{m}_{\alpha_j}^{-1}\hat{U}_\alpha\hat{m}_{\alpha_j}. \qquad \text{q.e.d.}$$

(4.3.9) *Define* $\hat{U}_\alpha{}^\lambda=\hat{U}_\alpha\cap\hat{K}^\lambda$ *and* $\hat{U}_\alpha{}^\lambda$ *is a proper normal subgroup of* $\hat{U}_\alpha$ *isomorphic to* $U_\alpha{}^\lambda$. *For any linearly independent roots* $\alpha,\beta\in\Phi$,

$$[\hat{U}_\alpha{}^\lambda,\hat{U}_\beta]\subset\prod_{\substack{p,q<0\\ p\alpha+q\beta\in\Phi}}\hat{U}^\lambda_{p\alpha+q\beta}.$$

PROOF For any linearly independent roots $\alpha,\beta\in\Phi$, there exists an element $\sigma\in W$ such that $\sigma(\alpha),\sigma(\beta)\in\Psi_{ij}$ for some i,j. So that by hypothesis, we have our assertion.

(4.3.10) $\hat{T}\hat{U}^+\cap\hat{U}^-=\{1\}$, $\hat{T}^\lambda\hat{K}^\lambda\hat{U}^{\lambda+}\cap\hat{K}^\lambda\hat{U}^-=\hat{K}^\lambda$ *for any* $\lambda\in\Lambda_0$.

PROOF Let $\hat{h}\hat{u}=\hat{v}\in\hat{T}\hat{U}^+\cap\hat{U}^-$, where $\hat{h}\in\hat{T}$, $\hat{u}\in\hat{U}^+$ and $\hat{v}\in\hat{U}^-$. Then, $\pi(\hat{h}\hat{u})=\pi(\hat{v})\in TU^+\cap U^-=\{1\}$. Since $\pi|_{U^-}$ is a bijection, we have $\hat{v}=1$ and $\hat{h}\hat{u}=1$. Thus $\hat{T}\hat{U}^+\cap\hat{U}^-=\{1\}$. Now, let $\hat{h}\hat{u}=\hat{k}\hat{v}\in\hat{K}^\lambda\hat{T}^\lambda\hat{U}^+\cap\hat{K}^\lambda\hat{U}^-$. Then $\hat{k}^{-1}\hat{h}\hat{u}=\hat{v}$ and $\pi(\hat{v})\in K^\lambda$. So we have $\hat{v}\in\hat{U}^-\cap\hat{K}^\lambda$ and $\hat{k}\hat{v}\in\hat{K}^\lambda$. Therefore, we have $\hat{K}^\lambda\hat{T}^\lambda\hat{U}^+\cap\hat{K}^\lambda\hat{U}^-=\hat{K}^\lambda$. q.e.d.

(4.3.11) *If the root data* $\{T,U_\alpha,M_\alpha\}_{\alpha\in\Phi}$ *is simple, then* $\{\hat{T}\hat{U}^+,\hat{N},S\}$ *is a Tits system with the Weyl group* W.

This completes the proof of Theorem 4.3.

THEOREM 4.4 *Let* G *be a group with a root data of type* Φ *which is strongly generative, strict and satisfies* (*DRG* 10) *and* Φ *is irreducible of rank* >1. *If the group* $\hat{G}$ *defined in Theorem* 4.3 *is also a group with a strict root data, then* $\hat{G}$ *is a central extension of* G.

$$\begin{array}{ccc} \hat{G} & \xrightarrow{\hat{\phi}} & \hat{G}' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ G & \xrightarrow[\phi]{} & G' \end{array}$$

(4.4.1) *Let G, G' be groups with root data $\{T^{\lambda}, U_{\alpha}^{\lambda}, M_{\alpha}\}_{\alpha\in\Phi,\lambda\in\Lambda}$, $\{T'^{\lambda}, U_{\alpha'}'^{\lambda}, M_{\alpha}'\}_{\alpha\in\Phi,\lambda\in\Lambda}$ of the same type Φ which is irreducible of rank >1 and over the same set Λ. Let ϕ: $G\to G'$ be a homomorphism of groups such that $\phi(U_{\alpha}^{\lambda})=U_{\alpha}'^{\lambda}$ for all $\alpha\in\Phi$ and $\lambda\in\Lambda$, then there exists a homorphism $\hat{\phi}$: $\hat{G}\to\hat{G}'$ such that $\hat{\phi}(\hat{U}_{\alpha}^{\lambda})=\hat{U}_{\alpha}'^{\lambda}$ for all $\alpha\in\Phi$ and $\lambda\in\Lambda$ and that $\phi\circ\pi=\pi'\circ\hat{\phi}$.*

PROOF The restriction ϕ_{α} of ϕ to U_{α} induces a homomorphism $\hat{\phi}_{\alpha}$ of $\hat{U}_{\alpha}$ into $\hat{U}_{\alpha}'$. Now, let $\hat{u}_{\alpha}, \hat{u}_{\beta}\in\hat{G}$ where α, β are such that $\beta\notin -\boldsymbol{Q}_{\alpha}^{+}$ and

$$[\hat{u}_{\alpha}, \hat{u}_{\beta}] = \prod_{p,q>0} \hat{u}_{p\alpha+q\beta}\,.$$

