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ON THE HYPERSPACE $\mathfrak{C}(X)$ OF CONTINUA

By

R. Escobedo, V. Sánchez-Gutierrez and J. Sánchez-Martínez

Abstract. Let X be a continuum. Let $C(X)$ be the hyperspace of all closed, connected and nonempty subsets of X, with the Hausdorff metric. For a mapping $f : X \to Y$ between continua, let $C(f) : C(X) \to C(Y)$ be the induced mapping by f, given by $C(f)(A) = f(A)$. In this paper we study the hyperspace $\mathfrak{C}(X) = \{C(A) : A \in C(X)\}$ as a subspace of $C(C(X))$, and define an induced function $\mathfrak{C}(f)$ between $\mathfrak{C}(X)$ and $\mathfrak{C}(Y)$. We prove some relationships between the functions f, $C(f)$ and $\mathfrak{C}(f)$ for the following classes of mapping: confluent, light, monotone and weakly confluent.

1. Introduction

A *continuum* is a nondegenerate compact connected metric space. Given a continumm X, denote by 2^X and $C(X)$ the hyperspace of all nonempty closed subsets and all subcontinua of X, respectively, equipped with the Hausdorff metric (see [10, Definition 0.1, p. 1]). It is well known that 2^X and $C(X)$ are continua (see [10, Theorem 1.13, p. 65]) and then $C(C(X))$ is a continuum. We consider $\mathfrak{C}(X) = \{C(A) : A \in C(X)\}$ as a subspace of $C(C(X))$. We study some properties of this hyperspace. Also we give a characterization of the arc and circumference using the structure of its hyperspaces $\mathfrak{C}(X)$.

A *mapping* means a continuous and not constant function. Given a mapping $f : X \to Y$ between continua, let $C(f) : C(X) \to C(Y)$ be the induced mapping by f, given by $C(f)(A) = f(A)$ for each $A \in C(X)$. We consider $\mathfrak{C}(f) : \mathfrak{C}(X) \to \mathfrak{C}(Y)$ given by $\mathfrak{C}(f)(C(A)) = C(f(A))$, for each $C(A) \in \mathfrak{C}(X)$. Let $\mathscr{M}$ be a class of mappings between continua. A general problem is to find all possible relationships among the following three statements:

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(1) $f \in \mathcal{M}$;
(2) $C(f) \in \mathcal{M}$;
(3) $\mathfrak{C}(f) \in \mathcal{M}$;

In this paper we study the interrelations among the statements (1)–(3), for the following classes of mappings: confluent, light, monotone and weakly confluent. Readers especially interested in this problem are referred to [1], [2], [3], [5] and [6].

The paper is divided into five sections. In Section 2, we give the basic definitions for understanding the paper. In Section 3, we give examples of geometric models of $\mathfrak{C}(X)$ for some continua X. In Section 4, we present some properties about topological structure of $\mathfrak{C}(X)$. Finally, Section 5 is devoted to the study of the relationships between the mappings f, $C(f)$ and $\mathfrak{C}(f)$.

2. Definitions and Preliminaries

The symbols $\mathbf{N}$ and $\mathbf{R}$ will denote the set of positive integers and real numbers, respectively. The symbol I will denote the closed interval $[0,1]$. An *arc* is any space which is homeomorphic to I. A *simple closed curve* is a space homeomorphic to $S^1 = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$.

Given a continuum Z, $A \subset Z$ and $\varepsilon > 0$, $\mathcal{V}_\varepsilon(A)$, $\mathrm{cl}_Z(A)$, $\mathrm{int}_Z(A)$ and $\partial_Z(A)$ denote the respective open ball about A of radius ε, closure, interior and boundary of A in Z.

In this paper, *dimension* means inductive dimension as defined in [10, (0.44), p. 21]. The symbol dim will be used to denote dimension.

Given a finite collection $K_1, \dots, K_r$ of subsets of X, $\langle K_1, \dots, K_r \rangle$, denotes the following subset of 2^X:

$$\left\{ A \in 2^X : A \subset \bigcup_{i=1}^{r} K_i, A \cap K_i \neq \emptyset \text{ for each } i \in \{1, \dots, r\} \right\}.$$

It is known that the family of all subsets of 2^X of the form $\langle K_1, \dots, K_r \rangle$, where each K_i is an open subset of X, forms a basis for a topology for 2^X (see [10, Theorem 0.11, p. 9]) called the *Vietoris Topology*. The Vietoris topology and the topology induced by the Hausdorff metric coincide (see [10, Theorem 0.13, p. 10]). The hyperspaces $C(X)$ and $F_n(X)$ are considered as subspaces of 2^X.

A continuum X is said to be a *dendrite* provided that it is locally connected and contains no circle. A *graph* is a continuum which can be written as the union of finitely many arcs, any two of which are either disjoint or intersect only in one or both of their end points. A *tree*, or acyclic graph, is a graph which contains no simple closed curve. *Hereditarily unicoherent* provided that for each pair of subcontinua A and B of X, $A \cap B$ is connected.

We need the following well known definition.

