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A NUMERICAL STUDY OF SIEGEL THETA SERIES OF VARIOUS DEGREES FOR THE 48-DIMENSIONAL EVEN UNIMODULAR EXTREMAL LATTICES

By

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Abstract. Salvati Manni showed that the difference of the Siegel theta series of degree 4 associated with the two even unimodular 48-dimensional extremal lattices is a constant multiple of the cube J^3 of the Schottky modular form J, which is a Siegel cusp form of degree 4 and weight 8. His result implies that the Siegel theta series of degree up to 3 is unique. But apparently his method does not supply us the process to compute the Fourier coefficients of these series.

In the present paper we show that the Fourier coefficients of the Siegel theta series associated with the even unimodular 48-dimensional extremal lattices of degrees 2 and 3 can be computed explicitly, and the Fourier coefficients of the Siegel theta series of degree 4 for those lattices are computed almost explicitly.

1. Introduction

In [11] we improved the method, which is initiated in [14], to compute the Fourier coefficients of Siegel theta series of various degrees for the class of 32-dimensional even unimodular extremal lattices. Siegel theta series of degree 4 associated with the 32-dimensional even unimodular extremal lattices depend on the individual lattices. In [10] we showed Siegel theta series of degree 4 associated with the five non-isometric 32-dimensional even unimodular extremal lattices constructed from the five doubly even self-dual binary extremal codes are different from each other.

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In the present paper we first show that the Fourier coefficients of Siegel theta series of degree up to 3 for the even unimodular 48 dimensional extremal lattices can be computed with the help of the so called Hecke-Schöneberg formulas. The method of computation in the present paper together with that of [11] gives a systematic way to compute the Fourier coefficients of Siegel theta series of lower degrees for the class of even unimodular extremal lattices of some tractable dimensions.

Next in degree 4 we show that Siegel theta series associated with the even unimodular 48 dimensional extremal lattice can be expressed as a linear combination of two specified Siegel modular forms. One is the cube J^3 of the Schottky modular form J, which is a Siegel cusp form of weight 8, and another is (we may call it) a pan theta series $P\Theta_{4,48}(Z)$ in 48 dimension. It turns out that $P\Theta_{4,48}(Z)$ does not depend on the particular extremal lattice. This feature of the present article may be regarded as a more materialization of a result of Salvati Manni [15] Theorem 3, (ii).

In combinatorial number theory or in the geometry of numbers extremal even unimodular lattices are a class of important objects (c.f. [2]). Combinatorial number theorists know the difficulties of constructing such lattices, and they realize that the problem that whether the constructed lattices are isometric or not is much more difficult one. In the dimension 48 case only 4 constructions of even unimodular extremal lattices are known and they are not isometric each other ([8]).

2. Basic Definitions

2.1. Some Definitions from Lattice Theory

Let $\mathbf{Z}$ be the ring of rational integers and $\mathbf{R}$ the field of real numbers. A finitely generated $\mathbf{Z}$-module L in $\mathbf{R}^g$ with a positive definite metric is called a positive definite quadratic lattice. Since we treat only the positive definite quadratic lattices, we shall omit the adjectives "positive definite quadratic". A lattice L is integral if L satisfies $(\mathbf{x}, \mathbf{y}) \in \mathbf{Z}$ for any $\mathbf{x}, \mathbf{y} \in L$ where $(\,,\,)$ is the bilinear form associated to the metric. Two integral lattices L_1 and L_2 are said to be isometric if and only if there exists a bijective linear mapping from L_1 to L_2 preserving the metric. The maximal number of linearly independent vectors over $\mathbf{R}$ in L is called the rank of L. The dual lattice $L^{\#}$ of L is defined by

$$L^{\#} = \{\mathbf{y} \in L \otimes_{\mathbf{Z}} \mathbf{Q} \mid (\mathbf{x}, \mathbf{y}) \in \mathbf{Z}, \forall \mathbf{x} \in L\}.$$

Here $\mathbf{Q}$ is the field of rational numbers. A lattice L is even if any element $\mathbf{x}$ of L has even norm $(\mathbf{x}, \mathbf{x})$. In an even lattice L, we say that $\mathbf{x}$ is a $2m$-vector if $(\mathbf{x}, \mathbf{x}) = 2m$ holds for some natural number m. Let $\Lambda_{2m}(L)$ be the set defined by

$$\Lambda_{2m}(L) = \{\mathbf{x} \in L \mid (\mathbf{x}, \mathbf{x}) = 2m\}.$$

A lattice L is called unimodular if $L = L^{\#}$. Even unimodular lattices exist only when $n \equiv 0 \pmod 8$. The minimal norm of a lattice is $\mathrm{Min}(L) = \min_{\mathbf{x} \in L \setminus \{0\}} (\mathbf{x}, \mathbf{x})$. When L is even unimodular of rank n it holds that (conf. [6])

$$\mathrm{Min}(L) \leq 2\left[\frac{n}{24}\right] + 2.$$

Such a lattice which attains the above maximum is said to be extremal.

In the present paper we are interested in any even unimodular extremal 48-dimensional lattice. We denote any one of such lattices as $\mathscr{L}_{48}$. We have $\mathrm{Min}(\mathscr{L}_{48}) = 6$. At present we know just four explicite non-isometric even unimodular extremal 48-dimensional lattices [8].

2.2. Theta Series of One Complex Variable Associated with the Even Unimodular Lattice

2.2.1. A Lattice Version

Let L be an even unimodular lattice of rank $8k$, then the (ordinary) theta series for L is defined by

$$\vartheta(z, L) = \sum_{\mathbf{x} \in L} \exp(\pi i (\mathbf{x}, \mathbf{x}) z),$$

where z is a complex variable with positive imaginary part. This series is rewritten as

$$\vartheta(z, L) = \sum_{m=0}^{\infty} a(2m, L) \exp(2\pi i m z),$$

where $a(2m, L) = |\Lambda_{2m}(L)|$.

When we consider the lattice $\mathscr{L}_{48}$, then the Fourier expansion of $\vartheta(z, \mathscr{L}_{48})$ is given by

$$\theta(\tau, \mathscr{L}_{48}) = 1 + 52416000q^6 + 39007332000q^8 + 6609020221440q^{10} + 437824977408000q^{12} + \cdots,$$

$$q = e^{\pi i z}.$$

2.2.2. A Quadratic Form Version

Let T be a positive definite symmetric square matrix with real entries of size g written by

$$T = \begin{pmatrix} t_{11} & t_{12}/2 & \cdots & t_{1g}/2 \\ t_{12}/2 & t_{22} & \cdots & t_{2g}/2 \\ \vdots & \vdots & \vdots & \vdots \\ t_{1g}/2 & t_{2g}/2 & \cdots & t_{gg} \end{pmatrix},$$

then associated with it a quadratic form $Q_T[\xi]$ is defined by

$$Q_T[\xi] = Q_T[\xi_1, \dots, \xi_g] = \sum_{1 \le i \le j \le g} t_{ij}\xi_i\xi_j,$$

where $\xi_1, \dots, \xi_g$ are real independent variables. The set of all positive definite symmetric matrices with real entries is denoted by $\mathscr{P}_g(\mathbf{R})$. Let $GL_g(\mathbf{Z})$ be the group of all unimodular square matrices of size g. An action of $U \in GL_g(\mathbf{Z})$ to an element of $T \in \mathscr{P}_g(\mathbf{R})$ is defined by $T \to UTU^t$, where U^t is the transposed matrix of U. Two elements $T_1, T_2 \in \mathscr{P}_g(\mathbf{R})$ are called integrally equivalent if there is a $U \in GL_g(\mathbf{Z})$ such that $T_2 = UT_1U^t$ holds. This is a well-known equivalence relation.

An element $T \in \mathscr{P}_g(\mathbf{R})$ is called semi-integral if the diagonal entries of $2T$ are all even integers and the off-diagonal entries are integers. The set of such elements is denoted by $\mathscr{P}_g^s(\mathbf{Z})$. Suppose T is in $\mathscr{P}_g^s(\mathbf{Z})$, then the theta series for $Q_T[\xi]$ is defined by

$$\vartheta(z, Q_T) = \sum_{\xi_1, \dots, \xi_g \in \mathbf{Z}} \exp\left(\pi i z\left(\sum_{1 \le i \le j \le g} t_{ij}\xi_i\xi_j\right)\right).$$

2.3. Hecke-Schöneberg Formulas Revisited

In [13] we gave the formulas for the inner product relations among the vectors in a 48-dimensional extremal even unimodular lattice $\mathscr{L}$. Here we reproduce them without proofs.

PROPOSITION 2.1. *Let* $\mathscr{L}_{48}$ *be an even unimodular* 48 *dimensional extremal lattice,* $\Lambda_6 = \Lambda_6(\mathscr{L}_{48})$ *and* $\boldsymbol{a} \in \mathscr{L}_{48} \otimes \mathbf{R}$, *then we have*

$$\sum_{\mathbf{x} \in \Lambda_6} (\mathbf{x}, \boldsymbol{a})^2 = 6552000(\boldsymbol{a}, \boldsymbol{a}) \tag{2.1}$$

$$\sum_{\mathbf{x} \in \Lambda_6} (\mathbf{x}, \boldsymbol{a})^4 = 2358720(\boldsymbol{a}, \boldsymbol{a})^2 \tag{2.2}$$

$$\sum_{\mathbf{x} \in \Lambda_6} (\mathbf{x}, \boldsymbol{a})^6 = 1360800(\boldsymbol{a}, \boldsymbol{a})^3 \tag{2.3}$$

$$\sum_{\mathbf{x} \in \Lambda_6} (\mathbf{x}, \boldsymbol{a})^8 = 1058400(\boldsymbol{a}, \boldsymbol{a})^4 \tag{2.4}$$

$$\sum_{\mathbf{x} \in \Lambda_6} (\mathbf{x}, \boldsymbol{a})^{10} = 1020600(\boldsymbol{a}, \boldsymbol{a})^5 \tag{2.5}$$

$$\sum_{\mathbf{x} \in \Lambda_6} (\mathbf{x}, \boldsymbol{a})^{14} - \frac{91 \cdot (\boldsymbol{a}, \boldsymbol{a})}{12} \sum_{\mathbf{x} \in \Lambda_6} (\mathbf{x}, \boldsymbol{a})^{12} = -7297290 \cdot (\boldsymbol{a}, \boldsymbol{a})^7 \tag{2.6}$$

Related to this proposition there is a simple fact, and we state as a lemma:

LEMMA 2.2. *Let the notations be the same with Proposition* 2.1. *Then it holds that*

$$\sum_{\mathbf{x} \in \Lambda_6} (\mathbf{x}, \boldsymbol{a})^0 = 52416000. \tag{2.0}$$

PROOF. The lefthand side of (2.0) is just the counting of vectors in Λ_6 which equals the righthand side. □

Similarly we can derive another formulas.

PROPOSITION 2.3. *Let* $\mathscr{L}_{48}$ *be an even unimodular* 48 *dimensional extremal lattice and* $\Lambda_8 = \Lambda_8(\mathscr{L}_{48})$, *then we have*

$$\sum_{\mathbf{x} \in \Lambda_8} (\mathbf{x}, \boldsymbol{a})^2 = 6501222000(\boldsymbol{a}, \boldsymbol{a}) \tag{2.7}$$

$$\sum_{\mathbf{x} \in \Lambda_8} (\mathbf{x}, \boldsymbol{a})^4 = 3120586560(\boldsymbol{a}, \boldsymbol{a})^2 \tag{2.8}$$

$$\sum_{\mathbf{x}\in\Lambda_8}(\mathbf{x},\boldsymbol{a})^6 = 2400451200(\boldsymbol{a},\boldsymbol{a})^3 \tag{2.9}$$

$$\sum_{\mathbf{x}\in\Lambda_8}(\mathbf{x},\boldsymbol{a})^8 = 2489356800(\boldsymbol{a},\boldsymbol{a})^4 \tag{2.10}$$

$$\sum_{\mathbf{x}\in\Lambda_8}(\mathbf{x},\boldsymbol{a})^{10} = 3200601600(\boldsymbol{a},\boldsymbol{a})^5 \tag{2.11}$$

$$\sum_{\mathbf{x}\in\Lambda_8}(\mathbf{x},\boldsymbol{a})^{14} - \frac{91\cdot(\boldsymbol{a},\boldsymbol{a})}{9}\sum_{\mathbf{x}\in\Lambda_8}(\mathbf{x},\boldsymbol{a})^{12} = -40683202560\cdot(\boldsymbol{a},\boldsymbol{a})^7 \tag{2.12}$$

2.4. Siegel Theta Series

A Siegel theta series of degree g ($g \geq 2$) attached to the even unimodular lattice L is defined by

$$\Theta_g(Z,L) = \sum_{\mathbf{x}_1,\dots,\mathbf{x}_g\in L} \exp(\pi i\sigma([\mathbf{x}_1,\dots,\mathbf{x}_g]Z)),$$

where Z is a variable on the Siegel upper-half space of degree g, $[\mathbf{x}_1,\dots,\mathbf{x}_g]$ is a g by g square matrix whose (i,j) entry is $(\mathbf{x}_i,\mathbf{x}_j)$ and σ is the trace of the matrix.

The Siegel theta series of degree g can be expanded to

$$\Theta_g(Z,L) = \sum_{T\in\hat{\mathscr{P}}_g^s(\mathbf{Z})} a(T,L)e^{2\pi i\sigma(TZ)}.$$

Here $\hat{\mathscr{P}}_g^s(\mathbf{Z})$ is the set of positive semi-definite semi-integral symmetric square matrices of degree g, and $a(T,L) = |\{\langle\mathbf{x}_1,\dots,\mathbf{x}_g\rangle\in L^g \mid [\mathbf{x}_1,\dots,\mathbf{x}_g] = 2T\}|$, and $|X|$ is the cardinality of a set X.

We shall say that $2T$ is represented by the lattice L if $a(T,L)\neq 0$. We quote one importatn property of $a(T,L)$ above as a proposition:

Proposition 2.4. *Let $a(T,L)$ be a Fourier coefficient of the Siegel theta series of degree g. Then we have*

$$a(UTU^t,L) = a(T,L),$$

where U is a unimodular matrix of size g.

For the proof of this we refer [21], Formula (48).

An easy but important property of Siegel theta series is the following lemma:

Lemma 2.5. *Let L be an even unimodular lattice, and* $\Theta_g(Z,L) = \sum_{T \in \hat{\mathscr{P}}_g^s(\mathbf{Z})} a(T,L) e^{2\pi i \sigma(TZ)}$ *be the Siegel theta series of degree g associated with the lattice L. If a $2T, T \in \hat{\mathscr{P}}_g^s$ is represented by the lattice L, then $2UTU^t$ is also represented by L.*

Proof. Suppose $2UTU^t$ is not represented by L. This means that $a(UTU^t, L) = 0$. But by our assumption we know $a(T,L) \neq 0$. This contradicts to Proposition 2.4. □

Let T be a symmetric square matrix of size g written by

$$T = \begin{pmatrix} t_{11} & t_{12}/2 & \cdots & t_{1g}/2 \\ t_{12}/2 & t_{22} & \cdots & t_{2g}/2 \\ \vdots & \vdots & \vdots & \vdots \\ t_{1g}/2 & t_{2g}/2 & \cdots & t_{gg} \end{pmatrix}.$$

We will use the convention $T = (t_{11}, t_{22}, \dots, t_{gg}, t_{12}, t_{13}, t_{23}, \dots, t_{1g}, t_{2g}, \dots, t_{g-1,g})$.

Fact: A Siegel theta series of degree g associated with an even unimodular lattice L of rank $2k$ ($2k$ is a multiple of 8) is a modular form of degree g and weight k.