If $\pi(\hat{u}_{\alpha})=u_{\alpha}$ then by definition of $\hat{G}$, $[u_{\alpha}, u_{\beta}]=\prod_{p,q>0} u_{p\alpha+q\beta}$. Also if $\phi(u_{\alpha})=u_{\alpha}'$, then we have $[u_{\alpha}', u_{\beta}']=\prod_{p,q>0} u'_{p\alpha+q\beta}$. Therefore, $[\hat{\phi}_{\alpha}, (\hat{u}_{\alpha}), \hat{\phi}_{\beta}(\hat{u}_{\beta})]=\prod_{p,q>0} \hat{\phi}_{p\alpha+q\beta}(\hat{u}_{p\alpha+q\beta})$, where $\hat{\phi}_{\alpha}$ is the homomorphism of $\hat{U}_{\alpha}$ into $\hat{U}_{\alpha}'$ induced by ϕ_{α}. Thus, $\{\hat{\phi}_{\alpha}\}_{\alpha\in\Phi}$ can be extended to a natural homomorphism $\hat{\phi}$: $\hat{G}\to\hat{G}'$ such that $\phi\circ\pi=\pi'\circ\hat{\phi}$. q.e.d.

(4.4.2) $(\hat{G})_{\lambda}\cong(G_{\lambda})^{\wedge}$

PROOF Let ϕ^{λ}: $G\to G/K^{\lambda}=G_{\lambda}$ be the natural homomorphism. Then by (4.4.1), there exists a homomorphism $\hat{\phi}^{\lambda}$: $\hat{G}\to(G_{\lambda})^{\wedge}$ such that $\phi^{\lambda}\circ\pi=\pi_{\lambda}\circ\hat{\phi}^{\lambda}$ where π_{λ}: $(G_{\lambda})^{\wedge}\to G_{\lambda}$ is the natural homomorphism. Since $\hat{K}^{\lambda}\subset Ker\,\hat{\phi}^{\lambda}$, $\hat{\phi}^{\lambda}$ induces a homomorphism of $\hat{G}/\hat{K}^{\lambda}=(\hat{G})_{\lambda}$ into $(G_{\lambda})^{\wedge}$ which we denote by the same symbol $\hat{\phi}^{\lambda}$. By (*DRG* 4-i), $U_{\alpha}\cap K^{\lambda}=U_{\alpha}^{\lambda}$. So the restriction of $\hat{\phi}_{\lambda}$ to $\hat{U}_{\alpha}\cap\hat{K}^{\lambda}$ is an isomorphism onto $U_{\alpha}\cap K^{\lambda}=U_{\alpha}^{\lambda}$. Therefore, $\hat{\phi}^{\lambda}$ induces an isomorphism $\hat{\phi}_{\alpha}^{\lambda}$: $\hat{U}_{\alpha}/\hat{U}_{\alpha}\cap\hat{K}^{\lambda}\to(U_{\alpha}/U_{\alpha}\cap K^{\lambda})^{\wedge}\cong U_{\alpha}/U_{\alpha}\cap K^{\lambda}$. The inverses of $\hat{\phi}_{\alpha}^{\lambda}$ for all $\alpha\in\Phi$ induce a homomorphism $\hat{\psi}^{\lambda}$: $(G_{\lambda})^{\wedge}\to(\hat{G})_{\lambda}$ which is the inverse of $\hat{\phi}^{\lambda}$. Thus we have $(G)_{\lambda}{}^{\wedge}\cong(\hat{G})_{\lambda}$. q.e.d.

(4.4.3) *The natural homomorphism $\hat{G}/\hat{K}^{\lambda}\to G/K^{\lambda}$ is a central extension.*

PROOF G/K^{λ} is a group with a simple root data. It is well known that $(G/K^{\lambda})^{\wedge}$ is a central extension which can be proved using the Bruhat decomposition.

(4.4.4) *The natural homorphism $\hat{G}/\hat{K}\to G/K$ is a central extension.*

PROOF Since $\hat{G}/\hat{K}$ (resp. G/K) can be imbedded in $\prod_{\lambda\in\Lambda_0}(\hat{G})_{\lambda}\cong\prod_{\lambda\in\Lambda_0}(G_{\lambda})^{\wedge}$ (resp. $\prod_{\lambda\in\Lambda_0}G_{\lambda}$) naturaly and by (4.4.3), the homomorphism $\prod_{\lambda\in\Lambda_0}(G_{\lambda})^{\wedge}\to\prod_{\lambda\in\Lambda_0}G_{\lambda}$ is central, we have that $\hat{G}/\hat{K}\to G/K$ is central. q.e.d.