Definition 2.1. Given a sequence $\{A_m\}_{m=1}^{\infty}$ of subsets of X define:

- $\limsup_{m\to\infty} A_m$ as the set of points $x \in X$ such that there exists a sequence of positive numbers $m_1 < m_2 < \cdots$ and there exists points $x_{m_k} \in A_{m_k}$ such that $\lim x_{m_k} = x$;
- $\liminf_{m\to\infty} A_m$ as the set of points $x \in X$ such that for each $n \in \mathbf{N}$ there exists a point $x_n \in A_n$ such that $\lim x_n = x$.

It is well known that a sequence $\{A_m\}_{m=1}^{\infty}$ in $C(X)$ converges to $A \in C(X)$ if and only if $\limsup_{m\to\infty} A_n \subset A \subset \liminf_{m\to\infty} A_m$.

Let X be a continuum. We define $C_X^* : C(X) \to C(C(X))$, given by $C_X^*(A) = C(A)$, for each $A \in C(X)$. Notice that $\mathfrak{C}(X) = C_X^*(C(X))$. Thus $C_X^* : C(X) \to \mathfrak{C}(X)$ is a biyective function.

Remark 2.2. C_X^* is continuous if for any sequence $\{A_i\}_{i\in\mathbf{N}}$ of subcontinua A_i of X converging to a subcontinuum A of X, any subcontinuum B of A is a limit of subcontinua B_i of A_i.

Definition 2.3. A continuum X is said to be **C^*-smooth at** $A \in C(X)$ provided that C_X^* is continuous at A. A continuum X is said to be **C^*-smooth** provided that C_X^* is continuous on $C(X)$, i.e., at each $A \in C(X)$.

It is well known that, each arc-like continuum is C^*-smooth (see [10, Theorem 15.13, p. 525]), C^*-smoothness implies hereditary unicoherence (see [4, Corollary 3.4, p. 203] and [10, Note 1, p. 530]). Thus each arcwise connected C^*-smooth continuum is a dendroid (see [10, Theorem 15.19, p. 528]). Further, a locally connected continuum is C^*-smooth if and only if it is a dendrite (see [10, Theorem 15.11, p. 522]).

Using Remark 2.2, is easy to show the following result.

Theorem 2.4. *For a continuum X, the following statements are equivalent*:

1) *X is C^*-smooth*;
2) *C_X^* is a homeomorphism*;
3) *$C(X)$ is homeomorphic to $\mathfrak{C}(X)$*;

4) $\mathfrak{C}(X)$ *is a continuum*;
5) $\mathfrak{C}(X)$ *is compact.*

Remark 2.5. Let X be a continuum. The union mapping $\mathscr{U} : 2^{2^X} \to 2^X$ is the function given by $\mathscr{U}(\mathscr{A}) = \bigcup \mathscr{A}$ for each $\mathscr{A} \in 2^{2^X}$ (see [8, Exercise 11.5, p. 91]). Denote by $\mathscr{U}_X = \mathscr{U}|_{\mathfrak{C}(X)}$. Notice that $(C_X^*)^{-1} = \mathscr{U}_X$. In addition, by [8, Exercise 11.5 (2), p. 91], $\mathscr{U}_X$ is a continuous function.

Remark 2.6. Let X be a continuum, by Remark 2.5, if $\{C(A_n)\}_{n \in \mathbf{N}}$ is a sequence in $\mathfrak{C}(X)$ converging to a point $C(A) \in \mathfrak{C}(X)$, then $\{A_n\}_{n \in \mathbf{N}}$ converges to A in $C(X)$.

A mapping $f : X \to Y$ between metric spaces is said to be:
- *confluent* if for each subcontinuum K of Y and for each component M of $f^{-1}(K)$, $f(M) = K$;
- *light* if $f^{-1}(y)$ is totally disconnected for each $y \in Y$;
- *monotone* if $f^{-1}(y)$ is connected in Y for each $y \in Y$;
- *weakly confluent* if for each subcontinuum K of Y, there exists a subcontinuum M of X such that $f(M) = K$.

A general study of these mappings can be found in [9].

3. Examples

From Theorem 2.4 and [10, Theorem 15.11, p. 522], we have the following two corolaries.

Corollary 3.1. *Let X be a locally connected continuum. $\mathfrak{C}(X)$ is homeomorphic to $C(X)$ if and only if X is a dendrite.*

Corollary 3.2. *Let X be a graph. $\mathfrak{C}(X)$ is homeomorphic to $C(X)$ if and only if X is a tree.*

Example 3.3. Notice that S^1 is C^*-smooth in $C(A)$ for each $A \in C(S^1) - \{S^1\}$, because in this case A is an arc. On the other hand, $C(S^1)$ is an isolated point in $\mathfrak{C}(S^1)$, in particular $\mathfrak{C}(S^1)$ is not connected and is not compact. On the contrary, suppose that there exists a sequence $\{A_1\}_{i \in \mathbf{N}}$, such that A_i is an arc contained in S^1 for each $i \in \mathbf{N}$ and $\{C(A_i)\}_{i \in \mathbf{N}}$ is a sequence in $\mathfrak{C}(S^1)$ converging to $C(S^1)$.

For each $i \in \mathbf{N}$, we denote by a_i and b_i the end points of A_i. So, we may assume that $\{a_i\}_{i \in \mathbf{N}}$ and $\{b_i\}_{i \in \mathbf{N}}$ both converge to a point $p \in S^1$.