3. Fourier Coefficients of Siegel Theta Series of Degree 2 for $\mathscr{L}_{48}$

We compute some Fourier coefficients of Siegel theta series of degree 2 for $\mathscr{L}_{48}$. For this we start from the equation

$$a(T, \mathscr{L}_{48}) = \sum_{\mathbf{x}_1, \mathbf{x}_2 \in \mathscr{L}_{48}, [\mathbf{x}_1, \mathbf{x}_2] = 2T} 1, \quad T \in \hat{\mathscr{P}}_2^s(\mathbf{Z}).$$

We prove a neccesary restriction on the inner products among the vectors in $\mathscr{L}_{48}$.

Lemma 3.1. (i) *For any two vectors $\mathbf{x}, \mathbf{y} \in \Lambda_6(\mathscr{L}_{48})$ we have*

$$(\mathbf{x}, \mathbf{y}) \in \{\pm 6, \pm 3, \pm 2, \pm 1, 0\}.$$

(ii) *For any vector $\mathbf{x} \in \Lambda_6(\mathscr{L}_{48})$ and for any vector $\mathbf{y} \in \Lambda_8(\mathscr{L}_{48})$ we have*

$$(\mathbf{x}, \mathbf{y}) \in \{\pm 4, \pm 3, \pm 2, \pm 1, 0\}.$$

(iii) *For any two vectors $\mathbf{x}, \mathbf{y} \in \Lambda_8(\mathscr{L}_{48})$ we have*

$$(\mathbf{x}, \mathbf{y}) \in \{\pm 8, \pm 5, \pm 4, \pm 3, \pm 2, \pm 1, 0\}.$$

Proof. Proof of (i). By the Schwartzian inequality:

$$(\mathbf{x},\mathbf{y})^2 \le (\mathbf{x},\mathbf{x})\cdot(\mathbf{y},\mathbf{y}) = 36,$$

we have

$$|(\mathbf{x},\mathbf{y})| \le 6.$$

Suppose $(\mathbf{x},\mathbf{y}) = \pm 5$, then we have $(\mathbf{x}\mp\mathbf{y},\mathbf{x}\mp\mathbf{y}) = 2$. This is impossible because $\mathscr{L}_{48}$ does not contain any vector of norm 2. Suppose $(\mathbf{x},\mathbf{y}) = \pm 4$, then we have $(\mathbf{x}\mp\mathbf{y},\mathbf{x}\mp\mathbf{y}) = 4$. This is impossible because $\mathscr{L}_{48}$ does not contain any vector of norm 4.

We omit the proofs of (ii) and (iii). □

For $\mathbf{y}\in\Lambda_6$ we put $\lambda_k(\mathbf{y}) = \#\{\mathbf{x}\in\Lambda_6 \mid (\mathbf{x},\mathbf{y}) = k\}$. Here k takes the values ± 6, ± 3, ± 2, ± 1, 0 by Lemma 3.1,(i). We easily see that $\lambda_{-6} = \lambda_6 = 1$, $\lambda_{-k} = \lambda_k$, $k = 1,2,3$. By Equation (2.1) we have

$$2\cdot 6^2\cdot\lambda_6(\mathbf{y}) + 2\cdot 3^2\cdot\lambda_3(\mathbf{y}) + 2\cdot 2^2\cdot\lambda_2(\mathbf{y}) + 2\cdot\lambda_1(\mathbf{y}) = 6552000\cdot 6.$$

Likewise from Equations (2.2) and (2.3) we have

$$2\cdot 6^4\cdot\lambda_6(\mathbf{y}) + 2\cdot 3^4\cdot\lambda_3(\mathbf{y}) + 2\cdot 2^4\cdot\lambda_2(\mathbf{y}) + 2\cdot\lambda_1(\mathbf{y}) = 2358720\cdot 6^2,$$

and

$$2\cdot 6^6\cdot\lambda_6(\mathbf{y}) + 2\cdot 3^6\cdot\lambda_3(\mathbf{y}) + 2\cdot 2^6\cdot\lambda_2(\mathbf{y}) + 2\cdot\lambda_1(\mathbf{y}) = 1360800\cdot 6^3.$$

Solving these equations we have $\lambda_3(\mathbf{y}) = 36848$, $\lambda_2(\mathbf{y}) = 1678887$, $\lambda_1(\mathbf{y}) = 12608784$. From the obvious equation:

$$\lambda_0(\mathbf{y}) + 2\lambda_1(\mathbf{y}) + 2\lambda_2(\mathbf{y}) + 2\lambda_3(\mathbf{y}) + 2\lambda_6(\mathbf{y}) = |\Lambda_6(\mathscr{L}_{48})| = 52416000,$$

we have $\lambda_0(\mathbf{y}) = 23766960$. We put $T_0 = (3,3,0)$, $T_1 = (3,3,1)$, $T_2 = (3,3,2)$, $T_3 = (3,3,3)$. Then can conclude that

$$a(T_0,\mathscr{L}_{48}) = 52416000\cdot 23766960,\quad a(T_1,\mathscr{L}_{48}) = 52416000\cdot 12608784,$$

$$a(T_2,\mathscr{L}_{48}) = 52416000\cdot 1678887,\quad a(T_3,\mathscr{L}_{48}) = 52416000\cdot 36848.$$

Next for $\mathbf{y}\in\Lambda_8$ we put $\mu_k(\mathbf{y}) = \#\{\mathbf{x}\in\Lambda_6 \mid (\mathbf{x},\mathbf{y}) = k\}$. Here k takes the values ± 4, ± 3, ± 2, ± 1, 0 by Lemma 3.1,(ii). We see $\mu_{-k} = \mu_k$, $k = 1,2,3,4$. Instead of determining μ_k's we directly compute $a(T,\mathscr{L}_{48})$'s. We begin from Equation (2.1) with $\boldsymbol{a} = \mathbf{y}\in\Lambda_8$:

$$2\cdot 4^2\mu_4(\mathbf{y}) + 2\cdot 3^2\mu_3(\mathbf{y}) + 2\cdot 2^2\mu_2(\mathbf{y}) + 2\cdot\mu_1(\mathbf{y}) = 655200\cdot 8.$$

From this we have

$$\sum_{\mathbf{y}\in\Lambda_8}(2\cdot 4^2\mu_4(\mathbf{y})+2\cdot 3^2\mu_3(\mathbf{y})+2\cdot 2^2\mu_2(\mathbf{y})+2\cdot\mu_1(\mathbf{y}))=655200\cdot 8\sum_{\mathbf{y}\in\Lambda_8}1.$$

After expansion the first term of the lefthand side of this equation is

$$\begin{aligned}2\cdot 4^2\sum_{\mathbf{y}\in\Lambda_8}\mu_4(\mathbf{y})&=2\cdot 4^2\sum_{\mathbf{x}\in\Lambda_6,\,\mathbf{y}\in\Lambda_8,\,[\mathbf{x},\mathbf{y}]=2(3,4,4)}1\\&=2\cdot 4^2a((3,4,4),\mathscr{L}_{48}).\end{aligned}$$

Similarly we have

$$\begin{aligned}2\cdot 3^2\sum_{\mathbf{y}\in\Lambda_8}\mu_3(\mathbf{y})&=2\cdot 3^2a((3,4,3),\mathscr{L}_{48}),\\2\cdot 2^2\sum_{\mathbf{y}\in\Lambda_8}\mu_2(\mathbf{y})&=2\cdot 2^2a((3,4,2),\mathscr{L}_{48}),\\2\cdot\sum_{\mathbf{y}\in\Lambda_8}\mu_1(\mathbf{y})&=2a((3,4,1),L).\end{aligned}$$

After all we have

$$\begin{aligned}&2\cdot 4^2a((3,4,4),\mathscr{L}_{48})+2\cdot 3^2a((3,4,3),\mathscr{L}_{48})\\&\quad+2\cdot 2^2a((3,4,2),\mathscr{L}_{48})+2a((3,4,1),\mathscr{L}_{48})\\&=655200\cdot 8\cdot 39007332000.\end{aligned}$$

Using Equation (2.2) with $\boldsymbol{\alpha}=\mathbf{y}\in\Lambda_8$ we have

$$\begin{aligned}&2\cdot 4^4a((3,4,4),\mathscr{L}_{48})+2\cdot 3^4a((3,4,3),\mathscr{L}_{48})\\&\quad+2\cdot 2^4a((3,4,2),\mathscr{L}_{48})+2a((3,4,1),\mathscr{L}_{48})\\&=2358720\cdot 8^2\cdot 39007332000,\end{aligned}$$

and from (2.3)

$$\begin{aligned}&2\cdot 4^6a((3,4,4),\mathscr{L}_{48})+2\cdot 3^6a((3,4,3),\mathscr{L}_{48})\\&\quad+2\cdot 2^6a((3,4,2),\mathscr{L}_{48})+2a((3,4,1),\mathscr{L}_{48})\\&=1360800\cdot 8^3\cdot 39007332000,\end{aligned}$$

and finally from (2.4)

$$2\cdot 4^8 a((3,4,4),\mathscr{L}_{48}) + 2\cdot 3^8 a((3,4,3),\mathscr{L}_{48}) + 2\cdot 2^8 a((3,4,2),\mathscr{L}_{48}) + 2a((3,4,1),\mathscr{L}_{48}) = 1058400\cdot 8^4\cdot 39007332000.$$

These four linear equations are enough to solve $a((3,4,4),\mathscr{L}_{48}),\dots,a((3,4,1),\mathscr{L}_{48})$. Indeed we have

$$a((3,4,4),\mathscr{L}_{48}) = 88000540992000, \qquad a((3,4,3),\mathscr{L}_{48}) = 7509379497984000,$$

$$a((3,4,2),\mathscr{L}_{48}) = 113344696797696000, \quad a((3,4,1),\mathscr{L}_{48}) = 499932945727488000.$$

As to $a((3,4,0),L)$ we may utilize a special case of the identity proved in [12]:

$$\sum_b a((3,4,b),\mathscr{L}_{48}) = a(3,\mathscr{L}_{48})a(4,\mathscr{L}_{48}) = |\Lambda_6|\cdot|\Lambda_8| = 52416000\cdot 39007332000,$$

where b runs over all integers so that the matrix $T=(3,4,b)$ is positive semi-definite. In this case $b=\pm 4,\pm 3,\pm 2,\pm 1,0$. It follows from this identity that

$$a((3,4,0),\mathscr{L}_{48}) = 802858268983680000.$$

Thirdly for $\mathbf{y}\in\Lambda_8$ we put $v_k(\mathbf{y}) = \#\{\mathbf{x}\in\Lambda_8 \mid (\mathbf{x},\mathbf{y})=k\}$. Here k takes the values ± 8, ± 5, ± 4, ± 3, ± 2, ± 1, 0 by Lemma 3.1,(iii). We see $v_{-k}=v_k$, $k=1,2,3,4,5,8$.

Equation (2.7) leads to

$$2\cdot 8^2 v_8(\mathbf{y}) + 2\cdot 5^2 v_5(\mathbf{y}) + 2\cdot 4^2 v_4(\mathbf{y}) + 2\cdot 3^2 v_3(\mathbf{y}) + 2\cdot 2^2 v_2(\mathbf{y}) + 2\cdot v_1(\mathbf{y}) = 6501222000\cdot 8.$$

By summing the both sides with respect to $\mathbf{y}\in\Lambda_8$ we have

$$\sum_{\mathbf{y}\in\Lambda_8}[2\cdot 8^2 v_8(\mathbf{y}) + 2\cdot 5^2 v_5(\mathbf{y}) + 2\cdot 4^2 v_4(\mathbf{y}) + 2\cdot 3^2 v_3(\mathbf{y}) + 2\cdot 2^2 v_2(\mathbf{y}) + 2\cdot v_1(\mathbf{y})]$$

$$= 2\cdot 8^2\sum_{\mathbf{y}\in\Lambda_8} v_8(\mathbf{y}) + 2\cdot 5^2\sum_{\mathbf{y}\in\Lambda_8} v_5(\mathbf{y}) + 2\cdot 4^2\sum_{\mathbf{y}\in\Lambda_8} v_4(\mathbf{y}) + 2\cdot 3^2\sum_{\mathbf{y}\in\Lambda_8} v_3(\mathbf{y}) + 2\cdot 2^2\sum_{\mathbf{y}\in\Lambda_8} v_2(\mathbf{y}) + 2\cdot\sum_{\mathbf{y}\in\Lambda_8} v_1(\mathbf{y})$$

$$= 6501222000\cdot 8\sum_{\mathbf{y}\in\Lambda_8} 1.$$

Thus we have

$$\begin{aligned}&2\cdot 8^2 39007332000+2\cdot 5^2 a((4,4,5),\mathscr{L}_{48})+2\cdot 4^2 a((4,4,4),\mathscr{L}_{48})\\&\quad+2\cdot 3^2 a((4,4,3),\mathscr{L}_{48})+2\cdot 2^2 a((4,4,2),\mathscr{L}_{48})+2a((4,4,1),\mathscr{L}_{48})\\&\quad=6501222000\cdot 8\cdot 39007332000.\end{aligned}$$

Likewise from Equations $(2.8),\dots,(2.11)$ we have

$$\begin{aligned}&2\cdot 8^4 39007332000+2\cdot 5^4 a((4,4,5),\mathscr{L}_{48})+2\cdot 4^4 a((4,4,4),\mathscr{L}_{48})\\&\quad+2\cdot 3^4 a((4,4,3),\mathscr{L}_{48})+2\cdot 2^4 a((4,4,2),\mathscr{L}_{48})+2a((4,4,1),\mathscr{L}_{48})\\&\quad=3120586560\cdot 8^2\cdot 39007332000,\\&2\cdot 8^6 39007332000+2\cdot 5^6 a((4,4,5),\mathscr{L}_{48})+2\cdot 4^6 a((4,4,4),\mathscr{L}_{48})\\&\quad+2\cdot 3^6 a((4,4,3),\mathscr{L}_{48})+2\cdot 2^6 a((4,4,2),\mathscr{L}_{48})+2a((4,4,1),\mathscr{L}_{48})\\&\quad=2400451200\cdot 8^3\cdot 39007332000,\\&2\cdot 8^8 39007332000+2\cdot 5^8 a((4,4,5),\mathscr{L}_{48})+2\cdot 4^8 a((4,4,4),\mathscr{L}_{48})\\&\quad+2\cdot 3^8 a((4,4,3),\mathscr{L}_{48})+2\cdot 2^8 a((4,4,2),\mathscr{L}_{48})+2a((4,4,1),\mathscr{L}_{48})\\&\quad=2489356800\cdot 8^4\cdot 39007332000,\\&2\cdot 8^{10} 39007332000+2\cdot 5^{10} a((4,4,5),\mathscr{L}_{48})+2\cdot 4^{10} a((4,4,4),\mathscr{L}_{48})\\&\quad+2\cdot 3^{10} a((4,4,3),\mathscr{L}_{48})+2\cdot 2^{10} a((4,4,2),\mathscr{L}_{48})+2a((4,4,1),\mathscr{L}_{48})\\&\quad=3200601600\cdot 8^5\cdot 39007332000.\end{aligned}$$

The above equations are enough to solve $a((4,4,5),\mathscr{L}_{48}), a((4,4,4),\mathscr{L}_{48}),\dots,$ $a((4,4,1),\mathscr{L}_{48})$ numerically and we have

$$\begin{aligned}a((4,4,5),\mathscr{L}_{48})&=7509379497984000,\\a((4,4,4),\mathscr{L}_{48})&=799235580046176000,\\a((4,4,3),\mathscr{L}_{48})&=17098857116909568000,\\a((4,4,2),\mathscr{L}_{48})&=121116904578109440000,\\a((4,4,1),\mathscr{L}_{48})&=363048461209534464000.\end{aligned}$$

As to the value of $a((4,4,0),\mathscr{L}_{48})$ we use a special case of the identity proved in [12]:

$$\sum_b a((4,4,b),\mathscr{L}_{48}) = a(4,\mathscr{L}_{48})a(4,\mathscr{L}_{48}) = |\Lambda_8|^2 = 39007332000^2,$$

where b runs over all integers so that the matrix $T = (4,4,b)$ is positive semi-definite. In this case $b = \pm 8, \pm 5, \pm 4, \pm 3, \pm 2, \pm 1, 0$. Thus we have

$$a((4,4,0),\mathscr{L}_{48}) = 517430013952014072000.$$

As a summary of the computation in this section we give the following table.