(4.4.5) *If G and $\hat{G}$ are groups with a strongly generative, strict root data, then π: $\hat{G}\to G$ is a central extension.*

PROOF Since, by (4.4.4), the homomorphism $\hat{\pi}: \hat{G}/\hat{K} \to G/K$ is a central extension, $Ker\ \hat{\pi} \subset \bigcap_{\lambda \in \Lambda_0} \hat{T}^{\lambda}\hat{K}^{\lambda}/\hat{K}$. Therefore, $Ker\ \pi \subset \bigcap_{\lambda \in \Lambda_0} \hat{T}^{\lambda}\hat{K}^{\lambda}$ which coincides with $\hat{T}\hat{K}$ by (*DRG* 9) for the group $\hat{G}$ and further $Ker\ \pi \subset \hat{U}^{+}\hat{T}\hat{U}^{-}$ by (*DRG* 8) for the group $\hat{G}$. First, we shall show that $Ker\ \pi \subset \hat{T}$. The map of $\hat{U}^{+} \times \hat{T} \times \hat{U}^{-}$ into $\hat{G}$ defined by $(\hat{u}, \hat{h}, \hat{v}) \longmapsto \hat{u}\hat{h}\hat{v}$ is a monomorphism. For, if $\hat{u}\hat{h}\hat{v} = \hat{u}_1\hat{h}_1\hat{v}_1$, then $\hat{h}_1^{-1}\hat{u}_1^{-1}\hat{u}\hat{h} = \hat{v}_1\hat{v}^{-1} \in \hat{T}\hat{U}^{+} \cap \hat{U}^{-} = \{1\}$. Thus, we have $\hat{v}_1 = \hat{v}$ and $\hat{u}_1^{-1}\hat{u} = \hat{h}_1\hat{h}^{-1} \in \hat{T} \cap \hat{U}^{+} = \{1\}$. Therefore, we have also $\hat{u}_1 = \hat{u}$ and $\hat{h}_1 = \hat{h}$. Let $\hat{x} = \hat{u}\hat{h}\hat{v} \in Ker\ \pi$. Then $\pi(\hat{x}) = uhv = 1$ and we see $u = h = v = 1$. Thus, $\hat{u} = \hat{v} = 1$. So that $\hat{x} = \hat{h} \in \hat{T}$. Now, let $\hat{h} \in Ker\ \pi$. Then $\pi(\hat{h}\hat{u}_\alpha\hat{h}^{-1}) = \pi(\hat{u}_\alpha)$ for all $\hat{u}_\alpha \in \hat{U}_\alpha$ and $\alpha \in \Phi$. Since the restruction of π onto $\hat{U}_\alpha$ is an isomorphism, we have $\hat{h}\hat{u}_\alpha\hat{h}^{-1} = \hat{u}_\alpha$ for all $\hat{u}_\alpha \in \hat{U}_\alpha$ and $\alpha \in \Phi$. Since $\hat{G}$ is generated by $\hat{U}_\alpha$ for all $\alpha \in \Phi$, $\hat{h} \in Z(\hat{G})$.

This completes the proof of Theorem 4.4.

COLLORALY 4.5 *Let G be a group with a strongly generative, strict root data of type Φ satisfying* (*DRG* 10), *where Φ is irreducible of rank* >1. *Assume that G is perfect and also assume that the root data in $\hat{G}$ is strict. Let $\rho: \tilde{G} \to G$ be a covering of G. If, for any root $\alpha \in \Psi$, there exists a subgroup $\tilde{U}_\alpha$ of $\tilde{G}$ isomorphic to U_α such that*

(i) $\tilde{m}_{\alpha_i}\tilde{U}_\alpha\tilde{m}_{\alpha_i}^{-1} = \tilde{U}_{\sigma_i(\alpha)}$ *for any* $\alpha \in \Psi_{ij}$,

(ii) $[\tilde{U}_\alpha^{\lambda}, \tilde{U}_\beta] \subset \prod_{p,q>0} \tilde{U}^{\lambda}_{p\alpha+q\beta}$ for any $\alpha, \beta \in \Psi_{ij}$ *such that* $\beta \notin -\boldsymbol{Q}_\alpha^{+}$ *and for any* $\lambda \in \Lambda$.

Then there exists a unique group homomorphism $\sigma: \hat{G} \to \tilde{G}$ such that $\rho \circ \sigma = \pi$.

In particular, applying the corollary for $G = \hat{G}$ and $\pi = \mathrm{id}$, we have a sufficient condition for $\hat{G}$ to be universal.

5. Application to Chevalley groups over commutative rings.

THOREM 5.1 *Let G be the elementary subgroup of a Chevally group over a commutative ring R with* 1 *of type Φ, where Φ is irreducible of rank* >1.

(i) *Assume* $\boldsymbol{Z}[R^*] = R$, *then the root data in $\hat{G}$ satisfies* (*DRG* 8).

(ii) *Assume R is semi-local, then the root data in $\hat{G}$ satisfies* (*DRG* 9).

Note that the assumption of (i) is fullfiled by semi-local rings with at most one residue field isomorphic to F_2.