Notice that, for each $0 < \varepsilon < 1$, there exists $N \in \mathbf{N}$ such that $a_i, b_i \in \mathrm{cl}_{S^1}(\mathscr{V}_\varepsilon(\{p\})) \cap S^1$, for all $i > N$. Thus, by the point

$$(\mathrm{cl}_{S^1}(\mathscr{V}_\varepsilon(\{p\})) \cap S^1) \in C(S^1),$$

there is not a sequence of elements in each $C(A_i)$ converging to

$$\mathrm{cl}_{S^1}(\mathscr{V}_\varepsilon(\{p\})) \cap S^1.$$

This is a contradiction, because $\{C(A_i)\}_{i \in \mathbf{N}}$ converges to $C(S^1)$.

We conclude that, $\mathfrak{C}(S^1)$ is homeomorphic to the union of the unit disk D in the plane $\mathbf{R}^2$ minus the point $(0,0)$ and $\{q\}$, where q is any point in $\mathbf{R}^2 - D$.

Regarding [4, Corollary 3.4, p. 203], we present the following example.

Example 3.4. Let

$$T = \{(x,0) \in \mathbf{R}^2 : -1 \le x \le 1\} \cup \{(0,y) \in \mathbf{R}^2 : 0 \le y \le 1\},$$

$p = (-1,0)$ and $q = (1,0)$. For each $n \in \mathbf{N}$ consider:

- $a_n = \left(\frac{-1}{n+1}, \frac{1}{n+1}\right)$, $b_n = \left(0, 1 + \frac{1}{n+1}\right)$, $c_n = \left(\frac{1}{n+1}, \frac{1}{n+1}\right)$ and $d_n = \left(1, \frac{1}{n+1}\right)$;
- I_n, J_n, K_n and L_n the linear segments joining p with a_n, a_n with b_n, b_n with c_n and c_n with d_n, respectively.

Define $X = T \cup (\bigcup_{n \in \mathbf{N}} I_n \cup J_n \cup K_n \cup L_n)$. It is clear that X is a hereditarily unicoherent continuum. The hyperspace $\mathfrak{C}(X)$ is not compact. In fact, if for each $n \in \mathbf{N}$ we consider $T_n = I_n \cup J_n \cup K_n \cup L_n$, then T_n is an arc and $\lim T_n = T$. Notice $\{C(T_n)\}_{n \in \mathbf{N}}$ does not converge to $C(T)$ because

$$A = \left\{(x,0) \in \mathbf{R}^2 : -\frac{1}{2} \le x \le \frac{1}{2}\right\} \cup \left\{(0,y) \in \mathbf{R}^2 : 0 \le y \le \frac{1}{2}\right\},$$

is an element in $C(T)$ which is not limit of points A_n in $C(T_n)$.

4. Properties of the Hyperspace $\mathfrak{C}(X)$

Remark 4.1. The spaces X, $\{\{x\} \in C(X) : x \in X\}$ and $\{\{\{x\}\} \in \mathfrak{C}(X)\}$ are mutually homeomorphic, for each continuum X.

Proposition 4.2. *For each continuum X, $int_{C(C(X))}(\mathfrak{C}(X)) = \varnothing$.*

Proof. Take $A \in C(X) - \{X\}$. Using [11, Exercise 5.25, p. 85], we can consider $\{A_n\}_{n \in \mathbf{N}}$ be a sequence in $C(X)$ such that $\lim A_n = A$ and $A \subsetneqq A_{n+1} \subsetneqq A_n \subsetneqq X$ for each $n \in \mathbf{N}$. Thus $\{C(A) \cup F_1(A_n)\}_{n \in \mathbf{N}}$ is a sequence in $C(C(X)) - \mathfrak{C}(X)$ converging to $C(A)$. Hence, $C(A) \notin \mathrm{int}_{C(C(X))}(\mathfrak{C}(X))$. Then, $\mathrm{int}_{C(C(X))}(\mathfrak{C}(X)) \subset \{C(X)\}$. Since $\{C(X)\}$ is a closed subset in the continuum $C(C(X))$, $\mathrm{int}_{C(C(X))}(\mathfrak{C}(X)) = \emptyset$. □

Theorem 4.3. *Let X be a continuum. Consider the following conditions*:
(1) *X is locally connected*;
(2) *$C(X)$ is locally connected*;
(3) *$\mathfrak{C}(X)$ is locally connected.*
Then (1) *and* (2) *are equivalent and* (3) *implies* (1) (*consequently* (3) *implies* (2)).

Proof. It is well known that (1) and (2) are equivalent (see [7, Exercise 2.17, p. 28]).