Table 1. Fourier coefficients of Siegel-theta series of degree 2 for the lattice $\mathscr{L}_{48}$

d_T	T	$a(T,\mathscr{L}_{48})$
*27	$(3,3,3)$	1931424768000
32	$(3,3,2)$	88000540992000
35	$(3,3,1)$	660902022144000
*36	$(3,3,0)$	1245768975360000
39	$(3,4,3)$	7509379497984000
44	$(3,4,2)$	113344696797696000
47	$(3,4,1)$	499932945727488000
48	$(3,4,0)$	802858268983680000
*48	$(4,4,4)$	799235580046176000
57	$(4,4,3)$	17098857116909568000
60	$(4,4,2)$	121116904578109440000
63	$(4,4,1)$	363048461209534464000
*64	$(4,4,0)$	517430013952014072000

REMARK 1. In the above table and succeeding Table 3 and Table 5, the star $*$ denotes the imprimitive forms.

4. Fourier Coefficients of Siegel Theta Series of Degree 3 for $\mathscr{L}_{48}$

We are going to compute the Fourier coefficients of Siegel theta series of degree 3. To do this we first search good candidates of indeces T of degree 3 by extending an appropriate index T_1 of degree 2, and next we compute the Fourier coefficients for a group of the extended indeces T by using the Hecke-Schöneberg formulas.

4.1. General Notations

Let $\mathfrak{T}$ be a positive semi-definite semi-integral symmetric matrix of size s. Let $a_1, \dots, a_s$ be s integers, and we denote by $(\mathfrak{T}, \{a_1/2, \dots, a_s/2\}, 3)$ the matrix of size $s+1$ defined by

$$\begin{pmatrix} & & a_1/2 \\ & \mathfrak{T} & \vdots \\ & & a_s/2 \\ a_1/2 \cdots a_s/2 & & 3 \end{pmatrix}.$$

For an s-tuple $\mathbf{x}_1, \dots, \mathbf{x}_s \in \Lambda_6$ satisfying $[\mathbf{x}_1, \dots, \mathbf{x}_s] = 2\mathfrak{T}$ we will use a subset of Λ_6 defined by

$$\Lambda_{a_1,\dots,a_s}(2\mathfrak{T}; \mathbf{x}_1, \dots, \mathbf{x}_s) = \{\mathbf{z} \in \Lambda_6 \mid (\mathbf{x}_i, \mathbf{z}) = a_i, i = 1, \dots, s\},$$

and

$$\lambda_{a_1,\dots,a_s}(2\mathfrak{T}; \mathbf{x}_1, \dots, \mathbf{x}_s) = |\Lambda_{a_1,\dots,a_s}(2\mathfrak{T}; \mathbf{x}_1, \dots, \mathbf{x}_s)|.$$

The Fourier coefficient at $(\mathfrak{T}, \{a_1/2, \dots, a_s/2\}, 3)$ is described by

$$a((\mathfrak{T}, \{a_1/2, \dots, a_s/2\}, 3), \mathscr{L}) = \sum_{\substack{\mathbf{x}_1,\dots,\mathbf{x}_s \in \Lambda_6 \\ [\mathbf{x}_1,\dots,\mathbf{x}_s] = 2\mathfrak{T}}} \lambda_{a_1,\dots,a_s}(2\mathfrak{T}; \mathbf{x}_1, \dots, \mathbf{x}_s). \tag{4.1}$$

4.2. Extension of Matrices of Degree 2 to Matrices of Degree 3

The following result is very usefull for judging whether the two ternary positive definite quadratic forms are equivalent or not.

Proposition 4.1 (Schiemann [18]). *Two ternary positive definite forms with real coefficients are integrally equivalent if and only if their theta series in one variable are equal.*

The next lemma serves to eliminate irrelevant cases for our present search.

Lemma 4.2. *Let $\mathscr{L}_{48}$ be any one of even unimodular 48 dimensional extremal lattices, and T be a positive definite semi-integral symmetric matrix of degree $g \geq 3$. If UTU^t is reduced in the sense of Minkowski reduction, and some of the diagonal entries of it are 1 or 2. Then the Fourier coefficient $a(T, \mathscr{L}_{48})$ for Siegel theta series of degree g associated with the lattice $\mathscr{L}_{48}$ is zero.*

PROOF. If a diagonal entry of the reduced matrix UTU^t is 1 or two and $2UTU^t$ is represented by the lattice $\mathscr{L}_{48}$. Then there is a vector $\mathbf{x} \in \mathscr{L}_{48}$ which satisfies $(\mathbf{x}, \mathbf{x}) = 2$ or 4. This contradicts to the extremality of the lattice $\mathscr{L}_{48}$. Thus we have $a(UTU^t, \mathscr{L}_{48}) = 0$ and by Proposition 2.4 $a(T, \mathscr{L}_{48}) = 0$. □

NOTATION. Since we will use the Minkowski reduced form of the T, where T is a positive definite semi-integral symmetric matrix of degree $g \geq 3$, we use $MR(T)$ to denote the Minkowski reduced form of T. For the Minkowski reduction one may refer [7], [22], [23]. For the degrees $g = 3, 4$ of T the tables of Minkowski reduced form of T are available at [1], [9].

4.2.1. First Case

Let

$$T_{2,1} = \begin{pmatrix} 3 & 3/2 \\ 3/2 & 3 \end{pmatrix}$$

be the positive definite symmetric semi-integral matrix of size 2.

First we explore all possible pairs of integers a, b under the conditions:

(i) $(T_{2,1}, \{a/2, b/2\}, 3)$ is positive semi-definite,

(ii) under the unimodular transformation

$$(T_{2,1}, \{a/2, b/2\}, 3) \mapsto U^t(T_{2,1}, \{a/2, b/2\}, 3)U,$$

the minimal value of the non-zero diagonal entries of the resulting matrix is reduced to 3.

REMARK 2. In view of Lemma 4.2 it is legitimate to impose the condition (ii) in computing the Fourier coefficients $a((T_{2,1}, \{a/2, b/2\}, 3), \mathscr{L}_{48})$.

The pair of integers $\langle a, b \rangle$ satisfying the conditions (i), (ii) are grouped into the sets according to the determinant of $2(T_{2,1}, \{a/2, b/2\}, 3)$ and the equivalence by the unimodular transformations. In the ternary quadratic forms $\det(2(T_{2,1}, \{a/2, b/2\}, 3))/2$ is called the discriminant of the matrix $T = (T_{2,1}, \{a/2, b/2\}, 3)$ (c.f. Section 3.1 in [11]). We denote it by d. We write $C_d(T_{2,1}, \langle a, b \rangle)$ to denote the set of ordered pairs $\langle a', b' \rangle$ such that $\det(2(T_{2,1}, \{a/2, b/2\}, 3))/2 = \det(2(T_{2,1}, \{a'/2, b'/2\}, 3))/2 = d$ and $(T_{2,1}, \{a/2, b/2\}, 3)$ is equivalent to $(T_{2,1}, \{a'/2, b'/2\}, 3)$. Most cases of $C_d(T_{2,1}, \langle a, b \rangle)$ are divided into the two subsets $C_d^+(T_{2,1}, \langle a, b \rangle)$ and the subset obtained from $C_d^+(T_{2,1}, \langle a, b \rangle)$ by multiplying each member of each pair by -1. The convention to use $C_d^+(\cdots)$ will be used at later occasions. The total sets thus defined are called admissible sets with respect to $T_{2,1}$. $C_{81}(T_{2,1}, \langle 0,0\rangle) = \{\langle 0,0\rangle\}$ and

$$C_{78}^{+}(T_{2,1}, \langle 1,1\rangle) = \{\langle 1,1\rangle, \langle 1,0\rangle, \langle 0,1\rangle\},$$

$$C_{72}^{+}(T_{2,1}, \langle 2,1\rangle) = \{\langle 2,1\rangle, \langle 1,2\rangle, \langle 1,-1\rangle\},$$

$$C_{69}^{+}(T_{2,1}, \langle 2,2\rangle) = \{\langle 2,2\rangle, \langle 2,0\rangle, \langle 0,2\rangle\},$$

$$C_{60}^{+}(T_{2,1}, \langle 3,2\rangle) = \{\langle 3,2\rangle, \langle 3,1\rangle, \langle 2,3\rangle, \langle 2,-1\rangle, \langle 1,3\rangle, \langle 1,-2\rangle\},$$

$$C_{54}^{+}(T_{2,1}, \langle 3,3\rangle) = \{\langle 3,3\rangle, \langle 3,0\rangle, \langle 0,3\rangle\},$$

$$C_{0}^{+}(T_{2,1}, \langle 6,3\rangle) = \{\langle 6,3\rangle, \langle 3,6\rangle, \langle 3,-3\rangle\}.$$

Remark 3. We should keep in mind that each set in the above determines a unique reduced ternary quadratic form. For instance $(T_{2,1}, \{0,0\}, 3)$ is integrally equivalent to the reduced ternary quadratic form $(3,3,3,3,0,0)$ in Table 2 of Section 4.2.5 below. This rule applies even for degree 4 extensions. See Table 2 and Table 4 below.

Proposition 4.3. *For a fixed pair* $\mathbf{x}_1, \mathbf{x}_2 \in \Lambda_6$ *which satisfy* $[\mathbf{x}_1, \mathbf{x}_2] = 2T_{2,1}$ *it can be verified that for any* $\mathbf{y} \in \Lambda_6$ *the pairs of values* $\langle a,b\rangle$ *such that* $(\mathbf{x}_1, \mathbf{y}) = a$ *and* $(\mathbf{x}_2, \mathbf{y}) = b$ *fall in one of* $C_{81}(T_{2,1}, \langle 0,0\rangle), C_{78}(T_{2,1}, \langle 1,1\rangle), \dots,$ $C_0(T_{2,1}, \langle 6, 3\rangle)$.

Proof. This statement is verified by a numerical search with the help of Lemmas 3.1 and 4.2. We omit the details. □

For a later purpose we use the cardinalities of the sets of pairs above as $c_{81}(T_{2,1}) = |C_{81}(T_{2,1}, \langle 0,0\rangle)| = 1, \dots, c_0(T_{2,1}) = |C_0(T_{2,1}, \langle 6,3\rangle)| = 6$.

4.2.2. Second Case

Let

$$T_{2,2} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

be another positive definite symmetric semi-integral matrix of size 2. This time we seek the pairs of integers $\langle a,b\rangle$ which satisfy the conditions (i) and (ii) but $T_{2,1}$ is replaced by $T_{2,2}$.

The following are all the possible sets. $C_{96}(T_{2,2},\langle 0,0\rangle)=\{\langle 0,0\rangle\}$ and

$$C_{93}^{+}(T_{2,2},\langle 0,1\rangle)=\{\langle 0,1\rangle,\langle 1,0\rangle\},$$
$$C_{92}^{+}(T_{2,2},\langle 1,1\rangle)=\{\langle 1,1\rangle\},$$
$$C_{88}^{+}(T_{2,2},\langle 1,-1\rangle)=\{\langle 1,-1\rangle\},$$
$$C_{85}^{+}(T_{2,2},\langle 2,1\rangle)=\{\langle 2,1\rangle,\langle 1,2\rangle\},$$
$$C_{84}^{+}(T_{2,2},\langle 2,0\rangle)=\{\langle 2,0\rangle,\langle 0,2\rangle\},$$
$$C_{80}^{+}(T_{2,2},\langle 2,2\rangle)=\{\langle 2,2\rangle\},$$
$$C_{77}^{+}(T_{2,2},\langle 2,-1\rangle)=\{\langle 2,-1\rangle,\langle 1,-2\rangle\},$$
$$C_{72}^{+}(T_{2,2},\langle 3,1\rangle)=\{\langle 3,1\rangle,\langle 1,3\rangle\},$$
$$C_{69}^{+}(T_{2,2},\langle 3,2\rangle)=\{\langle 3,2\rangle,\langle 3,0\rangle,\langle 2,3\rangle,\langle 0,3\rangle\},$$
$$C_{64}^{+}(T_{2,2},\langle 2,-2\rangle)=\{\langle 2,-2\rangle\},$$
$$C_{60}^{+}(T_{2,2},\langle 3,3\rangle)=\{\langle 3,3\rangle,\langle 3,-1\rangle,\langle 1,-3\rangle\},$$
$$C_{0}^{+}(T_{2,2},\langle 6,2\rangle)=\{\langle 6,2\rangle,\langle 2,6\rangle\},$$

PROPOSITION 4.4. *For a fixed pair* $\mathbf{x}_1,\mathbf{x}_2\in\Lambda_6$ *which satisfy* $[\mathbf{x}_1,\mathbf{x}_2]=2T_{2,2}$ *it can be verified that for any* $\mathbf{y}\in\Lambda_6$ *the pairs of values* $\langle a,b\rangle$ *such that* $(\mathbf{x}_1,\mathbf{y})=a$ *and* $(\mathbf{x}_2,\mathbf{y})=b$ *fall in one of* $C_{96}(T_{2,2},\langle 0,0\rangle),C_{92}(T_{2,2},\langle 1,1\rangle),\dots,$ $C_0(T_{2,2},\langle 6,2\rangle)$.

4.2.3. Third Case

We put

$$T_{2,3}=\begin{pmatrix}3 & 1/2\\ 1/2 & 3\end{pmatrix}$$

which is a positive definite symmetric semi-integral matrix of size 2.

We search the pairs of integers $\langle a,b\rangle$ which satisfy the conditions (i) and (ii) but $T_{2,1}$ is replaced by $T_{2,3}$.

We present only the results.

$C_{105}(T_{2,3},\langle 0,0\rangle)=\{\langle 0,0\rangle\}$ and

$$C_{102}^{+}(T_{2,3},\langle 1,0\rangle)=\{\langle 1,0\rangle,\langle 0,1\rangle\},$$
$$C_{100}^{+}(T_{2,3},\langle 1,1\rangle)=\{\langle 1,1\rangle\},$$

$$C^+_{98}(T_{2,3}, \langle 1, -1\rangle) = \{\langle 1, -1\rangle\},$$

$$C^+_{93}(T_{2,3}, \langle 2, 0\rangle) = \{\langle 2, 0\rangle, \langle 0, 2\rangle\},$$

$$C^+_{92}(T_{2,3}, \langle 2, 1\rangle) = \{\langle 2, 1\rangle, \langle 1, 2\rangle\},$$

$$C^+_{88}(T_{2,3}, \langle 2, -1\rangle) = \{\langle 2, -1\rangle, \langle 1, -2\rangle\},$$

$$C^+_{85}(T_{2,3}, \langle 2, 2\rangle) = \{\langle 2, 2\rangle\},$$

$$C^+_{78}(T_{2,3}, \langle 3, 1\rangle) = \{\langle 3, 1\rangle, \langle 3, 0\rangle, \langle 1, 3\rangle, \langle 0, 3\rangle\},$$

$$C^+_{77}(T_{2,3}, \langle 2, -2\rangle) = \{\langle 2, -2\rangle\},$$

$$C^+_{72}(T_{2,3}, \langle 3, 2\rangle) = \{\langle 3, 2\rangle, \langle 3, -1\rangle, \langle 2, 3\rangle, \langle 1, -3\rangle\},$$

$$C^+_{60}(T_{2,3}, \langle 3, 3\rangle) = \{\langle 3, 3\rangle, \langle 3, -2\rangle, \langle 2, -3\rangle\},$$

$$C^+_{0}(T_{2,3}, \langle 6, 1\rangle) = \{\langle 6, 1\rangle, \langle 1, 6\rangle\}.$$

Proposition 4.5. *For a fixed pair* $\mathbf{x}_1, \mathbf{x}_2 \in \Lambda_6$ *which satisfy* $[\mathbf{x}_1, \mathbf{x}_2] = 2T_{2,3}$ *it can be verified that for any* $\mathbf{y} \in \Lambda_6$ *the pairs of values* $\langle a, b\rangle$ *such that* $(\mathbf{x}_1, \mathbf{y}) = a$ *and* $(\mathbf{x}_2, \mathbf{y}) = b$ *fall in one of* $C_{105}(T_{2,3}, \langle 0, 0\rangle), C_{102}(T_{2,3}, \langle 1, 0\rangle), \dots,$ $C_0(T_{2,3}, \langle 6, 1\rangle)$.