PROOF (i) Let $a, b \in R^*$ and denote

$$\hat{w}_\alpha(a) = \hat{x}_\alpha(a)\hat{x}_{-\alpha}(-a^{-1})\hat{x}_\alpha(a), \quad \hat{h}_\alpha(a, b) = \hat{w}_\alpha(a)\hat{w}_\alpha(-b).$$

Then,

$$\begin{aligned}\hat{h}_\alpha(a, b) &= \hat{x}_\alpha(a)\hat{x}_{-\alpha}(-a^{-1})\hat{x}_\alpha(a)\hat{w}_\alpha(-b)\\ &= \hat{x}_\alpha(a-b)\hat{x}_\alpha(b)\hat{x}_{-\alpha}(-a^{-1}+b^{-1})\hat{x}_\alpha(-b)\hat{x}_\alpha(b)\hat{x}_{-\alpha}(-b^{-1})\hat{x}_\alpha(b)\hat{x}_\alpha(a-b)\hat{w}_\alpha(-b)\end{aligned}$$

$$= \hat{x}_\alpha(a-b)^{\hat{x}_\alpha(b)}\hat{x}_{-\alpha}(-a^{-1}+b^{-1})^{\hat{w}_\alpha(b)}\hat{x}_\alpha(a-b).^*$$

Therefore,

$$^{\hat{x}_\alpha(b)}\hat{x}_{-\alpha}(-a^{-1}+b^{-1}) = \hat{x}_\alpha(-a+b)\hat{h}_\alpha(a,b)\hat{x}_{-\alpha}(b^{-2}(a-b)).$$

For any $b \in R^*$ and $u \in J$, $b^{-1}-u=a \in R^*$. Hence, $u=b^{-1}-a$ for some $a \in R^*$. Since $1+bu \in R^*$, we see $(bu-1)^{-1} \in R^*$. Thus, $b-a^{-1}=b+b(bu-1)^{-1}=b(1+(bu-1)^{-1})=b^2(bu-1)^{-1}u \in J$ and we have

$$^{\hat{x}_{-\alpha}(b)}\hat{x}_\alpha(u) = \hat{x}_{-\alpha}(b-a^{-1})\hat{h}_{-\alpha}(a^{-1},b)\hat{x}_\alpha(b^{-2}(a^{-1}-b))$$

is an element of $\hat{U}^-(R,J)\hat{T}(R)\hat{U}^+(R,J)$, where

$$\hat{h}_{-\alpha}(a^{-1},b) = \hat{x}_{-\alpha}(a^{-1}-b)^{\hat{x}_{-\alpha}(b)}\hat{x}_\alpha(b^{-1}-a)\hat{x}_\alpha(b^{-2}(b-a^{-1}))$$

is an element of $\hat{K} \subset \hat{E}(R,J)$, namely $\hat{h}_{-\alpha}(a^{-1},b) \in \hat{T} \cap \hat{K} = \hat{T}_K$. If $\boldsymbol{Z}[R^*]=R$, then $^{\hat{x}_{-\alpha}(b)}\hat{x}_\alpha(u) \in \hat{U}_K{}^-\hat{T}_K\hat{U}_K{}^+$ for all $b \in R$. Thus we see $\hat{U}_K{}^-\hat{T}_K\hat{U}_K{}^+ \lhd \hat{G}$.

(ii) Let $\Lambda_0=\{\lambda_1,\cdots,\lambda_n\}$ and $x \in \bigcap_{i=1}^n \hat{T}^{\lambda_i}\hat{K}^{\lambda_i}$. Then x can be expressed by $x=h^{\lambda_1}k^{\lambda_1}$ where $h^{\lambda_1}=\prod_{i=1}^n h^{\lambda_1}_{\alpha_i}(t_i,u_i) \in \hat{T}^{\lambda_1}$, $t_i, u_i \in R-\mathfrak{m}_{\lambda_1}$ and $t_iu_i \equiv 1 \pmod{\mathfrak{m}_{\lambda_1}}$ for $1 \leqq i \leqq n$ and $k^{\lambda_1} \in K^{\lambda_1}$. By Chinese remainder theorem, for each $t_i, u_i \in R$, there exist elements t_i', u_i' of R such that

$$t_i' \equiv t_i \pmod{\mathfrak{m}_{\lambda_1}},\ t_i' \equiv 1 \pmod{\mathfrak{m}_{\lambda_j}} \quad (j>1),$$

$$u_i' \equiv u_i \pmod{\mathfrak{m}_{\lambda_1}},\ u_i' \equiv 1 \pmod{\mathfrak{m}_{\lambda_j}} \quad (j>1).$$

Therefore, we can replace $h^{\lambda_1}_{\alpha_i}(t_i,u_i)$ by $h^{\lambda_1}_{\alpha_i}(t_i',u_i')$ which is an element of $\hat{T}$ and $h^{\lambda_1^{-1}}x \in \bigcap_{i=1}^n \hat{T}^{\lambda_i}\hat{K}^{\lambda_i} \cap \hat{K}^{\lambda_1}$. By induction, there exist $h^{\lambda_2},\cdots,h^{\lambda_n} \in \hat{T}$ such that $h^{\lambda_n^{-1}}\cdots h^{\lambda_1^{-1}}x \in \bigcap_{i=1}^n \hat{K}^{\lambda_i}=\hat{K}$. Since $h^{\lambda_n^{-1}}\cdots h^{\lambda_1^{-1}} \in \hat{T}$, $x \in \hat{T}\hat{K}$. q.e.d.