To see (3) implies (2) let $p \in X$ and U be an open subset of X containing p. Notice that $\langle\langle U \rangle\rangle \cap \mathfrak{C}(X)$ is an open subset in $\mathfrak{C}(X)$ containing $C(\{p\})$. By hypothesis there exists $\mathfrak{W}$ be an open connected subset of $\mathfrak{C}(X)$ such that

$$C(\{p\}) \subset \mathfrak{W} \subset \mathrm{cl}_{\mathfrak{C}(X)}(\mathfrak{W}) \subset \langle\langle U \rangle\rangle \cap \mathfrak{C}(X).$$

By [10, Exercise 15.9 (2), p. 124] we have that $\mathscr{W} = \bigcup\{\mathscr{A} : \mathscr{A} \subset \mathrm{cl}_{C(C(X))}\, \mathfrak{W}\}$ is a subcontinuum in $C(X)$, observe that $\mathscr{W} \subset \langle U \rangle$. Again, by [10, Excercise 15.9 (2), p. 124] we obtain that $W = \bigcup\{A : A \in \mathscr{W}\}$ is a subcontiuum of X such that $p \in W \subset U$. Using that $C(\{p\}) \in \mathrm{int}_{\mathfrak{C}(X)}\, \mathfrak{W}$ we have that $p \in \mathrm{int}_X(W)$. So X is locally connected in p. □

Regarding Theorem 4.3, we present an example that show that (1) does not imply (3).

Example 4.4. For each $n \in \mathbf{N}$, let

$$S_n = \left\{(x, y) \in \mathbf{R}^2 : \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2}\right\},$$

and define $X = \bigcup_{n=1}^{\infty} S_n$, the continuum known as Hawaiian earring. It is clear that X is locally connected. We will prove that $\mathfrak{C}(X)$ is not locally connected in $C(\{(0,0)\})$. Let $\mathfrak{U}$ be an open subset of $\mathfrak{C}(X)$ containing $C(\{(0,0)\})$. Since $\lim C(S_n) = C(\{(0,0)\})$, there exists $m \in \mathbf{N}$ such that $C(S_m) \in \mathfrak{U}$. Consider the following two disjoint sets

$$\mathfrak{V} = \{C(A) \in \mathfrak{C}(X) : S_m \nsubseteq A\},$$

and

$$\mathfrak{W} = \{C(A) \in \mathfrak{C}(X) : S_m \subseteq A\}.$$

Notice that $C(\{(0,0)\}) \in \mathfrak{V} \cap \mathfrak{U}$, $C(S_m) \in \mathfrak{W} \cap \mathfrak{U}$ and $\mathfrak{C}(X) = \mathfrak{V} \cap \mathfrak{W}$.

On the other hand, if $\{C(A_n)\}_{n \in \mathbf{N}}$ is a sequence in $\mathfrak{C}(X)$ converging to a point $C(A) \in \mathfrak{C}(X)$ and $S_m \in C(A_n)$ for each n, then $S_m \in C(A)$ because $\lim A_n = A$. Thus, $\mathfrak{W}$ is closed in $\mathfrak{C}(X)$.

Whith an idea similar to that given in Example 3.3, we can prove that $\mathfrak{V}$ is closed in $\mathfrak{C}(X)$.

We conclude that $\mathfrak{U}$ is not connected. So, $\mathfrak{C}(X)$ is not locally connected at $C(\{(0,0)\})$.

Proposition 4.5. *If X and Y are two C^*-smooth continua and $C(X)$ is homeomorphic to $C(Y)$, then $\mathfrak{C}(X)$ is homeomorphic to $\mathfrak{C}(Y)$.*

Proof. Let $h : C(X) \to C(Y)$ be a homeomorphism. Consider $\hat{h} : \mathfrak{C}(X) \to \mathfrak{C}(Y)$ defined by $\hat{h} = C_Y^* \circ h \circ \mathscr{U}_X$. Since $\hat{h}^{-1} = C_X^* \circ h^{-1} \circ \mathscr{U}_Y$, we conclude that $\hat{h}$ is a homeomorphism. □

Remark 4.6. In general $C(X)$ homeomorphic to $C(Y)$ does not imply $\mathfrak{C}(X)$ homeomorphic to $\mathfrak{C}(Y)$. For example, if $X = I$ and $Y = S^1$, $C(X)$ and $C(Y)$ are both 2-cells and $\mathfrak{C}(X)$ is homeomorphic to $C(X)$ but $\mathfrak{C}(Y)$ is not homeomorphic to $C(Y)$.

Theorem 4.7. *Let X and Y be two continua. If X is C^*-smooth and $\mathfrak{C}(X)$ is homeomorphic to $\mathfrak{C}(Y)$, then $C(X)$ and $C(Y)$ are homeomorphic.*

Proof. Let $h : \mathfrak{C}(X) \to \mathfrak{C}(Y)$ be a homeomorphism. Notice that $\hat{h} = \mathscr{U}_Y \circ h \circ C_X^*$ is a bijective mapping between $C(X)$ and $C(Y)$. We conclude that $\hat{h}$ is a homeomorphism. □

We say that a continuum X *has unique hyperspace* $\mathfrak{C}(X)$ if for each continuum Y the condition $\mathfrak{C}(X)$ homeomorphic to $\mathfrak{C}(Y)$ implies that X is homeomorphic to Y.