4.2.4. Fourth Case

We put

$$T_{2,4} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

which is a positive definite symmetric half-integral matrix of size 2.

We search the pairs of integers $\langle a, b\rangle$ which satisfy the conditions (i) and (ii) but $T_{2,1}$ is replaced by $T_{2,4}$.

$C_{108}(T_{2,4}, \langle 0, 0\rangle) = \{\langle 0, 0\rangle\}$ and

$$C^+_{105}(T_{2,4}, \langle 1, 0\rangle) = \{\langle 1, 0\rangle, \langle 0, 1\rangle\},$$

$$C^+_{102}(T_{2,4}, \langle 1, 1\rangle) = \{\langle 1, 1\rangle, \langle 1, -1\rangle\},$$

$$C^+_{96}(T_{2,4}, \langle 2, 0\rangle) = \{\langle 2, 0\rangle, \langle 0, 2\rangle\},$$

$$C^+_{93}(T_{2,4}, \langle 2, 1\rangle) = \{\langle 2, 1\rangle, \langle 2, -1\rangle, \langle 1, 2\rangle, \langle 1, -2\rangle\},$$

$$C^+_{84}(T_{2,4}, \langle 2, 2\rangle) = \{\langle 2, 2\rangle, \langle 2, -2\rangle\},$$

$$C_{81}^{+}(T_{2,4}, \langle 3,0\rangle) = \{\langle 3,0\rangle, \langle 0,3\rangle\},$$

$$C_{78}^{+}(T_{2,4}, \langle 3,1\rangle) = \{\langle 3,1\rangle, \langle 3,-1\rangle, \langle 1,3\rangle, \langle 1,-3\rangle\},$$

$$C_{69}^{+}(T_{2,4}, \langle 3,2\rangle) = \{\langle 3,2\rangle, \langle 3,-2\rangle, \langle 2,3\rangle, \langle 2,-3\rangle\},$$

$$C_{54}^{+}(T_{2,4}, \langle 3,3\rangle) = \{\langle 3,3\rangle, \langle 3,-3\rangle\},$$

$$C_{0}^{+}(T_{2,4}, \langle 6,0\rangle) = \{\langle 6,0\rangle, \langle 0,6\rangle\}.$$

Proposition 4.6. *For a fixed pair* $\mathbf{x}_1, \mathbf{x}_2 \in \Lambda_6$ *which satisfy* $[\mathbf{x}_1, \mathbf{x}_2] = 2T_{2,4}$ *it can be verified that for any* $\mathbf{y} \in \Lambda_6$ *the pairs of values* $\langle a, b\rangle$ *such that* $(\mathbf{x}_1, \mathbf{y}) = a$ *and* $(\mathbf{x}_2, \mathbf{y}) = b$ *fall in one of* $C_{108}(T_{2,4}, \langle 0,0\rangle), C_{105}(T_{2,4}, \langle 1,0\rangle), \ldots,$ $C_0(T_{2,4}, \langle 6,0\rangle)$.

4.2.5. An Assembled Table of Ternary Quadratic Forms

Table 2. Table of extended ternary quadratic forms and its reduced forms

	extended ternary forms	reduced ternary forms
A_1	$C_{54}(T_{2,1}),\ C_{54}(T_{2,4})$	$(3,3,3,3,3,0)$
A_2	$C_{60}(T_{2,1}),\ C_{60}(T_{2,2}),\ C_{60}(T_{2,3})$	$(3,3,3,3,2,-1)$
A_3	$C_{64}(T_{2,2})$	$(3,3,3,2,2,-2)$
A_4	$C_{69}(T_{2,1}),\ C_{69}(T_{2,4})$	$(3,3,3,3,2,0)$
A_5	$C_{72}(T_{2,1}),\ C_{72}(T_{2,3})$	$(3,3,3,3,1,-1)$
A_6	$C_{77}(T_{2,2}),\ C_{77}(T_{2,3})$	$(3,3,3,2,2,-1)$
A_7	$C_{78}(T_{2,1}),\ C_{78}(T_{2,3}),\ C_{78}(T_{2,4})$	$(3,3,3,3,1,0)$
A_8	$C_{80}(T_{2,2})$	$(3,3,3,2,2,2)$
A_9	$C_{81}(T_{2,1}),\ C_{81}(T_{2,4})$	$(3,3,3,3,0,0)$
A_{10}	$C_{84}(T_{2,2}),\ C_{84}(T_{2,4})$	$(3,3,3,2,2,0)$
A_{11}	$C_{85}(T_{2,2}),\ C_{85}(T_{2,3})$	$(3,3,3,2,2,1)$
A_{12}	$C_{88}(T_{2,2}),\ C_{88}(T_{2,3})$	$(3,3,3,2,1,-1)$
A_{13}	$C_{92}(T_{2,2}),\ C_{92}(T_{2,3})$	$(3,3,3,2,1,1)$
A_{14}	$C_{93}(T_{2,2}),\ C_{93}(T_{2,3}),\ C_{93}(T_{2,4})$	$(3,3,3,2,1,0)$
A_{15}	$C_{96}(T_{2,2}),\ C_{96}(T_{2,4})$	$(3,3,3,2,0,0)$

Table 2 (continued)

	extended ternary forms	reduced ternary forms
A_{16}	$C_{98}(T_{2,3})$	$(3,3,3,1,1,-1)$
A_{17}	$C_{100}(T_{2,3})$	$(3,3,3,1,1,1)$
A_{18}	$C_{102}(T_{2,3}),\ C_{102}(T_{2,4})$	$(3,3,3,1,1,0)$
A_{19}	$C_{105}(T_{2,3}),\ C_{105}(T_{2,4})$	$(3,3,3,1,0,0)$
A_{20}	$C_{108}(T_{2,4})$	$(3,3,3,0,0,0)$

4.3. Computing the Fourier Coefficients of Siegel Theta Series of Degree 3 for L

4.3.1. The First Case (in Detail)

For a fixed pair $\mathbf{x}_1, \mathbf{x}_2 \in \Lambda_6$ which satisfy $[\mathbf{x}_1, \mathbf{x}_2] = 2T_{2,1}$, we set

$$\Lambda_{a,b}(2T_{2,1}; \mathbf{x}_1, \mathbf{x}_2) = \{\mathbf{z} \in \Lambda_6 \mid (\mathbf{x}_1, \mathbf{z}) = a, (\mathbf{x}_2, \mathbf{z}) = b\},$$

and

$$\lambda_{a,b}(2T_{2,1}; \mathbf{x}_1, \mathbf{x}_2) = |\Lambda_{a,b}(2T_{2,1}; \mathbf{x}_1, \mathbf{x}_2)|.$$

We take $\boldsymbol{\alpha} = u\mathbf{x}_1 + v\mathbf{x}_2$, where u and v are real independent variables, then we see that for $\mathbf{y} \in \Lambda_6$

$$(\mathbf{y}, \boldsymbol{\alpha}) = au + bv.$$

Then by the formula (2.1) we have

$$\begin{aligned}
&\sum_{\mathbf{y} \in \Lambda_6} \lambda_{a,b}(2T_{2,1}; \mathbf{x}_1, \mathbf{x}_2)(au+bv)^2 \\
&\quad = \sum_{\langle a,b\rangle \in C_{78}(T_{2,1}, \langle 1,1\rangle)} \lambda_{a,b}(2T_{2,1}; \mathbf{x}_1, \mathbf{x}_2)(au+bv)^2 \\
&\qquad + \sum_{\langle a,b\rangle \in C_{72}(T_{2,1}, \langle 2,1\rangle)} \lambda_{a,b}(2T_{2,1}; \mathbf{x}_1, \mathbf{x}_2)(au+bv)^2 \\
&\qquad + \sum_{\langle a,b\rangle \in C_{69}(T_{2,1}, \langle 2,2\rangle)} \lambda_{a,b}(2T_{2,1}; \mathbf{x}_1, \mathbf{x}_2)(au+bv)^2 \\
&\qquad + \sum_{\langle a,b\rangle \in C_{60}(T_{2,1}, \langle 3,2\rangle)} \lambda_{a,b}(2T_{2,1}; \mathbf{x}_1, \mathbf{x}_2)(au+bv)^2
\end{aligned}$$

$$+ \sum_{\langle a,b\rangle \in C_{54}(T_{2,1},\langle 3,3\rangle)} \lambda_{a,b}(2T_{2,1};\mathbf{x}_1,\mathbf{x}_2)(au+bv)^2$$

$$+ \sum_{\langle a,b\rangle \in C_{0}(T_{2,1},\langle 6,3\rangle)} \lambda_{a,b}(2T_{2,1};\mathbf{x}_1,\mathbf{x}_2)(au+bv)^2$$

$$= 6552000(\boldsymbol{a},\boldsymbol{a})$$

$$= 6552000(6u^2+6uv+6v^2).$$

We note that $\lambda_{a,b}(2T_{2,1};\mathbf{x}_1,\mathbf{x}_2) = 1$ for any $\langle a,b\rangle \in C_0(T_{2,1}\langle 6,3\rangle)$, since $\mathbf{x}_1$ and $\mathbf{x}_2$ are linealy independent and $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{y}$ with $(\mathbf{x}_1,\mathbf{y}) = a, (\mathbf{x}_2,\mathbf{y}) = b$ are linearly dependent.

The general formula (4.1) is specified to

$$a((T_{2,1},\{a/2,b/2\},3),\mathscr{L}_{48}) = \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2]=2T_{2,1}}} \lambda_{a,b}(2T_{2,1};\mathbf{x}_1,\mathbf{x}_2). \tag{4.2}$$

Taking the sum over $\mathbf{x}_1,\mathbf{x}_2 \in \lambda_6$ in the above equation we have the equation (for a spacing problem we use $\lambda_{a,b}(2T_{2,1})$ instead of $\lambda_{a,b}(2T_{2,1};\mathbf{x}_1,\mathbf{x}_2)$):

$$\sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2]=2T_{2,1}}} \sum_{\langle a,b\rangle \in C_{78}(T_{2,1},\langle 1,1\rangle)} \lambda_{a,b}(2T_{2,1})(au+bv)^2$$

$$+ \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2]=2T_{2,1}}} \sum_{\langle a,b\rangle \in C_{72}(T_{2,1},\langle 2,1\rangle)} \lambda_{a,b}(2T_{2,1})(au+bv)^2$$

$$+ \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2]=2T_{2,1}}} \sum_{\langle a,b\rangle \in C_{69}(T_{2,1},\langle 2,2\rangle)} \lambda_{a,b}(2T_{2,1})(au+bv)^2$$

$$+ \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2]=2T_{2,1}}} \sum_{\langle a,b\rangle \in C_{60}(T_{2,1},\langle 3,2\rangle)} \lambda_{a,b}(2T_{2,1})(au+bv)^2$$

$$+ \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2]=2T_{2,1}}} \sum_{\langle a,b\rangle \in C_{54}(T_{2,1},\langle 3,3\rangle)} \lambda_{a,b}(2T_{2,1})(au+bv)^2$$

$$+ \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2]=2T_{2,1}}} \sum_{\langle a,b\rangle \in C_{0}(T_{2,1},\langle 6,3\rangle)} (au+bv)^2$$

$$= 6552000(6u^2+6uv+6v^2) \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2]=2T_{2,1}}} 1.$$

We note that

$$\sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2]=2T_{2,1}}} \sum_{\langle a,b\rangle\in C_{78}(T_{2,1},\langle 1,1\rangle)} \lambda_{a,b}(2T_{2,1})(au+bv)^2$$
$$= \sum_{\langle a,b\rangle\in C_{78}(T_{2,1},\langle 1,1\rangle)} \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2]=2T_{2,1}}} \lambda_{a,b}(2T_{2,1})(au+bv)^2$$
$$= \sum_{\langle a,b\rangle\in C_{78}(T_{2,1},\langle 1,1\rangle)} a((T_{2,1},\{a/2,b/2\},3),\mathscr{L}_{48})(au+bv)^2$$
$$= a((T_{2,1},\{1/2,1/2\},3),\mathscr{L}_{48}) \sum_{\langle a,b\rangle\in C_{78}(T_{2,1},\langle 1,1\rangle)} (au+bv)^2$$

The last transition is supported by the fact that for each meber $\langle a,b\rangle$ of $C_{78}(T_{2,1},\langle 1,1\rangle)$ the matrix $(T_{2,1},\{a/2,b/2\},3)$ is integarally equivalent and by Proposition 2.4. Likewise we can show that

$$\sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2]=2T_{2,1}}} \sum_{\langle a,b\rangle\in C_{72}(T_{2,1},\langle 2,1\rangle)} \lambda_{a,b}(2T_{2,1})(au+bv)^2$$
$$= \sum_{\langle a,b\rangle\in C_{78}(T_{2,1},\langle 1,1\rangle)} \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2]=2T_{2,1}}} \lambda_{a,b}(2T_{2,1})(au+bv)^2$$
$$= a((T_{2,1},\{1,1/2\},3),\mathscr{L}_{48}) \sum_{\langle a,b\rangle\in C_{72}(T_{2,1},\langle 2,1\rangle)} (au+bv)^2$$

and etc. To save space we will use the shorthand: $A_7 = a((T_{2,1},\{1/2,1/2\},3),\mathscr{L}_{48}),\ldots,A_1 = a((T_{2,1},\{3/3,3/2\},3),\mathscr{L}_{48})$. (Conf. Table 2.) We have the equation

$$A_7\sum\nolimits_{\langle a,b\rangle\in C_{78}(T_{2,1},\langle 1,1\rangle)}(au+bv)^2 + A_5\sum\nolimits_{\langle a,b\rangle\in C_{72}(T_{2,1},\langle 2,1\rangle)}(au+bv)^2$$
$$+ A_4\sum\nolimits_{\langle a,b\rangle\in C_{69}(T_{2,1},\langle 2,2\rangle)}(au+bv)^2 + A_2\sum\nolimits_{\langle a,b\rangle\in C_{60}(T_{2,1},\langle 3,2\rangle)}(au+bv)^2$$
$$+ A_1\sum\nolimits_{\langle a,b\rangle\in C_{54}(T_{2,1},\langle 3,3\rangle)}(au+bv)^2$$
$$+ a(T_{2,1},\mathscr{L}_{48})\sum\nolimits_{\langle a,b\rangle\in C_{0}(T_{2,1},\langle 6,3\rangle)}(au+bv)^2$$
$$= 6552000(6u^2+6uv+6v^2)a(T_{2,1},\mathscr{L}_{48}).$$

The above equation is a polynomial identity with respect to the variables u, v and we hve linear constraints concerning $A_7, \dots, A_1$.