COROLLARY 5.2 *Let G be the elementary subgroup of a universal Chevalley group over a commutative ring R with* 1. *Under the assumption of theorem* 5.1, π: *$\hat{G} \to G$ is a central extension.*

THEOREM 5.3 *Let G be the elementary subgroup of Chevalley group over a commutative ring with an identity of type Φ where Φ is irreducible of rank $\geqq 3$. Assume R is semi-local, $\boldsymbol{Z}[R^*]=R$ and further*

(i) *$\{u^2-1 : u \in R^*\}$ generates a unit ideal if Φ is of type (A_3), (B_3) or (D_4).*

(ii) *There exists no maximal ideal $\mathfrak{m}$ such that $R/\mathfrak{m} \cong F_2$ if Φ is of type (C_3) or (F_4).*

Then G and $\hat{G}$ satisfy the assumption of Corollary 4.5 *and $\hat{G}$ is a universal cover-*

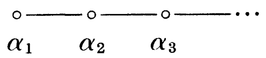
---

* For any elements x, y of a group, we denote $^xy=xyx^{-1}$.

ing of G.

PROOF We shall omit the detailed proof, for we can prove this by the same way as those of M. Stein [6]. We can only simplify the proof by Corollary 4.5.

We shall show only the case of type A_l ($l \geqq 3$). Let the following is the Dynkin deagram of the fundamental roots of Φ.



Let ρ: $\tilde{G} \to G$ be a covering of G. We shall show that there exists a unique homomorphism of groups ϕ: $\hat{G} \to \tilde{G}$ such that $\rho \circ \sigma = \pi$. Take

$$\begin{aligned}
\tilde{x}_{\alpha_1}(t) &= [\tilde{x}_{\alpha_1+\alpha_2}(t), \tilde{x}_{-\alpha_2}(N_{\alpha_1+\alpha_2,-\alpha_2})] \\
\tilde{x}_{\alpha_1+\alpha_2}(t) &= \tilde{m}_{\alpha_2}\tilde{x}_{\alpha_1}(N_{\alpha_1\alpha_2}t)\tilde{m}_{\alpha_2}^{-1} \\
&= [\tilde{x}_{\alpha_1}(-N_{-\alpha_2,\alpha_1+\alpha_2}t), \tilde{x}_{\alpha_2}(N_{\alpha_1\alpha_2}N_{\alpha_1+\alpha_2,\alpha_2})] \\
\tilde{x}_{\alpha_2}(t) &= \tilde{m}_{\alpha_1}\tilde{x}_{\alpha_1+\alpha_2}(N_{\alpha_1\alpha_2}N_{\alpha_1,\alpha_1+\alpha_2}t)\tilde{m}_{\alpha_1}^{-1} \\
&= [\tilde{x}_{-\alpha_1}(N_{-\alpha_2,\alpha_1+\alpha_2}N_{-\alpha_1,\alpha_1+\alpha_2}t), \tilde{x}_{\alpha_1+\alpha_2}(N_{\alpha_1+\alpha_2,-\alpha_2})]
\end{aligned}$$

and for any $\beta \in \Phi$, $\tilde{x}_\beta(t)$ can be defined as a conjugate of $\tilde{x}_{\alpha_1}(u)$ for some $u \in R$. These are independent of the choice of representatives. We shall show that these elements satisfy the relations in the root subgroup and the commutator relations as in G.
(5.3.1) Set

$$\begin{aligned}
Z(\alpha_1) &= \{\gamma \in \Phi\,;\ \alpha_1+\gamma \notin \Phi, \gamma \neq -\alpha_1\} \\
&= \{\alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, -\alpha_2, -\alpha_2-\alpha_3, \pm\alpha_3, \cdots\} \\
Z(\alpha_2) &= \{\gamma \in \Phi\,;\ \alpha_2+\gamma \notin \Phi, \gamma \neq -\alpha_2\} \\
&= \{\alpha_2+\alpha_2, \alpha_2+\alpha_3, -\alpha_1, -\alpha_3, \pm(\alpha_1+\alpha_2+\alpha_3), \cdots\}
\end{aligned}$$

and define $Z(U_{\alpha_i}) = \langle U_\gamma\,;\ \gamma \in Z(\alpha_i)\rangle$ and $Z_0(U_{\alpha_i}) = \langle U_\gamma\,;\ \pm\gamma \in Z(\alpha_i)\rangle$, $i=1,2$. For a root $\alpha \in Z(\alpha_i)$ the map

$$\phi\colon Z(U_\alpha) \longrightarrow Z(\tilde{G})$$

defined by $\phi(z) = [\tilde{x}_{\alpha_i}, \tilde{z}]$ is a homomorphism. Since $Z(\tilde{G})$ is abelian, $[Z(U_\alpha), Z(U_\alpha)] \subset Ker\,\phi$. Therefore, $\phi(z)=1$ if $z \in [Z(U_\alpha), Z(U_\alpha)]$. Since $x_{\alpha_1+\alpha_2}(u) \in [U_{\alpha_1+\alpha_2+\alpha_3}, U_{-\alpha_3}]$, we have

$$[\tilde{U}_{\alpha_1}, \tilde{U}_{\alpha_1+\alpha_2}]=1 \quad \text{and} \quad [\tilde{U}_{\alpha_2}, \tilde{U}_{\alpha_1+\alpha_2}]=1\,.$$