Theorem 4.8. *If X is homeomorphic to I or S^1, then X has unique hyperspace $\mathfrak{C}(X)$.*

Proof. Suppose that $X = I$ or $X = S^1$, and let Y be a continuum such that $\mathfrak{C}(X)$ is homeomorphic to $\mathfrak{C}(Y)$. Since $\mathfrak{C}(X)$ is homeomorphic to unit disk D in the plane $\mathbf{R}^2$ or the union of D minus the point $(0,0)$ and the set $\{q\}$, where q is any point in $\mathbf{R}^2 - D$, then by Theorem 4.3, Y is locally connected, further Y is arcwise connected. By [8, Theorem 70.1, p. 337], Y does not contain simple triods. By [11, Proposition 9.5, p. 142], Y is an arc or a simple closed curve. Since $\mathfrak{C}(I)$ is connected and $\mathfrak{C}(S^1)$ is not, we conclude that Y is an arc if $X = I$, or Y is a simple closed curve if $X = S^1$. □

Theorem 4.9. *Let X be a continuum. The following conditions are equivalent*:

1) *X is homeomorphic to $[0,1]$;*
2) *$\mathfrak{C}(X)$ is homeomorphic to $[0,1]^2$;*
3) *$\mathfrak{C}(X)$ is homeomorphic to $[0,1]^n$, for some $n \in \mathbf{N}$;*
4) *$\mathfrak{C}(X)$ is the finite product of locally connected continua.*

Proof. 1) implies 2) follows from Corollary 3.1 and Remark 4.6. It is clear that 2) $\Rightarrow$ 3) $\Rightarrow$ 4) hold.

Now, to see that 4) implies 1), notice that $\mathfrak{C}(X)$ is a continuum, thus by Theorem 2.4, $C(X)$ is homeomorphic to $\mathfrak{C}(X)$. By [10, (10.1)], X is an arc or a simple closed curve. By Example 3.3, we conclude that X is an arc. □

Using Example 3.3, Theorem 4.8 and [8, Theorem 70.1, p. 337], we have the following result.

Theorem 4.10. *Let X be a continuum. The following conditions are equivalent*:

1) *X is homeomorphic to S^1;*
2) *$\mathfrak{C}(X)$ is homeomorphic to the union of a unit disk D in the plane $\mathbf{R}^2$ minus the point $(0,0)$ and $\{q\}$, where q is any point in $\mathbf{R}^2 - D$;*
3) *$\mathfrak{C}(X)$ is a locally connected, two-dimensional and nonconnected space.*

5. On Induced Function $\mathfrak{C}(f)$

Given a mapping $f : X \to Y$ between continua, we consider the induced mapping $C(f) : C(X) \to C(Y)$, given by $C(f)(A) = f(A)$ for each $A \in C(X)$ (see [5]). In a similar way we define the function $\mathfrak{C}(f) : \mathfrak{C}(X) \to \mathfrak{C}(Y)$, given by

$\mathfrak{C}(f)(C(A)) = C(f(A))$ for each $C(A) \in \mathfrak{C}(X)$. It is clear that $\mathfrak{C}(f)$ is well defined and the following diagrams

$$\begin{array}{ccc} C(X) & \xrightarrow{C(f)} & C(Y) \\ {\scriptstyle C_X^*}\downarrow & & \downarrow{\scriptstyle C_Y^*} \\ \mathfrak{C}(X) & \xrightarrow[\mathfrak{C}(f)]{} & \mathfrak{C}(Y) \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{C}(X) & \xrightarrow{\mathfrak{C}(f)} & \mathfrak{C}(Y) \\ {\scriptstyle \mathcal{U}_X}\downarrow & & \downarrow{\scriptstyle \mathcal{U}_Y} \\ C(X) & \xrightarrow[C(f)]{} & C(Y) \end{array}$$

are commutative.

We begin with some simple results.

Theorem 5.1. *Let $f : X \to Y$ be a mapping between continua. Then the following conditions are equivalent*:

(1) f *is* 1-1;

(2) $C(f)$ *is* 1-1;

(3) $\mathfrak{C}(f)$ *is* 1-1.

Theorem 5.2. *Let $f : X \to Y$ be a mapping between continua. Then the following conditions are equivalent*:

(1) f *is weakly confluent*;

(2) $C(f)$ *is surjective*;

(3) $\mathfrak{C}(f)$ *is surjective.*

Example 5.3. In general, if f is a continuous function between continua it is not necessarily true that $\mathfrak{C}(f)$ is a continuos function. For example, let $f : [0,1] \to S^1$, given by $f(x) = (\cos 2\pi x, \sin 2\pi x)$. Notice that f is a mapping, but $\mathfrak{C}(f)$ is not a continuous function because $\{C([\frac{1}{n}, 1 - \frac{1}{n}])\}_{n \in \mathbf{N}}$ converges to $C([0,1])$ in $\mathfrak{C}([0,1])$ because $[0,1]$ is C^*-smooth and

$$\left\{\mathfrak{C}(f)\left(C\left(\left[\frac{1}{n}, 1 - \frac{1}{n}\right]\right)\right)\right\}_{n \in \mathbf{N}} = \left\{C\left(f\left(\left[\frac{1}{n}, 1 - \frac{1}{n}\right]\right)\right)\right\}_{n \in \mathbf{N}},$$

does not converge to $C(S^1)$ in $\mathfrak{C}(S^1)$ (see Example 3.3).