Similarly from the formula (2.2)~(2.4) we have the equations:

$$\begin{aligned} A_7 & \sum\nolimits_{\langle a,b\rangle \in C_{78}(T_{2,1},\langle 1,1\rangle)} (au+bv)^4 + A_5 \sum\nolimits_{\langle a,b\rangle \in C_{72}(T_{2,1},\langle 2,1\rangle)} (au+bv)^4 \\ & + A_4 \sum\nolimits_{\langle a,b\rangle \in C_{69}(T_{2,1},\langle 2,2\rangle)} (au+bv)^4 + A_2 \sum\nolimits_{\langle a,b\rangle \in C_{60}(T_{2,1},\langle 3,2\rangle)} (au+bv)^4 \\ & + A_1 \sum\nolimits_{\langle a,b\rangle \in C_{54}(T_{2,1},\langle 3,3\rangle)} (au+bv)^4 \\ & + a(T_{2,1},\mathscr{L}_{48}) \sum\nolimits_{\langle a,b\rangle \in C_{0}(T_{2,1},\langle 6,3\rangle)} (au+bv)^4 \\ = {} & 2358720(6u^2+6uv+6v^2)^2 a(T_{2,1},\mathscr{L}_{48}), \end{aligned}$$

$$\begin{aligned} A_7 & \sum\nolimits_{\langle a,b\rangle \in C_{78}(T_{2,1},\langle 1,1\rangle)} (au+bv)^6 + A_5 \sum\nolimits_{\langle a,b\rangle \in C_{72}(T_{2,1},\langle 2,1\rangle)} (au+bv)^6 \\ & + A_4 \sum\nolimits_{\langle a,b\rangle \in C_{69}(T_{2,1},\langle 2,2\rangle)} (au+bv)^6 + A_2 \sum\nolimits_{\langle a,b\rangle \in C_{60}(T_{2,1},\langle 3,2\rangle)} (au+bv)^6 \\ & + A_1 \sum\nolimits_{\langle a,b\rangle \in C_{54}(T_{2,1},\langle 3,3\rangle)} (au+bv)^6 \\ & + a(T_{2,1},\mathscr{L}_{48}) \sum\nolimits_{\langle a,b\rangle \in C_{0}(T_{2,1},\langle 6,3\rangle)} (au+bv)^6 \\ = {} & 1360800(6u^2+6uv+6v^2)^3 a(T_{2,1},\mathscr{L}_{48}), \end{aligned}$$

$$\begin{aligned} A_7 & \sum\nolimits_{\langle a,b\rangle \in C_{78}(T_{2,1},\langle 1,1\rangle)} (au+bv)^8 + A_5 \sum\nolimits_{\langle a,b\rangle \in C_{72}(T_{2,1},\langle 2,1\rangle)} (au+bv)^8 \\ & + A_4 \sum\nolimits_{\langle a,b\rangle \in C_{69}(T_{2,1},\langle 2,2\rangle)} (au+bv)^8 + A_2 \sum\nolimits_{\langle a,b\rangle \in C_{60}(T_{2,1},\langle 3,2\rangle)} (au+bv)^8 \\ & + A_1 \sum\nolimits_{\langle a,b\rangle \in C_{54}(T_{2,1},\langle 3,3\rangle)} (au+bv)^8 \\ & + a(T_{2,1},\mathscr{L}_{48}) \sum\nolimits_{\langle a,b\rangle \in C_{0}(T_{2,1},\langle 6,3\rangle)} (au+bv)^8 \\ = {} & (6u^2+6uv+6v^2)^4 a(T_{2,1},\mathscr{L}_{48}). \end{aligned}$$

The above four polynomial identities are enough to solve $A_1, \dots, A_7$. We have

$$A_7 = 5365440\, a(T_{2,1},\mathscr{L}_{48}),$$

$$A_5 = 922185\, a(T_{2,1},\mathscr{L}_{48}),$$

$$A_4 = 361584\, a(T_{2,1},\mathscr{L}_{48}),$$

$$
\begin{aligned}
A_2 &= 16767\, a(T_{2,1}, \mathscr{L}_{48}),\\
&= a((T_{2,1}, \{3/2, 1/2\}, 3), \mathscr{L}_{48})\\
&= a(T_{3,2}, \mathscr{L}_{48}) = 32384199085056000\\
&\quad \text{(we will use this value at Proposition 5.4)},\\
A_1 &= 1656\, a(T_{2,1}, \mathscr{L}_{48})\\
&= a((T_{2,1}, \{3/2, 3/2\}, 3), \mathscr{L}_{48})\\
&= a(T_{3,1}, \mathscr{L}_{48}) = 3198439415808000\\
&\quad \text{(we will use this value at Proposition 5.3)}.
\end{aligned}
$$

Here $a(T_{2,1}, \mathscr{L}_{48}) = 1931424768000$. As to $a((T_{2,1}, \{0,0\}, 3), \mathscr{L}_{48})$ we use the formula (2.0). The lefthand side of (2.0) is slightly different from the formulas (2.1)~(2.5). Skipping the earlier equations we reach

$$
\begin{aligned}
&a((T_{2,1}, \{0,0\}, 3), \mathscr{L}_{48})c_{81}(T_{2,1}) + A_7 \cdot c_{78}(T_{2,1}) + A_5 \cdot c_{72}(T_{2,1}) + A_4 \cdot c_{64}(T_{2,1})\\
&\qquad + A_2 \cdot c_{60}(T_{2,1}) + A_1 \cdot c_{54}(T_{2,1}) + a(T_{2,1}, \mathscr{L}_{48}) \cdot c_0(T_{2,1})\\
&\quad = a(T_{2,1}, \mathscr{L}_{48}) \cdot 52416000.
\end{aligned}
$$

From this we have $a((T_{2,1}, \{0,0\}, 3), \mathscr{L}_{48})) = 23775066324172800000$.

4.3.2. The Second Case (a Summary)

For a fixed pair $\mathbf{x}_1, \mathbf{x}_2 \in \Lambda_6$ which satisfy $[\mathbf{x}_1, \mathbf{x}_2] = 2T_{2,2}$, we set

$$\lambda_{a,b}(2T_{2,2}; \mathbf{x}_1, \mathbf{x}_2) = |\{\mathbf{z} \in \Lambda_6 \mid (\mathbf{x}_1, \mathbf{z}) = a, (\mathbf{x}_2, \mathbf{z}) = b\}|.$$

We take $\boldsymbol{\alpha} = u\mathbf{x}_1 + v\mathbf{x}_2$, where u and v are real independent variables, then we see that for $\mathbf{y} \in \Lambda_6$

$$(\mathbf{y}, \boldsymbol{\alpha}) = au + bv,$$

and

$$(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 6u^2 + 4uv + 6v^2.$$

Applying the formulas (2.1)~(2.5) in this settings we have

$$
\begin{aligned}
A_{14} & \sum\nolimits_{\langle a,b\rangle \in C_{93}(T_{2,2},\langle 1,0\rangle)} (au+bv)^{2k} \\
& + A_{13} \sum\nolimits_{\langle a,b\rangle \in C_{92}(T_{2,2},\langle 1,1\rangle)} (au+bv)^{2k} \\
& + A_{12} \sum\nolimits_{\langle a,b\rangle \in C_{88}(T_{2,2},\langle 1,-1\rangle)} (au+bv)^{2k} \\
& + A_{11} \sum\nolimits_{\langle a,b\rangle \in C_{85}(T_{2,2},\langle 2,1\rangle)} (au+bv)^{2k} \\
& + A_{10} \sum\nolimits_{\langle a,b\rangle \in C_{84}(T_{2,2},\langle 2,0\rangle)} (au+bv)^{2k} \\
& + A_{8} \sum\nolimits_{\langle a,b\rangle \in C_{80}(T_{2,2},\langle 2,2\rangle)} (au+bv)^{2k} \\
& + A_{6} \sum\nolimits_{\langle a,b\rangle \in C_{77}(T_{2,2},\langle 2,-1\rangle)} (au+bv)^{2k} \\
& + A_{5} \sum\nolimits_{\langle a,b\rangle \in C_{72}(T_{2,2},\langle 3,1\rangle)} (au+bv)^{2k} \\
& + A_{4} \sum\nolimits_{\langle a,b\rangle \in C_{69}(T_{2,2},\langle 3,2\rangle)} (au+bv)^{2k} \\
& + A_{3} \sum\nolimits_{\langle a,b\rangle \in C_{64}(T_{2,2},\langle 2,-2\rangle)} (au+bv)^{2k} \\
& + A_{2} \sum\nolimits_{\langle a,b\rangle \in C_{60}(T_{2,2},\langle 3,3\rangle)} (au+bv)^{2k} \\
& + a(T_{2,2},\mathscr{L}_{48}) \sum\nolimits_{\langle a,b\rangle \in C_{0}(T_{2,2},\langle 6,2\rangle)} (au+bv)^{2k} \\
= {} & \rho_k (6u^2+4uv+6v^2)^k \, a(T_{2,2},\mathscr{L}_{48}),
\end{aligned}
$$

where

$$
\rho_k = \begin{cases} 6552000 & \text{if } k=1, \\ 2358720 & \text{if } k=2, \\ 1360800 & \text{if } k=3, \\ 1058400 & \text{if } k=4, \\ 1020600 & \text{if } k=5. \end{cases}
$$

Again we have linear constraints on $A_{14}\dots,A_2$ and we solve them

$$A_{14} = 5623040\, a(T_{2,2},\mathscr{L}_{48}),$$

$$A_{13} = 4432560\, a(T_{2,2},\mathscr{L}_{48}),$$

$$A_{12} = 1667040\, a(T_{2,2},\mathscr{L}_{48}),$$

$$A_{11} = 777216\, a(T_{2,2},\mathscr{L}_{48}),$$

$$A_{10} = 599104\, a(T_{2,2}, \mathscr{L}_{48}),$$

$$A_8 = 204792\, a(T_{2,2}, \mathscr{L}_{48}),$$

$$A_6 = 88320\, a(T_{2,2}, \mathscr{L}_{48}),$$

$$A_5 = 20240\, a(T_{2,2}, \mathscr{L}_{48}),$$

$$A_4 = 7936\, a(T_{2,2}, \mathscr{L}_{48}),$$

$$A_3 = 1518\, a(T_{2,2}, \mathscr{L}_{48}),$$

$$A_2 = 368\, a(T_{2,2}, \mathscr{L}_{48}).$$

Here $a(T_{2,2}, \mathscr{L}_{48}) = 88000540992000$. As for $a((T_{2,2}, \{0,0\}, 3), \mathscr{L}_{48}))$ by taking a similar way to the last part of Section 4.3.1 we obtain

$$a((T_{2,2}, \{0,0\}, 3), \mathscr{L}_{48})) = 9950045168883456000000.$$

Remark 4. In our present computations we do not refer the formula (2.6), since this formula gives us no new information other than the ones from the formulas (2.1)~(2.5). The same comment is also applied to the later sections.

4.3.3. The Third Case (a Summary)

For a fixed pair $\mathbf{x}_1, \mathbf{x}_2 \in \Lambda_6$ which satisfy $[\mathbf{x}_1, \mathbf{x}_2] = 2T_{2,3}$, we set

$$\lambda_{a,b}(2T_{2,3}; \mathbf{x}_1, \mathbf{x}_2) = |\{\mathbf{z} \in \Lambda_6 \mid (\mathbf{x}_1, \mathbf{z}) = a, (\mathbf{x}_2, \mathbf{z}) = b\}|.$$

We take $\boldsymbol{\alpha} = u\mathbf{x}_1 + v\mathbf{x}_2$, where u and v are real independent variables, then we see that for $\mathbf{y} \in \Lambda_6$

$$(\mathbf{y}, \boldsymbol{\alpha}) = au + bv,$$

and

$$(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 6u^2 + 2uv + 6v^2.$$

Applying the formulas (2.1)~(2.5) in this settings we have

$$A_{18} \sum\nolimits_{\langle a,b\rangle \in C_{102}(T_{2,3}, \langle 1,0\rangle)} (au+bv)^{2k}$$

$$+ A_{17} \sum\nolimits_{\langle a,b\rangle \in C_{100}(T_{2,3}, \langle 1,1\rangle)} (au+bv)^{2k}$$

$$+ A_{16} \sum\nolimits_{\langle a,b\rangle \in C_{98}(T_{2,3}, \langle 1,-1\rangle)} (au+bv)^{2k}$$

$$+ A_{14} \sum\nolimits_{\langle a,b\rangle \in C_{93}(T_{2,3}, \langle 2,0\rangle)} (au+bv)^{2k}$$

$$+ A_{13} \sum_{\langle a,b\rangle \in C_{92}(T_{2,3},\langle 2,1\rangle)} (au+bv)^{2k}$$

$$+ A_{12} \sum_{\langle a,b\rangle \in C_{88}(T_{2,3},\langle 2,-1\rangle)} (au+bv)^{2k}$$

$$+ A_{11} \sum_{\langle a,b\rangle \in C_{85}(T_{2,3},\langle 2,2\rangle)} (au+bv)^{2k}$$

$$+ A_{7} \sum_{\langle a,b\rangle \in C_{78}(T_{2,3},\langle 3,1\rangle)} (au+bv)^{2k}$$

$$+ A_{6} \sum_{\langle a,b\rangle \in C_{77}(T_{2,3},\langle 2,-2\rangle)} (au+bv)^{2k}$$

$$+ A_{5} \sum_{\langle a,b\rangle \in C_{72}(T_{2,3},\langle 3,2\rangle)} (au+bv)^{2k}$$

$$+ A_{2} \sum_{\langle a,b\rangle \in C_{60}(T_{2,3},\langle 3,3\rangle)} (au+bv)^{2k}$$

$$+ a(T_{2,3},\mathscr{L}_{48}) \sum_{\langle a,b\rangle \in C_{0}(T_{2,3},\langle 6,1\rangle)} (au+bv)^{2k}$$

$$= \rho_k (6u^2+2uv+6v^2)^k \, a(T_{2,3},\mathscr{L}_{48}), \quad k=1,\dots,5.$$

We have

$$A_{18} = 5713400 \, a(T_{2,3},\mathscr{L}_{48}),$$

$$A_{17} = 3695433 \, a(T_{2,3},\mathscr{L}_{48}),$$

$$A_{16} = 2369400 \, a(T_{2,3},\mathscr{L}_{48}),$$

$$A_{14} = 748720 \, a(T_{2,3},\mathscr{L}_{48}),$$

$$A_{13} = 590205 \, a(T_{2,3},\mathscr{L}_{48}),$$

$$A_{12} = 221970 \, a(T_{2,3},\mathscr{L}_{48}),$$

$$A_{11} = 103488 \, a(T_{2,3},\mathscr{L}_{48}),$$

$$A_{7} = 15680 \, a(T_{2,3},\mathscr{L}_{48}),$$

$$A_{6} = 11760 \, a(T_{2,3},\mathscr{L}_{48}),$$

$$A_{5} = 2695 \, a(T_{2,3},\mathscr{L}_{48}),$$

$$A_{2} = 49 \, a(T_{2,3},\mathscr{L}_{48}).$$

Here $a(T_{2,3},\mathscr{L}_{48}) = 660902022144000$. As for the value $a((T_{2,3},\{0,0\},3),\mathscr{L}_{48})$ a similar effort to Sections 4.3.1 and 4.3.2 works and we have

$$a((T_{2,3},\{0,0\},3),\mathscr{L}_{48}) = 7145249686126755840000.$$

4.3.4. The Fourth Case (a Summary)

For a fixed pair $\mathbf{x}_1, \mathbf{x}_2 \in \Lambda_6$ which satisfy $[\mathbf{x}_1, \mathbf{x}_2] = 2T_{2,4}$, we set

$$\lambda_{a,b}(2T_{2,4}; \mathbf{x}_1, \mathbf{x}_2) = |\{\mathbf{z} \in \Lambda_6 \mid (\mathbf{x}_1, \mathbf{z}) = a, (\mathbf{x}_2, \mathbf{z}) = b\}|.$$