If $l=3$, then $Z_0(U_{\alpha_1}) = \langle \tilde{U}_{\pm\alpha_3}\rangle$ is perfect if $\{u^2-1\,;\ u \in R^*\}$ generates the unit ideal. If $l>3$, then *rank* $Z_0(U_{\alpha_i}) \geqq 2$ and $Z_0(U_{\alpha_i})$ is perfect. In this case, we have

$$[\tilde{U}_{\alpha_i}, \tilde{U}_{\alpha_3}]=1 \qquad (i=1,2)\,.$$

By definition, if $l>3$, we can see that $[\tilde{U}_\alpha, \tilde{U}_\beta]=1$ for any $\alpha, \beta \in \Phi$ such that $\alpha+\beta \notin \Phi$.

$$
\begin{aligned}
(5.3.2)\qquad \tilde{x}_{\alpha_1}(t+u) &= [\tilde{x}_{\alpha_1+\alpha_2}(t+u), \tilde{x}_{-\alpha_2}(N_{\alpha_1+\alpha_2,-\alpha_2})] \\
&= [\tilde{x}_{\alpha_1+\alpha_2}(t)\tilde{x}_{\alpha_1+\alpha_2}(u), \tilde{x}_{-\alpha_2}(N_{\alpha_1+\alpha_2,-\alpha_2})] \\
&= {}^{\tilde{x}_{\alpha_1+\alpha_2}(t)}[\tilde{x}_{\alpha_1+\alpha_2}(u), \tilde{x}_{-\alpha_2}(N_{\alpha_1+\alpha_2,-\alpha_2})][\tilde{x}_{\alpha_1+\alpha_2}(t), \tilde{x}_{-\alpha_2}(N_{\alpha_1+\alpha_2,-\alpha_2})] \\
&= [\tilde{x}_{\alpha_1+\alpha_2}(u), \tilde{x}_{-\alpha_2}(N_{\alpha_1+\alpha_2,-\alpha_2})\tilde{x}_{\alpha_1}(\pm N_{\alpha_1+\alpha_2,-\alpha_2}t)][\tilde{x}_{\alpha_1+\alpha_2}(t), \tilde{x}_{-\alpha_2}(N_{\alpha_1+\alpha_2,-\alpha_2})] \\
&= [\tilde{x}_{\alpha_1+\alpha_2}(u), \tilde{x}_{-\alpha_2}(N_{\alpha_1+\alpha_2,-\alpha_2})][\tilde{x}_{\alpha_1+\beta_2}(t), \tilde{x}_{-\alpha_2}(N_{\alpha_1+\alpha_2,-\alpha_2})] \\
&= \tilde{x}_{\alpha_1}(t)\tilde{x}_{\alpha_1}(u)\,.
\end{aligned}
$$

From the definition, we see that the above relation holds for any root $\alpha\in\Phi$.

$$
\begin{aligned}
(5.3.3)\qquad \tilde{x}_{\alpha_1+\alpha_2}(N_{\alpha_1\alpha_2}tu) &= {}^{\tilde{m}_{\alpha_2+\alpha_3}}\tilde{x}_{\alpha_1+\alpha_2}(N_{\alpha_1\alpha_2}tu) \\
&= {}^{\tilde{m}_{\alpha_2+\alpha_3}}[\tilde{x}_{\alpha_1}(N_{\alpha_1\alpha_2}N_{\alpha_1+\alpha_2,-\alpha_2}tu), \tilde{x}_{\alpha_2}(N_{\alpha_1\alpha_2}N_{\alpha_1+\alpha_2,-\alpha_2})] \\
&= [\tilde{x}_{\alpha_1+\alpha_2+\alpha_3}(-N_{\alpha_2+\alpha_3,\alpha_1}N_{\alpha_1+\alpha_2,-\alpha_2}N_{\alpha_1\alpha_2}tu), \tilde{x}_{-\alpha_3}(-N_{-\alpha_2-\alpha_3,\alpha_2}N_{\alpha_1\alpha_2}N_{\alpha_1+\alpha_2,-\alpha_2})] \\
&= [\tilde{x}_{\alpha_1+\alpha_2+\alpha_3}(N_{\alpha_1,\alpha_2+\alpha_3}tu), \tilde{x}_{-\alpha_3}(N_{\alpha_2+\alpha_3,-\alpha_2})]
\end{aligned}
$$

for $N_{\alpha_1\alpha_2}N_{\alpha_1+\alpha_2,-\alpha_2}=1, N_{\alpha,\beta}=-N_{-\alpha,-\beta}$.