THEOREM 5.4. *Let $f : X \to Y$ be a mapping between continua. Then the following conditions are equivalent*:

(1) *f is a homeomorphism*;

(2) *$C(f)$ is a homeomorphism*;

(3) *$\mathfrak{C}(f)$ is a homeomorphism.*

PROOF. Conditions (1) and (2) are equivalent by [2, Theorem 3.11, p. 199]. To see (2) implies (3), by Theorems 5.1 and 5.2, $\mathfrak{C}(f)$ is a surjective function. To show the continuity of $\mathfrak{C}(f)$, let $\{C(A_n)\}_{n\in\mathbf{N}}$ be a sequence in $\mathfrak{C}(X)$ converging to $C(A) \in \mathfrak{C}(X)$. It suffices to show that $\limsup_{n\to\infty} C(f(A_n)) \subset C(f(A)) \subset \liminf_{n\to\infty} C(f(A_n))$.

Let $B \in \limsup_{n\to\infty} C(f(A_n))$, thus there is a sequence $\{n_k\}_{k\in\mathbf{N}}$ in $\mathbf{N}$ such that for each $k \in \mathbf{N}$ there exists $B_{n_k} \in C(f(A_{n_k}))$ and $\lim B_{n_k} = B$. Since $C(f)$ is a homeomorphism, $\lim C(f)^{-1}(B_{n_k}) = C(f)^{-1}(B)$, i.e., $\lim f^{-1}(B_{n_k}) = f^{-1}(B)$. Notice that $f^{-1}(B_{n_k}) \in C(A_{n_k})$ and $f^{-1}(B) \in C(X)$, therefore $f^{-1}(B) \in C(A)$ and then $B \in C(f(A))$.

On the other hand, if $E \in C(f(A))$, then $f^{-1}(E) \in C(A)$. For each $n \in \mathbf{N}$, let $E_n \in C(A_n)$ such that $\lim E_n = f^{-1}(E)$. By hypothesis, $\lim C(f)(E_n) = C(f)(f^{-1}(E)) = E$. Thus, $E \in \liminf_{n\to\infty} C(f(A_n))$. We conclude that $C(f)(A) \subset \liminf_{n\to\infty} C(f(A_n))$.

(3) implies (1) follows from Theorems 5.1 and 5.2. □

PROPOSITION 5.5. *If $f : X \to Y$ is a mapping between continua and Y is C^*-smooth, then $\mathfrak{C}(f)$ is continuous.*

PROOF. This proposition follows from the fact that

$$\mathfrak{C}(f) = C_Y^* \circ C(f) \circ \mathscr{U}_X. \qquad \square$$

The following result is immediate from Theorem 5.4.

COROLLARY 5.6. *C^*-smoothness is a topological property.*

DEFINITION 5.7. A mapping $f : X \to Y$ is said to be $\mathfrak{C}$-*mapping* if $\mathfrak{C}(f)$ is a mapping.

By the class of monotone mappings we have the following theorem.

THEOREM 5.8. *Let $f : X \to Y$ be a surjective $\mathfrak{C}$-mapping between continua. Consider the following conditions*:

(1) *f is monotone*;

(2) *$C(f)$ is monotone*;

(3) *$\mathfrak{C}(f)$ is monotone.*

Then (1) *and* (2) *are equivalent*, *and* (3) *implies* (2).

PROOF. It is known that (1) and (2) are equivalent (see [6, Theorem 3.2, p. 241]).

Now, suppose that $\mathfrak{C}(f)$ is monotone and let $B \in C(Y)$. Notice that $\mathfrak{C}(f)^{-1}(C(B))$ is a connected set in $\mathfrak{C}(X)$. Thus

$$C(f)^{-1}(B) = \mathscr{U}_X(\mathfrak{C}(f)^{-1}(C(B)))$$

is a connected set in X. Then (3) implies (2). □

Regarding Theorem 5.8, we have that (1) does not imply (3).

EXAMPLE 5.9. Let

$$X = S^1 \cup \{(x, 0) \in \mathbf{R}^2 : 1 \le x \le 2\},$$

$$Y = \{(x, 0) \in \mathbf{R}^2 : 1 \le x \le 2\}$$

and $f : X \to Y$ be the quotient mapping such that $f(S^1) = \{(1, 0)\}$ and the identity in Y. Notice that f is a monotone $\mathfrak{C}$-mapping by Proposition 5.5. Since $\mathfrak{C}(f)^{-1}(C(\{(0, 1)\})) = \mathfrak{C}(S^1)$, we have that $\mathfrak{C}(f)$ is not monotone.

THEOREM 5.10. *Let $f : X \to Y$ be a monotone $\mathfrak{C}$-mapping between continua, if X is C^*-smooth then $\mathfrak{C}(f)$ is monotone.*

PROOF. It follows from the fact that, for each $C(B) \in \mathfrak{C}(Y)$ we have that

$$\mathfrak{C}(f)^{-1}(C(B)) = C_X^*(C(f)^{-1}(B)). \qquad \square$$

With respect the class of confluent mappings we have the following results.

THEOREM 5.11. *Let $f : X \to Y$ be a $\mathfrak{C}$-mapping between continua. Consider the following conditions*:

(1) *f is confluent*;

(2) *$C(f)$ is confluent*;

(3) *$\mathfrak{C}(f)$ is confluent.*

Then each of (2) *and* (3) *implies* (1).