We take $\boldsymbol{\alpha} = u\mathbf{x}_1 + v\mathbf{x}_2$, where u and v are real independent variables, then we see that for $\mathbf{y} \in \Lambda_6$

$$(\mathbf{y}, \boldsymbol{\alpha}) = au + bv,$$

and

$$(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 6u^2 + 6v^2.$$

Applying the formulas (2.1)~(2.5) in this settings we have

$$\begin{aligned}
A_{19} & \sum\nolimits_{\langle a,b\rangle \in C_{105}(T_{2,4}, \langle 1,0\rangle)} (au+bv)^{2k} + A_{18} \sum\nolimits_{\langle a,b\rangle \in C_{102}(T_{2,4}, \langle 1,1\rangle)} (au+bv)^{2k} \\
& + A_{15} \sum\nolimits_{\langle a,b\rangle \in C_{96}(T_{2,4}, \langle 2,0\rangle)} (au+bv)^{2k} + A_{14} \sum\nolimits_{\langle a,b\rangle \in C_{93}(T_{2,4}, \langle 2,1\rangle)} (au+bv)^{2k} \\
& + A_{10} \sum\nolimits_{\langle a,b\rangle \in C_{84}(T_{2,4}, \langle 2,2\rangle)} (au+bv)^{2k} + A_{9} \sum\nolimits_{\langle a,b\rangle \in C_{81}(T_{2,4}, \langle 3,0\rangle)} (au+bv)^{2k} \\
& + A_{7} \sum\nolimits_{\langle a,b\rangle \in C_{78}(T_{2,4}, \langle 3,1\rangle)} (au+bv)^{2k} + A_{4} \sum\nolimits_{\langle a,b\rangle \in C_{69}(T_{2,4}, \langle 3,2\rangle)} (au+bv)^{2k} \\
& + A_{1} \sum\nolimits_{\langle a,b\rangle \in C_{54}(T_{2,4}, \langle 3,3\rangle)} (au+bv)^{2k} \\
& + a(T_{2,4}, \mathscr{L}_{48}) \sum\nolimits_{\langle a,b\rangle \in C_{0}(T_{2,4}, \langle 6,0\rangle)} (au+bv)^{2k} \\
= {} & \rho_k (6u^2 + 6v^2)^k \, a(T_{2,4}, \mathscr{L}_{48}), \quad k = 1, \ldots, 5.
\end{aligned}$$

From these we have linear equations on $A_{19}, \ldots, A_1$. This time we can not solve the linear equations. We only obtain relations among $A_{19}, \ldots, A_1$.

$$\begin{aligned}
A_{19} &= 5736384 \, a(T_{2,4}, \mathscr{L}_{48}) - 300 \, A_1, \\
A_{18} &= 3030480 \, a(T_{2,4}, \mathscr{L}_{48}) + 225 \, A_1, \\
A_{15} &= 798399 \, a(T_{2,4}, \mathscr{L}_{48}) + 120 \, A_1, \\
A_{14} &= 397440 \, a(T_{2,4}, \mathscr{L}_{48}) - 90 \, A_1, \\
A_{10} &= 42228 \, a(T_{2,4}, \mathscr{L}_{48}) + 36 \, A_1, \\
A_{9} &= 19136 \, a(T_{2,4}, \mathscr{L}_{48}) - 20 \, A_1,
\end{aligned}$$

$$A_7 = 8280\, a(T_{2,4}, \mathscr{L}_{48}) + 15\, A_1,$$

$$A_4 = 576\, a(T_{2,4}, \mathscr{L}_{48}) - 6\, A_1.$$

Here $a(T_{2,4}, \mathscr{L}_{48}) = 1245768975360000$, and A_1 is computed in the previous sections. $A_1 = 3198439415808000$. The value $a((T_{2,4}, \{0,0\}, 3), \mathscr{L}_{48})$ can be deduced from $A_{19}, \dots, A_1$ likewisely, and we have

$$a((T_{2,4}, \{0,0\}, 3), \mathscr{L}_{48}) = 132800803764056064000000.$$

4.3.5. A Table

Table 3. Fourier coefficints of Siegel theta series of degree 3 for the 48 dimensional extremal lattice

num	D	reduced T	$a(T, \mathcal{L}_{48})$
1	*54	$(3,3,3,3,3,0)$	3198439415808000
2	60	$(3,3,3,3,2,-1)$	32384199085056000
3	64	$(3,3,3,2,2,-2)$	133584821225856000
4	69	$(3,3,3,3,2,0)$	698372293312512000
5	72	$(3,3,3,3,1,-1)$	1781130949678080000
6	77	$(3,3,3,2,2,-1)$	7772207780413440000
7	78	$(3,3,3,3,1,0)$	10362943707217920000
8	80	$(3,3,3,2,2,2)$	18021806790833664000
9	*81	$(3,3,3,3,0,0)$	23775066324172800000
10	84	$(3,3,3,2,2,0)$	52721476110471168000
11	85	$(3,3,3,2,2,1)$	68395428467638272000
12	88	$(3,3,3,2,1,-1)$	146700421855303680000
13	92	$(3,3,3,2,1,1)$	390067677979499520000
14	93	$(3,3,3,2,1,0)$	494830562019655680000
15	96	$(3,3,3,2,0,0)$	995004516888345600000
16	98	$(3,3,3,1,1,-1)$	1565941251267993600000
17	100	$(3,3,3,1,1,1)$	2442319142397668352000
18	102	$(3,3,3,1,1,0)$	3775997613317529600000
19	105	$(3,3,3,1,0,0)$	7145249686126755840000
20	*108	$(3,3,3,0,0,0)$	13280080376405606400000

REMARK 5. In Table 3 we only give the values of the Fourier coefficients $a(T, \mathscr{L}_{48})$ with the restriction that the diagonal entries of $MR(T)$ are all 3. But our method enables us to compute $a(T, \mathscr{L}_{48})$ for the indeces T such that the diagonal entries of $MR(T)$ are ≥ 3.

5. Fourier Coefficients of Siegel Theta Series of Degree 4 for $\mathscr{L}$

Our plan to compute the Fourier coefficients or to find the relations among the Fourier coefficients is as follows.

5.1. Extending the Matrices of Degree 3 to the Matrices of Degree 4

The following result which is due to Schiemann is quite useful.

PROPOSITION 5.1 (Schiemann [17]). *Except one pair of forms two positive definite quaternary even integral quadratic forms with the same discriminant* ≤ 3000 *are integrally equivalent if and only if their theta series of one variable equal.*
The only exceptional pair of forms is given at the discriminant 1729.

By Proposition 5.1 we can quickly and effectively distinguish whether two quaternary quadratic forms are integrally equivalent or not.

5.1.1. First Case

We put

$$2 \cdot T_{3,1} = \begin{pmatrix} 6 & 3 & 3 \\ 3 & 6 & 3 \\ 3 & 3 & 6 \end{pmatrix}.$$

For a fixed triple $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ satisfying $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2 \cdot T_{3,1}$ we seek to find out all ordered triples $\langle a, b, c \rangle$ of integers satisfying the two conditions

(1) $(T_{3,1}, \{a/2, b/2, c/2\}, 3)$ is positive semi-definite,

(2) when $(T_{3,1}, \{a/2, b/2, c/2\}, 3)$ is reduced under the unimodular transformation

$U^t(T_{3,1}, \{a/2, b/2, c/2\}, 3)U$, the minimal value of the non-zero diagonal entries of the resulting matrix is 3.

It is rather easy to find all triples $\langle a, b, c \rangle$ satisfying the condition (1) only. Next from all such triples we form the sets of triples with two further conditions (i) in a set each triple $\langle a, b, c \rangle$ has identical $\det(2(T_{3,1}, \{a/2, b/2, c/2\}, 3))$ (this is called the discriminant of $(T_{3,1}, \{a/2, b/2, c/2\}, 3)$), (ii) each member of triple $\langle a, b, c \rangle$ in a set determines the unique integrally equivalent form $2(T_{3,1}, \{a/2, b/2, c/2\}, 3)$. To check (ii) we use Proposition 5.1. We find out that $C_{648}(T_{3,1}, \langle 0, 0, 0 \rangle)$ and

$$C^+_{621}(T_{3,1}, \langle 1, 1, 1 \rangle) = \{\langle 1, 1, 1 \rangle, \langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle\}$$

$$C^+_{612}(T_{3,1}, \langle 1, 1, 0 \rangle) = \{\langle 1, 1, 0 \rangle, \langle 1, 0, 1 \rangle, \langle 0, 1, 1 \rangle\}$$

$$C^+_{576}(T_{3,1}, \langle 2, 1, 1 \rangle) = \{\langle 2, 1, 1 \rangle, \langle 1, 2, 1 \rangle, \langle 1, 1, 2 \rangle, \langle 1, 0, -1 \rangle, \langle 1, -1, 0 \rangle, \langle 0, 1, -1 \rangle\}$$

$$C^+_{549}(T_{3,1},\langle 2,2,1\rangle) = \{\langle 2,2,1\rangle, \langle 2,1,2\rangle, \langle 2,1,0\rangle, \langle 2,0,1\rangle, \langle 1,2,2\rangle, \langle 1,2,0\rangle, \langle 1,1,-1\rangle, \langle 1,0,2\rangle, \langle 1,-1,1\rangle, \langle 1,-1,-1\rangle, \langle 0,2,1\rangle, \langle 0,1,2\rangle\},$$

$$C^+_{540}(T_{3,1},\langle 2,2,2\rangle) = \{\langle 2,2,2\rangle, \langle 2,0,0\rangle, \langle 0,2,0\rangle, \langle 0,0,2\rangle\},$$

$$C^+_{504}(T_{3,1},\langle 2,2,0\rangle) = \{\langle 2,2,0\rangle, \langle 2,0,2\rangle, \langle 0,2,2\rangle\},$$

$$C^+_{477}(T_{3,1},\langle 3,2,2\rangle) = \{\langle 3,2,2\rangle, \langle 3,1,1\rangle, \langle 2,3,2\rangle, \langle 2,2,3\rangle, \langle 2,0,-1\rangle, \langle 2,-1,0\rangle, \langle 1,3,1\rangle, \langle 1,1,3\rangle, \langle 1,0,-2\rangle, \langle 1,-2,0\rangle, \langle 0,2,-1\rangle, \langle 0,1,-2\rangle\},$$

$$C^+_{468}(T_{3,1},\langle 3,2,1\rangle) = \{\langle 3,2,1\rangle, \langle 3,1,2\rangle, \langle 2,3,1\rangle, \langle 2,1,3\rangle, \langle 2,1,-1\rangle, \langle 2,-1,1\rangle, \langle 1,3,2\rangle, \langle 1,2,3\rangle, \langle 1,2,-1\rangle, \langle 1,-1,2\rangle, \langle 1,-1,-2\rangle, \langle 1,-2,-1\rangle\},$$

$$C^+_{432}(T_{3,1},\langle 3,3,2\rangle) = \{\langle 3,3,2\rangle, \langle 3,2,3\rangle, \langle 3,1,0\rangle, \langle 3,0,1\rangle, \langle 2,3,3\rangle, \langle 2,-1,-1\rangle, \langle 1,3,0\rangle, \langle 1,1,-2\rangle, \langle 1,0,3\rangle, \langle 1,-2,1\rangle, \langle 0,3,1\rangle, \langle 0,1,3\rangle\},$$

$$C^+_{405,1}(T_{3,1},\langle 3,3,3\rangle) = \{\langle 3,3,3\rangle, \langle 3,0,0\rangle, \langle 0,3,0\rangle, \langle 0,0,3\rangle\},$$

$$C^+_{405,2}(T_{3,1},\langle 3,3,1\rangle) = \{\langle 3,3,1\rangle, \langle 3,2,0\rangle, \langle 3,1,3\rangle, \langle 3,0,2\rangle, \langle 2,3,0\rangle, \langle 2,0,3\rangle, \langle 2,-1,2\rangle, \langle 1,3,3\rangle, \langle 1,-2,-2\rangle, \langle 0,3,2\rangle, \langle 0,2,3\rangle\},$$

$$C^+_{324}(T_{3,1},\langle 3,3,0\rangle) = \{\langle 3,3,0\rangle, \langle 3,0,3\rangle, \langle 3,0,-1\rangle, \langle 3,-1,0\rangle, \langle 1,0,-3\rangle, \langle 1,-3,0\rangle, \langle 0,3,3\rangle, \langle 0,3,-1\rangle, \langle 0,1,-3\rangle\},$$

$$C^+_{0}(T_{3,1},\langle 6,3,3\rangle) = \{\langle 6,3,3\rangle, \langle 3,6,3\rangle, \langle 3,3,6\rangle, \langle 3,3,-2\rangle, \langle 3,0,-3\rangle, \langle 3,-2,3\rangle, \langle 3,-3,0\rangle, \langle 2,-3,-3\rangle, \langle 0,3,-3\rangle\}.$$

PROPOSITION 5.2. *For a fixed triple* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6$ *which satisfy* $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,1}$ *it can be verified that for any* $\mathbf{y} \in \Lambda_6$ *the triples of values* $\langle a,b,c\rangle$ *such that* $(\mathbf{x}_1,\mathbf{y}) = a$, $(\mathbf{x}_2,\mathbf{y}) = b$ *and* $(\mathbf{x}_3,\mathbf{y}) = c$ *fall in one of* $C_{648}(T_{3,1},\langle 0,0,0\rangle)$, $C_{621}(T_{3,1},\langle 3,3,0\rangle), \dots, C_0(T_{3,1},\langle 6,3,3\rangle)$.

5.1.2. Second Case

We begin with a ternary matrix:

$$2 \cdot T_{3,2} = \begin{pmatrix} 6 & 3 & 3 \\ 3 & 6 & 1 \\ 3 & 1 & 6 \end{pmatrix}.$$

We explore all triples $\langle a, b, c\rangle$ which satisfy

(3) $(T_{3,2}, \{a/2, b/2, c/2\}, 3)$ is positive semi-definite,

(4) when $(T_{3,2}, \{a/2, b/2, c/2\}, 3)$ is reduced under the unimodular transformation:

$U^t(T_{3,2}, \{a/2, b/2, c/2\}, 3)U$, the minimal value of the non-zero diagonal entries of the resulting matrix is 3.

Next from all such triples we form the sets of triples with two further conditions (i) in a set each triple $\langle a, b, c\rangle$ has identical $\det(2(T_{3,1}, \{a/2, b/2, c/2\}, 3))$, (ii) each member of triple $\langle a, b, c\rangle$ in a set determines the unique integrally equivalent form $2(T_{3,2}, \{a/2, b/2, c/2\}, 3)$. We find out that $C_{720}(T_{3,2}, \langle 0, 0, 0\rangle) = \{\langle 0, 0, 0\rangle\}$ and

$$C^+_{693}(T_{3,2}, \langle 0, 1, 0\rangle) = \{\langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle\},$$

$$C^+_{688}(T_{3,2}, \langle 1, 1, 0\rangle) = \{\langle 1, 1, 0\rangle, \langle 1, 0, 1\rangle\},$$

$$C^+_{685}(T_{3,2}, \langle 1, 1, 1\rangle) = \{\langle 1, 1, 1\rangle, \langle 1, 0, 0\rangle\},$$

$$C^+_{672}(T_{3,2}, \langle 0, 1, -1\rangle) = \{\langle 0, 1, -1\rangle\},$$

$$C^+_{660}(T_{3,2}, \langle 0, 1, 1\rangle) = \{\langle 0, 1, 1\rangle\},$$

$$C^+_{640}(T_{3,2}, \langle 2, 1, 1\rangle) = \{\langle 2, 1, 1\rangle\},$$

$$C^+_{637}(T_{3,2}, \langle 1, 2, 0\rangle) = \{\langle 1, 2, 0\rangle, \langle 1, 1, -1\rangle, \langle 1, 0, 2\rangle, \langle 1, -1, 1\rangle\},$$

$$C^+_{628}(T_{3,2}, \langle 1, 2, 1\rangle) = \{\langle 1, 2, 1\rangle, \langle 1, 1, 2\rangle, \langle 1, 0, -1\rangle, \langle 1, -1, 0\rangle\},$$

$$C^+_{613}(T_{3,2}, \langle 2, 2, 1\rangle) = \{\langle 2, 2, 1\rangle, \langle 2, 1, 2\rangle, \langle 2, 1, 0\rangle, \langle 2, 0, 1\rangle\},$$

$$C^+_{612}(T_{3,2}, \langle 0, 2, 0\rangle) = \{\langle 0, 2, 0\rangle, \langle 0, 0, 2\rangle\},$$

$$C^+_{597}(T_{3,2}, \langle 0, 2, -1\rangle) = \{\langle 0, 2, -1\rangle, \langle 0, 1, -2\rangle\},$$