On the other hand,

$$
\begin{aligned}
[\tilde{x}_{\alpha_1}(t), \tilde{x}_{\alpha_2}(u)] &= [\tilde{x}_{\alpha_1}(t), [\tilde{x}_{\alpha_2+\alpha_3}(u), \tilde{x}_{-\alpha_3}(N_{\alpha_2+\alpha_3,-\alpha_3})]] \\
&= [[\tilde{x}_{\alpha_1}(t), \tilde{x}_{\alpha_2+\alpha_3}(u)], \tilde{x}_{-\alpha_3}(N_{\alpha_2+\alpha_3,-\alpha_3})] \\
&= [\tilde{x}_{\alpha_1+\alpha_2+\alpha_3}(N_{\alpha_1,\alpha_2+\alpha_3}tu), \tilde{x}_{-\alpha_3}(N_{\alpha_2+\alpha_3,-\alpha_3})]\,.
\end{aligned}
$$

Thus, we have $[\tilde{x}_{\alpha_1}(t), \tilde{x}_{\alpha_2}(u)]=\tilde{x}_{\alpha_1+\alpha_2}(N_{\alpha_1\alpha_2}tu)$. From the definition, we see that the above relation holds for any roots $\alpha, \beta\in\Phi$ such that $\alpha+\beta\in\Phi$. Therefore, the map $\hat{x}_\alpha(t)\longmapsto\tilde{x}_\alpha(t)$ defines a unique homomorphism σ: $\hat{G}\to\tilde{G}$ such that $\rho\circ\sigma=\pi$ under the assumption of the theorem.

Similarly, we can prove the theorem for groups of types E_6, E_7, E_8 and D_l ($l\geqq 5$), and for the groups of type D_4 under the assumption that $\{u^2-1;\ u\in R^*\}$ generates the unit ideal.

As for the groups of type B_l, C_l ($l\geqq 3$) or F_4, first define $\tilde{x}_\alpha(t)$ for a long or short root α and then $\tilde{x}_\beta(u)$ for a short or long root β. Then, we can define $\tilde{x}_\gamma(v)$ for any $\gamma\in\Phi$ as a conjugate of $\tilde{x}_\alpha(t)$ or $\tilde{x}_\beta(u)$. Among the root subgroups, we can prove the same relations as in the groups G under the assumption of the theorem.

THEOREM 5.4 *Let G be the elementary subgroup of a twisted Chevalley group over a commutative ring R with an identity and with an involution σ and of type Φ, where Φ is irreducible of rank >1. Assume that the intersection of all σ-invariant maximal ideals of R coincides with its Jacobson radical, R is semi-local and $\boldsymbol{Z}[R^*]=R$. If Φ is of type A_{2n} ($n>1$), we assume further the following*: $\mathcal{R}^*\neq\phi$. *Let*

$$S=\{b\in R^*;\ b+\bar{b}=a\bar{a} \text{ for some } a\in R\}$$
$$S_1=\{b\in S\colon b\equiv 1 \pmod{J}\}$$
$$S_2=\{c\bar{c}b;\ c\in R^*, b\in S_1\}$$

and let $\langle S_2\rangle$ *be the set of all b-components of the subgroup of* $\mathcal{R}^*$ *(with respect to* $\dot{+}$*) generated by the elements whose b-components are contained in* S_2*. Assume* $\langle S_2\rangle = S$.

Then, G is a group with a strict root data.

Poof We shall prove (*DRG* 8) for the group of type A_{2n}. As for the other statement, we can prove in the same way as the case of Chevalley groups.

Let $\xi=(a,b)\in\mathcal{R}^*$. By definition,

$$\hat{w}_\alpha(\xi)=\hat{x}_\alpha(\xi)\hat{x}_{-\alpha_1}(-\bar{b}\longrightarrow\xi)\hat{x}(b\bar{b}^{-1}\longrightarrow\xi)$$

$$\hat{w}_\alpha(\xi)^{-1}=\hat{w}_\alpha(b\bar{b}^{-1}\longrightarrow\xi^*).$$

Now, we define $\hat{h}_\alpha(\xi,\eta)=\hat{w}_\alpha(\xi)\hat{w}_\alpha(\eta)$ for $\xi=(a,b)$, $\eta=(c,d)\in\mathcal{R}^*$, Then