PROOF. It is known that (2) implies (1) (see [6, Theorem 6.3, p. 246]).

To see (3) implies (1), suppose that $\mathfrak{C}(f)$ is confluent and let $B \in C(Y)$ and K be a component of $f^{-1}(B)$.

Consider $\mathfrak{B} = \{\{\{x\}\} : x \in B\} \subset \mathfrak{C}(Y)$. It is clear that $\mathfrak{B}$ is a subcontinuum of $\mathfrak{C}(Y)$. Choose any point $p \in K$. Notice that $\{\{p\}\} \in \mathfrak{C}(f)^{-1}(\mathfrak{B})$. Let $\mathfrak{K}$ be the component of $\mathfrak{C}(f)^{-1}(\mathfrak{B})$ containing $\{\{p\}\}$. Observe that $\bigcup\{A \in C(X) : C(A) \in \mathfrak{K}\} = K$. In fact, since

$$\mathscr{U}_X(\mathfrak{K}) = \{A \in C(X) : C(A) \in \mathfrak{K}\},$$

and $\mathscr{U}_X$ is a mapping and $\{p\} \in \mathscr{U}_X(\mathfrak{K})$, by [6, Lemma 3.1, p. 241], $\bigcup\{A \in C(X) : C(A) \in \mathfrak{K}\}$ is a connected set contained in $f^{-1}(B)$ cointaining p, then

$$\bigcup\{A \in C(X) : C(A) \in \mathfrak{K}\} \subset K,$$

therefore $\bigcup\{A \in C(X) : C(A) \in \mathfrak{K}\} = K$ because $\{\{\{k\}\} : k \in K\} \subset \mathfrak{K}$. It is clear that $f(K) \subset B$. Let $x \in B$. Since $\mathfrak{C}(f)$ is confluent, there exists $C(D) \in \mathfrak{K}$ such that $\mathfrak{C}(f)(C(D)) = \{\{x\}\}$. Thus there exists $d \in D \subset \bigcup\{A \in C(X) : C(A) \in \mathfrak{K}\} = K$ such that $f(d) = x$. We conclude that $f(K) = B$. □

Regarding Theorem 5.11, (1) and (2) are equivalent when Y is locally connected (see [6, Theorem 6.3, p. 246]), and (1) does not imply (2) (see [6, p. 247]).

THEOREM 5.12. *Let $f : X \to Y$ be a $\mathfrak{C}$-mapping between continua, if X is C^*-smooth and $C(f)$ is confluent, then $\mathfrak{C}(f)$ is confluent.*

PROOF. Let $\mathfrak{B}$ be a subcontinuum of $\mathfrak{C}(Y)$ and $\mathfrak{D}$ be a component of $\mathfrak{C}(f)^{-1}(\mathfrak{B})$. Since X is C^*-smooth, we have that $\mathscr{U}_X(\mathfrak{D})$ is a component of $C(f)^{-1}(\mathscr{U}_Y(\mathfrak{B}))$. Thus, using that $C(f)$ is confluent, we obtain that $\mathfrak{C}(f)(\mathfrak{D}) = \mathfrak{B}$. □

EXAMPLE 5.13. Let $X = S^1$, $Y = [-1, 1]$ and $f : X \to Y$ the projection onto the first coordinate. By [6, Theorem 6.3, p. 246], $C(f)$ is confluent and by Proposition 5.5 $\mathfrak{C}(f)$ is a mapping. Is clear that $\mathfrak{C}(f)^{-1}(\mathfrak{C}(Y)) = \mathfrak{C}(X)$ and

$\{C(X)\}$ is a component of $\mathfrak{C}(f)^{-1}(\mathfrak{C}(Y))$. Notice that $\mathfrak{C}(f)(\{C(X)\}) = \{C(Y)\} \neq \mathfrak{C}(Y)$ and then $\mathfrak{C}(f)$ is not confluent.

THEOREM 5.14. *Let $f : X \to Y$ be a mapping between continua. If Y is C^*-smooth and $\mathfrak{C}(f)$ is confluent then $C(f)$ is confluent.*

PROOF. Let $\mathscr{A}$ be a subcontinuum of $C(Y)$ and $\mathscr{B}$ be a component of $C(f)^{-1}(A)$. Using that Y is C^*-smooth, we have that $C_Y^*(\mathscr{A})$ is a subcontinuum of $\mathfrak{C}(Y)$. Notice that $C_X^*(\mathscr{B}) \subset \mathfrak{C}(f)^{-1}(C_Y^*(\mathscr{A}))$, thus there exists $\mathfrak{D}$ a component of $\mathfrak{C}(f)^{-1}(C_Y^*(\mathscr{A}))$ such that $\mathfrak{D} \cap C_X^*(\mathscr{B}) \neq \varnothing$. Since $\mathscr{U}_X(\mathfrak{D})$ is a connected set contained in $C(f)^{-1}(\mathscr{A})$ and $\mathscr{B} \cap \mathscr{U}_X(\mathfrak{D}) \neq \varnothing$, we conclude that $\mathscr{U}_X(\mathfrak{D}) \subset \mathscr{B}$. The fact that $\mathfrak{C}(f)(\mathfrak{D}) = C_Y^*(\mathscr{A})$ implies that $C(f)(\mathscr{B}) = \mathscr{A}$. □

THEOREM 5.15. *Let $f : X \to Y$ be a $\mathfrak{C}$-mapping between continua. Consider the following conditions*:

(1) *f is weakly confluent*;

(2) *$C(f)$ is weakly confluent*;

(3) *$\mathfrak{C}(f)$ is weakly confluent.*

Then each of (2) *and* (3) *implies* (1).