$$C^+_{592}(T_{3,2}, \langle 2, 2, 0\rangle) = \{\langle 2, 2, 0\rangle, \langle 2, 0, 2\rangle, \langle 1, 2, -1\rangle, \langle 1, -1, 2\rangle\},$$

$$C^+_{580}(T_{3,2}, \langle 2, 2, 2\rangle) = \{\langle 2, 2, 2\rangle, \langle 2, 0, 0\rangle\},$$

$$C^+_{573}(T_{3,2}, \langle 0, 2, 1\rangle) = \{\langle 0, 2, 1\rangle, \langle 0, 1, 2\rangle\},$$

$$C^+_{565}(T_{3,2},\langle 1,2,2\rangle) = \{\langle 1,2,2\rangle,\langle 1,-1,-1\rangle\},$$

$$C^+_{532}(T_{3,2},\langle 2,3,1\rangle) = \{\langle 2,3,1\rangle,\langle 2,1,3\rangle,\langle 2,1,-1\rangle,\langle 2,-1,1\rangle,\langle 1,3,0\rangle,\langle 1,1,-2\rangle,\langle 1,0,3\rangle,\langle 1,-2,1\rangle\},$$

$$C^+_{528}(T_{3,2},\langle 3,2,1\rangle) = \{\langle 3,2,1\rangle,\langle 3,1,2\rangle,\langle 0,2,-2\rangle\},$$

$$C^+_{525}(T_{3,2},\langle 3,2,2\rangle) = \{\langle 3,2,2\rangle,\langle 3,1,1\rangle\},$$

$$C^+_{517}(T_{3,2},\langle 2,3,0\rangle) = \{\langle 2,3,0\rangle,\langle 2,2,-1\rangle,\langle 2,0,3\rangle,\langle 2,-1,2\rangle,\langle 1,3,1\rangle,\langle 1,1,3\rangle,\langle 1,0,-2\rangle,\langle 1,-2,0\rangle\},$$

$$C^+_{493}(T_{3,2},\langle 2,3,2\rangle) = \{\langle 2,3,2\rangle,\langle 2,2,3\rangle,\langle 2,0,-1\rangle,\langle 2,-1,0\rangle,\langle 1,3,-1\rangle,\langle 1,2,-2\rangle,\langle 1,-1,3\rangle,\langle 1,-2,2\rangle\},$$

$$C^+_{480}(T_{3,2},\langle 0,2,2\rangle) = \{\langle 0,2,2\rangle\},$$

$$C^+_{477}(T_{3,2},\langle 3,3,1\rangle) = \{\langle 3,3,1\rangle,\langle 3,2,0\rangle,\langle 3,1,3\rangle,\langle 3,0,2\rangle,\langle 0,3,0\rangle,\langle 0,0,3\rangle\},$$

$$C^+_{468}(T_{3,2},\langle 3,3,2\rangle) = \{\langle 3,3,2\rangle,\langle 3,2,3\rangle,\langle 3,1,0\rangle,\langle 3,0,1\rangle,\langle 0,3,-1\rangle,\langle 0,1,-3\rangle\},$$

$$C^+_{448}(T_{3,2},\langle 2,3,-1\rangle) = \{\langle 2,3,-1\rangle,\langle 2,-1,3\rangle,\langle 1,3,2\rangle,\langle 1,2,3\rangle,\langle 1,-1,-2\rangle,\langle 1,-2,-1\rangle\},$$

$$C^+_{432}(T_{3,2},\langle 3,3,0\rangle) = \{\langle 3,3,0\rangle,\langle 3,0,3\rangle,\langle 0,3,1\rangle,\langle 0,1,3\rangle\},$$

$$C^+_{405}(T_{3,2},\langle 3,3,3\rangle) = \{\langle 3,3,3\rangle,\langle 3,0,0\rangle,\langle 0,3,-2\rangle,\langle 0,2,-3\rangle\},$$

$$C^+_{400}(T_{3,2},\langle 2,3,3\rangle) = \{\langle 2,3,3\rangle,\langle 2,-1,-1\rangle,\langle 1,3,-2\rangle,\langle 1,-2,3\rangle\},$$

$$C^+_{0}(T_{3,2},\langle 6,3,3\rangle) = \{\langle 6,3,3\rangle,\langle 3,6,1\rangle,\langle 3,2,-3\rangle,\langle 3,1,6\rangle,\langle 3,-3,2\rangle\},$$

5.1.3. An Assembled Table of Quaternary Quadratic Forms

Table 4. Table of extended quaternary quadratic forms and its reduced forms

extended quaternary forms	reduced quaternary forms
$C_{324}(T_{3,1})$	$(3,3,3,3,0,0,0,3,3,3)$
$C_{400}(T_{3,2})$	$(3,3,3,3,2,2,2,-3,-3,1)$
$C_{405,1}(T_{3,1})$	$(3,3,3,3,3,0,0,3,0,3)$

Table 4 (continued)

extended quaternary forms	reduced quaternary forms
$C_{405}(T_{3,2}),\ C_{405,2}(T_{3,1})$	$(3,3,3,3,1,0,0,3,3,3)$
$C_{432}(T_{3,1}),\ C_{432}(T_{3,2})$	$(3,3,3,3,3,1,0,1,3,3)$
$C_{448}(T_{3,2})$	$(3,3,3,3,2,2,-2,1,1,3)$
$C_{468}(T_{3,1}),\ C_{468}(T_{3,2})$	$(3,3,3,3,1,1,0,3,3,3)$
$C_{477}(T_{3,1}),\ C_{477}(T_{3,2})$	$(3,3,3,3,2,1,0,0,3,3)$
$C_{480}(T_{3,2})$	$(3,3,3,3,2,0,0,2,-2,-3)$
$C_{493}(T_{3,2})$	$(3,3,3,3,3,2,0,1,2,3)$
$C_{504}(T_{3,1})$	$(3,3,3,3,3,2,0,3,0,0)$
$C_{517}(T_{3,2})$	$(3,3,3,3,3,2,-1,0,2,1)$
$C_{525}(T_{3,2})$	$(3,3,3,3,3,2,-1,2,-1,1)$
$C_{528}^{(1)}(T_{3,2})$	$(3,3,3,3,3,2,-1,2,2,2)$
$C_{528}^{(2)}(T_{3,2})$	$(3,3,3,3,3,2,-1,-1,2,-1)$
$C_{532}(T_{3,2})$	$(3,3,3,3,3,1,-1,-1,2,1)$
$C_{540}(T_{3,1})$	$(3,3,3,3,2,0,0,0,3,3)$
$C_{549}(T_{3,1})$	$(3,3,3,3,3,2,0,3,0,1)$
$C_{565}(T_{3,2})$	$(3,3,3,3,2,2,-1,3,3,1)$
$C_{573}(T_{3,2})$	$(3,3,3,3,2,2,-1,3,3,0)$
$C_{576}(T_{3,1})$	$(3,3,3,3,3,1,-1,3,0,1)$
$C_{580}(T_{3,2})$	$(3,3,3,3,3,2,-1,0,-2,2)$
$C_{592}^{(1)}(T_{3,2})$	$(3,3,3,3,3,2,0,-1,2,0)$
$C_{592}^{(2)}(T_{3,1})$	$(3,3,3,3,3,2,-1,1,2,1)$
$C_{597}(T_{3,2})$	$(3,3,3,3,2,2,1,3,3,0)$
$C_{612}^{(1)}(T_{3,1})$	$(3,3,3,3,3,2,-1,0,0,2)$
$C_{612}^{(2)}(T_{3,2})$	$(3,3,3,3,1,1,0,1,3,3)$
$C_{613}(T_{3,2})$	$(3,3,3,3,2,1,0,3,3,2)$
$C_{621}(T_{3,1})$	$(3,3,3,3,1,0,0,3,0,3)$
$C_{628}(T_{3,2})$	$(3,3,3,3,3,2,-1,0,-1,2)$

Table 4 (continued)

extended quaternary forms	reduced quaternary forms
$C_{637}(T_{3,2})$	$(3,3,3,3,1,1,-1,2,-3,0)$
$C_{640}(T_{3,2})$	$(3,3,3,3,1,1,1,3,3,2)$
$C_{648}^{(1)}(T_{3,1})$	$(3,3,3,3,0,0,0,3,3,0)$
$C_{660}(T_{3,2})$	$(3,3,3,3,3,2,-1,-1,-1,1)$
$C_{672}(T_{3,2})$	$(3,3,3,3,1,1,-1,3,0,3)$
$C_{685}(T_{3,2})$	$(3,3,3,3,3,2,-1,0,1,-1)$
$C_{688}(T_{3,2})$	$(3,3,3,3,3,2,-1,0,1,0)$
$C_{693}(T_{3,2})$	$(3,3,3,3,1,1,0,3,3,0)$
$C_{720}(T_{3,2})$	$(3,3,3,3,3,2,-1,0,0,0)$

5.2. Trials to Compute the Fourier Coefficients of Siegel Theta Series of Degree 4 for $\mathscr{L}$

5.2.1. First Case

We specialize the general notations in Section 4.1 to the case $s = 3$. For a fixed triple $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6$ satisfying $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,1}$ we set

$$\lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = |\{\mathbf{y} \in \Lambda_6 \mid (\mathbf{x}_1, \mathbf{y}) = a, (\mathbf{x}_2, \mathbf{y}) = b, (\mathbf{x}_3, \mathbf{y}) = c\}|.$$

Put $\boldsymbol{\alpha} = u\mathbf{x}_1 + v\mathbf{x}_2 + t\mathbf{x}_3$, where u, v, t are algebraically independent real variables.

$$\sum\nolimits_{\mathbf{y} \in \Lambda_6} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k}$$
$$= \sum\nolimits_{\langle a,b,c\rangle \in C_{621}(T_{3,1}, \langle 1,1,1\rangle)} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k}$$
$$+ \sum\nolimits_{\langle a,b,c\rangle \in C_{612}(T_{3,1}, \langle 1,1,0\rangle)} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k}$$
$$+ \sum\nolimits_{\langle a,b,c\rangle \in C_{576}(T_{3,1}, \langle 2,1,1\rangle)} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k}$$
$$+ \sum\nolimits_{\langle a,b,c\rangle \in C_{549}(T_{3,1}, \langle 2,2,1\rangle)} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k}$$
$$+ \sum\nolimits_{\langle a,b,c\rangle \in C_{540}(T_{3,1}, \langle 2,2,2\rangle)} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k}$$
$$+ \sum\nolimits_{\langle a,b,c\rangle \in C_{504}(T_{3,1}, \langle 2,2,0\rangle)} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k}$$

$$+\sum\nolimits_{\langle a,b,c\rangle\in C_{477}(T_{3,1},\langle 3,2,2\rangle)}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$
$$+\sum\nolimits_{\langle a,b,c\rangle\in C_{468}(T_{3,1},\langle 3,2,1\rangle)}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$
$$+\sum\nolimits_{\langle a,b,c\rangle\in C_{432}(T_{3,1},\langle 3,3,2\rangle)}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$
$$+\sum\nolimits_{\langle a,b,c\rangle\in C_{405,1}(T_{3,1},\langle 3,3,3\rangle)}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$
$$+\sum\nolimits_{\langle a,b,c\rangle\in C_{405,2}(T_{3,1},\langle 3,3,1\rangle)}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$
$$+\sum\nolimits_{\langle a,b,c\rangle\in C_{324}(T_{3,1},\langle 3,3,0\rangle)}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$
$$+\sum\nolimits_{\langle a,b,c\rangle\in C_{0}(T_{3,1},\langle 6,3,3\rangle)}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$
$$=\rho_k(6u^2+6v^2+6t^2+6uv+6ut+6vt)^k,\quad 1\le k\le 5.$$

By taking a sum over $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_6$ with $[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2T_{3,1}$ in the last equation we have

$$(5.1)\qquad \sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2T_{3,1}}}\ \sum_{\substack{\langle a,b,c\rangle\in\\ C_{621}(T_{3,1},\langle 1,1,1\rangle)}}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$
$$+\sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2T_{3,1}}}\ \sum_{\substack{\langle a,b,c\rangle\in\\ C_{612}(T_{3,1},\langle 1,1,0\rangle)}}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$
$$+\sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2T_{3,1}}}\ \sum_{\substack{\langle a,b,c\rangle\in\\ C_{576}(T_{3,1},\langle 2,1,1\rangle)}}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$
$$+\sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2T_{3,1}}}\ \sum_{\substack{\langle a,b,c\rangle\in\\ C_{549}(T_{3,1},\langle 2,2,1\rangle)}}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$
$$+\sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2T_{3,1}}}\ \sum_{\substack{\langle a,b,c\rangle\in\\ C_{540}(T_{3,1},\langle 2,2,2\rangle)}}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$
$$+\sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2T_{3,1}}}\ \sum_{\substack{\langle a,b,c\rangle\in\\ C_{504}(T_{3,1},\langle 2,2,0\rangle)}}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$
$$+\sum_{\substack{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\in\Lambda_6\\ [\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]=2T_{3,1}}}\ \sum_{\substack{\langle a,b,c\rangle\in\\ C_{477}(T_{3,1},\langle 3,2,2\rangle)}}\lambda_{a,b,c}(2T_{3,1};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+ \sum_{\substack{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6 \\ [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,1}}} \sum_{\substack{\langle a,b,c \rangle \in \\ C_{468}(T_{3,1}, \langle 3,2,1 \rangle)}} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+ \sum_{\substack{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6 \\ [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,1}}} \sum_{\substack{\langle a,b,c \rangle \in \\ C_{432}(T_{3,1}, \langle 3,3,2 \rangle)}} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+ \sum_{\substack{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6 \\ [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,1}}} \sum_{\substack{\langle a,b,c \rangle \in \\ C_{405,1}(T_{3,1}, \langle 3,3,3 \rangle)}} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+ \sum_{\substack{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6 \\ [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,1}}} \sum_{\substack{\langle a,b,c \rangle \in \\ C_{405,2}(T_{3,1}, \langle 3,3,1 \rangle)}} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+ \sum_{\substack{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6 \\ [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,1}}} \sum_{\substack{\langle a,b,c \rangle \in \\ C_{324}(T_{3,1}, \langle 3,3,0 \rangle)}} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+ \sum_{\substack{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6 \\ [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,1}}} \sum_{\substack{\langle a,b,c \rangle \in \\ C_{0}(T_{3,1}, \langle 6,3,3 \rangle)}} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au+bv+ct)^{2k}$$

$$= \sum_{\substack{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6 \\ [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,1}}} \rho_k (6u^2 + 6v^2 + 6t^2 + 6uv + 6ut + 6vt)^k.$$

We observe that

$$\sum_{\substack{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6 \\ [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,1}}} \sum_{\substack{\langle a,b,c \rangle \in \\ C_{621}(T_{3,1}, \langle 1,1,1 \rangle)}} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au+bv+ct)^{2k}$$

$$= \sum_{\substack{\langle a,b,c \rangle \in \\ C_{621}(T_{3,1}, \langle 1,1,1 \rangle)}} \sum_{\substack{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6 \\ [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,1}}} \lambda_{a,b,c}(2T_{3,1}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au+bv+ct)^{2k}$$

$$= \sum_{\substack{\langle a,b,c \rangle \in \\ C_{621}(T_{3,1}, \langle 1,1,1 \rangle)}} a((T_{3,1}, \{a/2, b/2, c/2\}, 3), \mathscr{L}_{48})(au+bv+ct)^{2k}.$$

$$= a((T_{3,1}, \{1/2, 1/2, 1/2\}, 3), \mathscr{L}_{48}) \sum_{\substack{\langle a,b,c \rangle \in \\ C_{621}(T_{3,1}, \langle 1,1,1 \rangle)}} (au+bv+ct)^{2k}.$$

The last transition is certified by Proposition 2.4, since each triple $\langle a,b,c \rangle \in C_{621}(T_{3,1}, \langle 1,1,1 \rangle)$ induces the integrally equivalent matrix $(T_{3,1}, \{a/2, b/2, c/2\}, 3)$ to each other.