$$\begin{aligned}\hat{h}_\alpha(\xi,\eta)&=\hat{x}_\alpha(\xi)\hat{x}_{-\alpha}(-\bar{b}\longrightarrow\xi)\hat{w}_\alpha(b\bar{b}^{-1}\longrightarrow\xi)\hat{w}_\alpha(\eta)\\&=\hat{x}_\alpha(\xi\dot{+}d\bar{d}^{-1}\longrightarrow\eta))\hat{x}_\alpha(d\bar{d}^{-1}\longrightarrow\eta^*)\hat{x}_\alpha((-\bar{b}\longrightarrow\xi)\dot{+}(-\tilde{d}\longrightarrow\eta))\\&\quad\hat{x}_\alpha(d\bar{d}^{-1}\longrightarrow\eta)\hat{x}_\alpha(d\bar{d}^{-1}\longrightarrow\eta^*)\hat{x}_{-\alpha}(-\bar{d}\longrightarrow\eta^*)\hat{x}_\alpha(\eta^*)\hat{x}_\alpha(\eta\dot{+}b\bar{b}^{-1}\longrightarrow\xi))\hat{w}_\alpha(\eta)\\&=\hat{x}_\alpha(\xi\dot{+}(\hat{d}\bar{d}^{-1}\longrightarrow\eta))^{x_\alpha(d\bar{d}^{-1}\longrightarrow\eta^*)}\hat{x}_{-\alpha}((-\bar{b}\longrightarrow\xi)\dot{+}(-d\longrightarrow\eta))^{\hat{w}_\alpha(\eta)^{-1}}\hat{x}_\alpha(\eta\dot{+}(b\bar{b}^{-1}\longrightarrow\xi)).\end{aligned}$$

Thus, we have

$$\begin{aligned}&{}^{\hat{x}_\alpha(d\bar{d}^{-1}\longrightarrow\eta^*)}\hat{x}_{-\alpha}((-\bar{b}\longrightarrow\xi)\dot{+}(-\bar{d}\longrightarrow\eta))\\&\qquad=\hat{x}_{-\alpha}((d\bar{d}^{-1}\longrightarrow\eta^*)\dot{+}\xi^*)\hat{h}_\alpha(\xi,\eta)\hat{x}_{-\alpha}((-\bar{d}^{-1}db\bar{b}^{-1}\longrightarrow\xi^*)\dot{+}(-\bar{d}^{-2}d\longrightarrow\eta^*))\end{aligned}$$

In other words,

$$\begin{aligned}&{}^{\hat{x}_{-\alpha}(\eta)}\hat{x}_\alpha(\xi\mathring{+}(-d^{-1}\bar{d}^2\longrightarrow\eta^*))\\&\quad=\hat{x}_\alpha(\eta\dot{+}(-\bar{b}^{-1}\longrightarrow\xi^*))\hat{h}_{-\alpha}(-\bar{b}^{-1}\longrightarrow\xi,\ d^{-1}\bar{d}\longrightarrow\eta^*)\hat{x}_\alpha((\bar{d}^{-2}d\bar{b}^{-2}b\longrightarrow\xi^*)\dot{+}(-\bar{d}^{-1}\longrightarrow\eta)).\end{aligned}$$

Now, for any $\eta=(c,d)\in\mathcal{R}^*$ and $\gamma=(u,v)\in\mathcal{R}$ where $u,v\in J$, the equation

$$(u,v)=(a+d^{-1}\bar{d}^2c,\quad b+d\bar{d}^2+\bar{a}cd^{-1}\bar{d}^2)$$

has a solution $(a,b)\in\mathcal{R}^*$. In fact,

$$a=u-d^{-1}\bar{d}^2c\in R,\quad b=v-\bar{u}cd^{-1}\bar{d}^2+d^2\bar{d}\in R^*$$

and we see $a\bar{a}=b+\bar{b}$. Here, $b\equiv d^2\bar{d}\pmod{J}$, $a\equiv -d^{-1}\bar{d}^2c\pmod{J}$ and

$$\hat{h}_{-\alpha}(-\bar{b}^{-1}\longrightarrow\xi,\ d^{-1}\bar{d}\longrightarrow\eta^*)\in\hat{K}\Longleftrightarrow(d\bar{d})^2\equiv 1\pmod{J}$$

Apply the above equation for $\eta=(c,d)$, $d\equiv 1\pmod{J}$ and $\gamma=(u,v)\in\mathcal{R}$, where $u,v\in J$, and we have

$${}^{\hat{x}_{-\alpha}(\eta)}\hat{x}_\alpha(\gamma)\in\hat{U}_K^-\hat{T}_K\hat{U}_K^+.$$

Thus we have (*DRG* 8) from the assumption of the theorem.

The following theorem can be proved similarly and we shall omit the proof.

THEOREM 5.5 *Let $\hat{G}$ be the elementary subgroup of the twisted Chevalley group over a commutative ring with an identity and with an involution. Assume the same condition as in Theorem* 5.4. *Then $\hat{G}$ is a universal covering of G if the rank $\Phi_\sigma \geqq 5$.*

If rank $\Phi_\sigma=4$, when G is of the first type, assume further $\{u-1;\ u\in R^*\}$ generate the unit ideal R and when G is of the second type, assume further there exists a unit $u\in R^*$ such that $u-1$, $u\bar{u}-1\in R^*$; if rank $\Phi_\sigma=3$, when G is of the first type, assume further $\{u-1;\ u\in R_0^*\}$ generates the unit ideal R_0 and when G is of the second type, assume further there exists a unit $u\in R^*$ such that $u=bd$ for some b, $d\in S^*$ and $u^{-1}\bar{u}^2-1$, $u\bar{u}-1$ are units of R, then $\hat{G}$ is also a universal covering of G.

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