PROOF. If $C(f)$ or $\mathfrak{C}(f)$ are weakly confluent, then $C(f)$ is surjective which is equivalent to the weak confluence of f. □

Regarding Theorem 5.15, (1) does not imply (2) (see [3, Example 6.8, p. 149]).

With similar proofs to those of Theorems 5.12 and 5.14, we have the following two results.

COROLLARY 5.16. *Let $f : X \to Y$ be a $\mathfrak{C}$-mapping between continua, if X is C^*-smooth and $C(f)$ is weakly confluent, then $\mathfrak{C}(f)$ is weakly confluent.*

COROLLARY 5.17. *Let $f : X \to Y$ be a mapping between continua. If Y is C^*-smooth and $\mathfrak{C}(f)$ is weakly confluent then $C(f)$ is weakly confluent.*

For the class of light mappings we have the following results.

THEOREM 5.18. *Let $f : X \to Y$ be a $\mathfrak{C}$-mapping. Consider the following conditions*:

(1) *f is light;*
(2) *$C(f)$ is lightv;*
(3) *$\mathfrak{C}(f)$ is light.*

Then (2) *implies* (3) *and* (3) *implies* (1). *Consequently* (2) *implies* (1).

PROOF. To see (2) implies (3), suppose on the contrary that there exists $C(B) \in \mathfrak{C}(Y)$ and $\mathfrak{D}$ is a nondegenerated component of $\mathfrak{C}(f)^{-1}(\mathfrak{D})$. Since $\mathscr{U}_X$ is a bijective mapping, $\mathscr{U}_X(\mathfrak{D})$ is a nondegenerate connected subset of $C(f)^{-1}(B)$, which is a contradiction.

On the other hand, to see (3) implies (1), let $\mathfrak{X} = \{\{\{x\}\} \in \mathfrak{C}(X) : x \in X\}$ and notice that $\mathfrak{X}$ is homeomorphic to X, so $\mathfrak{C}(f)|_{\mathfrak{X}}$ is light and

$$\mathfrak{C}(f)(\mathfrak{Y}) = \{\{\{y\}\} \in \mathfrak{C}(Y) : y \in Y\}.$$

Consider the homeomorphisms $g : X \to \mathfrak{X}$ and $h : \mathfrak{C}(f)(\mathfrak{Y}) \to Y$, given by $g(x) = \{\{x\}\}$ and $h(\{\{y\}\}) = y$ for each $x \in X$ and $y \in Y$, respectively. Since $f = h \circ \mathfrak{C}(f)|_{\mathfrak{X}} \circ g$, we conclude that f is light. □

Regarding Theorem 5.18, (1) does not imply (2) (see [1, Theorem 3.10, p. 184]).

THEOREM 5.19. *If $f : X \to Y$ is a $\mathfrak{C}$-mapping, X is C^*-smooth and $\mathfrak{C}(f)$ is light, then $C(f)$ is light.*

PROOF. On the contrary, suppose that there exists $B \in C(Y)$ and $\mathscr{D}$ is a nondegenerated componnent of $C(f)^{-1}(B)$. Notice that $C_X^*(\mathscr{D})$ is a nondegenerated connected set contained in $\mathfrak{C}(f)^{-1}(C(B))$ which is a contradiction. Then $C(f)$ is light. □

By [1, Theorem 3.7, p. 183] and Theorems 5.18 and 5.19, we have the following result.

COROLLARY 5.20. *Let $f : X \to Y$ be a $\mathfrak{C}$-mapping. If X is C^*-smooth, then the following conditions are equivalent:*

(1) *$C(f)$ is light;*
(2) *for every $A, B \in C(X)$ the condition $A \subsetneq B$ implies the condition $f(A) \subsetneq f(B)$;*
(3) *$\mathfrak{C}(f)$ is light;*
(4) *for every $C(A), C(B) \in C(X)$ the condition $C(A) \subsetneq C(B)$ implies the condition $\mathfrak{C}(f)(C(A)) \subsetneq \mathfrak{C}(f)(C(B))$.*

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R. Escobedo and V. Sánchez
Facultad de Ciencias Físico Matemáticas
Benemérita Universidad Autónoma de Puebla
Ave. San Claudio y Río Verde
Ciudad Universitaria
San Manuel Puebla, Pue, C.P. 72570, México
E-mail: escobedo@fcfm.buap.mx

E-mail: rompc_19_87@hotmail.com

J. Sánchez-Martínez
Facultad de Ciencias en Física y Matemáticas
Universidad Autónoma de Chiapas
Carretera San Fernando Km. 8.5
Ciudad Universitaria
Terán, Tuxtla Gutieérrez, Chiapas, C. P. 29050, México
E-mail: jsanchezm@unach.mx