Other terms in (5.1) are treated in the same way. We rewrite (5.1) as

$$(5.2)\quad a((T_{3,1},\{1/2,1/2,1/2\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{621}(T_{3,1},\langle 1,1,1\rangle)}}(au+bv+ct)^{2k}$$
$$+a((T_{3,1},\{1/2,1/2,0\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{612}(T_{3,1},\langle 1,1,0\rangle)}}(au+bv+ct)^{2k}$$
$$+a((T_{3,1},\{1,1/2,1/2\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{576}(T_{3,1},\langle 2,1,1\rangle)}}(au+bv+ct)^{2k}$$
$$+a((T_{3,1},\{1,1,1/2\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{549}(T_{3,1},\langle 2,2,1\rangle)}}(au+bv+ct)^{2k}$$
$$+a((T_{3,1},\{1,1,1\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{540}(T_{3,1},\langle 2,2,2\rangle)}}(au+bv+ct)^{2k}$$
$$+a((T_{3,1},\{1,1,0\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{504}(T_{3,1},\langle 2,2,0\rangle)}}(au+bv+ct)^{2k}$$
$$+a((T_{3,1},\{3/2,1,1\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{477}(T_{3,1},\langle 3,2,2\rangle)}}(au+bv+ct)^{2k}$$
$$+a((T_{3,1},\{3/2,1,1/2\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{468}(T_{3,1},\langle 3,2,1\rangle)}}(au+bv+ct)^{2k}$$
$$+a((T_{3,1},\{3/2,3/2,1\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{432}(T_{3,1},\langle 3,3,2\rangle)}}(au+bv+ct)^{2k}$$
$$+a((T_{3,1},\{3/2,3/2,3/2\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{405,1}(T_{3,1},\langle 3,3,3\rangle)}}(au+bv+ct)^{2k}$$
$$+a((T_{3,1},\{3/2,3/2,1/2\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{405,2}(T_{3,1},\langle 3,3,1\rangle)}}(au+bv+ct)^{2k}$$
$$+a((T_{3,1},\{3/2,3/2,0\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{324}(T_{3,1},\langle 3,3,0\rangle)}}(au+bv+ct)^{2k}$$

$$+ \sum_{\substack{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6 \\ [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,1}}} \sum_{\substack{\langle a, b, c \rangle \in \\ C_0(T_{3,1}, \langle 6, 3, 3 \rangle)}} (au + bv + ct)^{2k}$$

$$= \sum_{\substack{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6 \\ [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,1}}} \rho_k (6u^2 + 6v^2 + 6t^2 + 6uv + 6ut + 6vt)^k.$$

The above equations are polynomial identities concerning the independent real variables u, v, t, and by comparing the coefficients we get many linear equations on the quantities $a((T_{3,1}, \{1/2, 1/2, 1/2\}, 3), \mathscr{L}_{48}), \dots, a((T_{3,1}, \{3/2, 3/2, 0\}, 3), \mathscr{L}_{48})$. The linear equations are not many enough to solve them separately, but we have strong relations among them. We summarize our effort as a proposition.

PROPOSITION 5.3. *Let $T_{4,*}$ be some indeces given in Table 5 below. Then the Fourier coefficients $a(T_{4,*}, L)$ are expressed as*

(i) $a(T_{4,1}, \mathscr{L}_{48}) = 278\, a(T_{3,1}, \mathscr{L}_{48}) - a(T_{4,3}, \mathscr{L}_{48})$,

(ii) $a(T_{4,4}, \mathscr{L}_{48}) = -552\, a(T_{3,1}, \mathscr{L}_{48}) + 3\, a(T_{4,3}, \mathscr{L}_{48})$,

(iii) $a(T_{4,5}, \mathscr{L}_{48}) = 1929\, a(T_{3,1}, \mathscr{L}_{48}) - 3\, a(T_{4,3}, \mathscr{L}_{48})$,

(iv) $a(T_{4,7}, \mathscr{L}_{48}) = 8622\, a(T_{3,1}, \mathscr{L}_{48}) - 9\, a(T_{4,3}, \mathscr{L}_{48})$,

(v) $a(T_{4,8}, \mathscr{L}_{48}) = 6768\, a(T_{3,1}, \mathscr{L}_{48}) + 9\, a(T_{4,3}, \mathscr{L}_{48})$,

(vi) $a(T_{4,11}, \mathscr{L}_{48}) = 35064\, a(T_{3,1}, \mathscr{L}_{48}) - 18\, a(T_{4,3}, \mathscr{L}_{48})$,

(vii) $a(T_{4,17}, \mathscr{L}_{48}) = 140952\, a(T_{3,1}, \mathscr{L}_{48}) - 30\, a(T_{4,3}, \mathscr{L}_{48})$,

(viii) $a(T_{4,18}, \mathscr{L}_{48}) = 179352\, a(T_{3,1}, \mathscr{L}_{48}) + 36\, a(T_{4,3}, \mathscr{L}_{48})$,

(ix) $a(T_{4,21}, \mathscr{L}_{48}) = 546237\, a(T_{3,1}, \mathscr{L}_{48}) - 54\, a(T_{4,3}, \mathscr{L}_{48})$,

(x) $a(T_{4,27}, \mathscr{L}_{48}) = 1983762\, a(T_{3,1}, \mathscr{L}_{48}) - 99\, a(T_{4,3}, \mathscr{L}_{48})$,

(xi) $a(T_{4,29}, \mathscr{L}_{48}) = 2647392\, a(T_{3,1}, \mathscr{L}_{48}) + 111\, a(T_{4,3}, \mathscr{L}_{48})$,

(xii) $a(T_{4,33}, \mathscr{L}_{48}) = 6732912\, a(T_{3,1}, \mathscr{L}_{48}) - 164\, a(T_{4,3}, \mathscr{L}_{48})$.

5.2.2. Second Case

For a fixed triple $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \Lambda_6$ satisfying $[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] = 2T_{3,2}$ we set

$$\lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = |\{\mathbf{y} \in \Lambda_6 \mid (\mathbf{x}_1, \mathbf{y}) = a, (\mathbf{x}_2, \mathbf{y}) = b, (\mathbf{x}_3, \mathbf{y}) = c\}|.$$

Put $\boldsymbol{\alpha} = u\mathbf{x}_1 + v\mathbf{x}_2 + t\mathbf{x}_3$, where u, v, t are algebraically independent real variables. We verify that $(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 6u^2 + 6v^2 + 6t^2 + 6uv + 6ut + 2vt$ and $(\boldsymbol{\alpha}, \mathbf{y}) = au + bv + ct$ with certain integers a, b, c. By applying the formulas (2.1)~(2.5) we have

$$
\begin{aligned}
&\sum\nolimits_{\mathbf{y} \in \Lambda_6} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\quad = \sum\nolimits_{\langle a,b,c\rangle \in C_{693}(T_{3,2}, \langle 0,1,0\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{688}(T_{3,2}, \langle 1,1,0\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{685}(T_{3,2}, \langle 1,1,1\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{672}(T_{3,2}, \langle 0,1,-1\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{660}(T_{3,2}, \langle 0,1,1\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{640}(T_{3,2}, \langle 2,1,1\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{637}(T_{3,2}, \langle 1,2,0\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{628}(T_{3,2}, \langle 1,2,1\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{613}(T_{3,2}, \langle 2,2,1\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{612}(T_{3,2}, \langle 0,2,0\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{597}(T_{3,2}, \langle 0,2,-1\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{592}(T_{3,2}, \langle 2,2,0\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{580}(T_{3,2}, \langle 1,2,0\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{573}(T_{3,2}, \langle 0,2,1\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{565}(T_{3,2}, \langle 1,2,2\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{532}(T_{3,2}, \langle 2,3,1\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{528}(T_{3,2}, \langle 3,2,1\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k} \\
&\qquad + \sum\nolimits_{\langle a,b,c\rangle \in C_{525}(T_{3,2}, \langle 3,2,2\rangle)} \lambda_{a,b,c}(2T_{3,2}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)(au + bv + ct)^{2k}
\end{aligned}
$$

$$+\sum_{\langle a,b,c\rangle\in C_{517}(T_{3,2},\langle 2,3,0\rangle)}\lambda_{a,b,c}(2T_{3,2};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+\sum_{\langle a,b,c\rangle\in C_{493}(T_{3,2},\langle 2,3,2\rangle)}\lambda_{a,b,c}(2T_{3,2};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+\sum_{\langle a,b,c\rangle\in C_{480}(T_{3,2},\langle 0,2,2\rangle)}\lambda_{a,b,c}(2T_{3,2};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+\sum_{\langle a,b,c\rangle\in C_{477}(T_{3,2},\langle 3,3,1\rangle)}\lambda_{a,b,c}(2T_{3,2};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+\sum_{\langle a,b,c\rangle\in C_{468}(T_{3,2},\langle 3,3,2\rangle)}\lambda_{a,b,c}(2T_{3,2};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+\sum_{\langle a,b,c\rangle\in C_{448}(T_{3,2},\langle 2,3,-1\rangle)}\lambda_{a,b,c}(2T_{3,2};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+\sum_{\langle a,b,c\rangle\in C_{432}(T_{3,2},\langle 3,3,0\rangle)}\lambda_{a,b,c}(2T_{3,2};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+\sum_{\langle a,b,c\rangle\in C_{405}(T_{3,2},\langle 3,3,3\rangle)}\lambda_{a,b,c}(2T_{3,2};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+\sum_{\langle a,b,c\rangle\in C_{400}(T_{3,2},\langle 2,3,3\rangle)}\lambda_{a,b,c}(2T_{3,2};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$

$$+\sum_{\langle a,b,c\rangle\in C_{0}(T_{3,2},\langle 6,3,3\rangle)}\lambda_{a,b,c}(2T_{3,2};\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)(au+bv+ct)^{2k}$$

$$=\rho_k(6u^2+6v^2+6t^2+6uv+6ut+2vt)^k,\quad 1\le k\le 5.$$

From here the arguement is quite similar to that of Section 5.2.1 and we have

$$a((T_{3,2},\{0,1/2,0\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{693}(T_{3,2},\langle 0,1,0\rangle)}}(au+bv+ct)^{2k}$$

$$+\,a((T_{3,2},\{1/2,1/2,0\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{688}(T_{3,2},\langle 1,1,0\rangle)}}(au+bv+ct)^{2k}$$

$$+\,a((T_{3,2},\{1/2,1/2,1/2\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{685}(T_{3,2},\langle 1,1,1\rangle)}}(au+bv+ct)^{2k}$$

$$+\,a((T_{3,2},\{0,1/2,-1/2\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{672}(T_{3,2},\langle 0,1,-1\rangle)}}(au+bv+ct)^{2k}$$

$$+\,a((T_{3,2},\{0,1/2,1/2\},3),\mathscr{L}_{48})\sum_{\substack{\langle a,b,c\rangle\in\\ C_{660}(T_{3,2},\langle 0,1,1\rangle)}}(au+bv+ct)^{2k}$$

$$+ a((T_{3,2},\{1,1/2,1/2\},3),\mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle\in\\ C_{640}(T_{3,2},\langle 2,1,1\rangle)}} (au+bv+ct)^{2k}$$

$$+ a((T_{3,2},\{1/2,1,0\},3),\mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle\in\\ C_{637}(T_{3,2},\langle 1,2,0\rangle)}} (au+bv+ct)^{2k}$$

$$+ a((T_{3,2},\{1/2,1,1/2\},3),\mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle\in\\ C_{628}(T_{3,2},\langle 1,2,1\rangle)}} (au+bv+ct)^{2k}$$

$$+ a((T_{3,2},\{1,1,1/2\},3),\mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle\in\\ C_{613}(T_{3,2},\langle 2,2,1\rangle)}} (au+bv+ct)^{2k}$$

$$+ a((T_{3,2},\{0,1,0\},3),\mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle\in\\ C_{612}(T_{3,2},\langle 0,2,0\rangle)}} (au+bv+ct)^{2k}$$

$$+ a((T_{3,2},\{0,1,-1/2\},3),\mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle\in\\ C_{597}(T_{3,2},\langle 0,2,-1\rangle)}} (au+bv+ct)^{2k}$$

$$+ a((T_{3,2},\{1,1,0\},3),\mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle\in\\ C_{592}(T_{3,2},\langle 2,2,0\rangle)}} (au+bv+ct)^{2k}$$

$$+ a((T_{3,2},\{1,1,1\},3),\mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle\in\\ C_{580}(T_{3,2},\langle 2,2,2\rangle)}} (au+bv+ct)^{2k}$$

$$+ a((T_{3,2},\{1/2,1,0\},3),\mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle\in\\ C_{573}(T_{3,2},\langle 0,2,1\rangle)}} (au+bv+ct)^{2k}$$

$$+ a((T_{3,2},\{1/2,1,1\},3),\mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle\in\\ C_{565}(T_{3,2},\langle 1,2,2\rangle)}} (au+bv+ct)^{2k}$$

$$+ a((T_{3,2},\{1,3/2,1\},3),\mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle\in\\ C_{532}(T_{3,2},\langle 2,3,1\rangle)}} (au+bv+ct)^{2k}$$

$$+ a((T_{3,2},\{3/2,1,1/2\},3),\mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle\in\\ C_{528}(T_{3,2},\langle 3,2,1\rangle)}} (au+bv+ct)^{2k}$$

$$+ a((T_{3,2},\{3/2,1,1\},3),\mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle\in\\ C_{525}(T_{3,2},\langle 3,2,2\rangle)}} (au+bv+ct)^{2k}$$

$$+ a((T_{3,2}, \{1, 3/2, 0\}, 3), \mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle \in \\ C_{517}(T_{3,2}, \langle 2,3,0\rangle)}} (au + bv + ct)^{2k}$$

$$+ a((T_{3,2}, \{1, 3/2, 1\}, 3), \mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle \in \\ C_{493}(T_{3,2}, \langle 2,3,2\rangle)}} (au + bv + ct)^{2k}$$

$$+ a((T_{3,2}, \{0, 1, 1\}, 3), \mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle \in \\ C_{480}(T_{3,2}, \langle 0,2,2\rangle)}} (au + bv + ct)^{2k}$$

$$+ a((T_{3,2}, \{3/2, 3/2, 1/2\}, 3), \mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle \in \\ C_{477}(T_{3,2}, \langle 3,3,1\rangle)}} (au + bv + ct)^{2k}$$

$$+ a((T_{3,2}, \{3/2, 3/2, 1\}, 3), \mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle \in \\ C_{468}(T_{3,2}, \langle 3,3,2\rangle)}} (au + bv + ct)^{2k}$$

$$+ a((T_{3,2}, \{1, 3/2, -1/2\}, 3), \mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle \in \\ C_{448}(T_{3,2}, \langle 2,3,-1\rangle)}} (au + bv + ct)^{2k}$$

$$+ a((T_{3,2}, \{0, 1, -1/2\}, 3), \mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle \in \\ C_{432}(T_{3,2}, \langle 3,3,0\rangle)}} (au + bv + ct)^{2k}$$

$$+ a((T_{3,2}, \{3/2, 3/2, 3/2\}, 3), \mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle \in \\ C_{405}(T_{3,2}, \langle 3,3,3\rangle)}} (au + bv + ct)^{2k}$$

$$+ a((T_{3,2}, \{1, 3/2, 3/2\}, 3), \mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle \in \\ C_{400}(T_{3,2}, \langle 2,3,3\rangle)}} (au + bv + ct)^{2k}$$

$$+ a(T_{3,2}, \mathscr{L}_{48}) \sum_{\substack{\langle a,b,c\rangle \in \\ C_{0}(T_{3,2}, \langle 6,3,3\rangle)}} (au + bv + ct)^{2k}$$

$$= a(T_{3,2}, \mathscr{L}_{48})\rho_k(6u^2 + 6v^2 + 6t^2 + 6uv + 6ut + 2vt)^k, \quad 1 \le k \le 5.$$

Proposition 5.4. *Let $T_{4,*}$ be some indeces given in Table 5 below. Then the Fourier coefficients $a(T_{4,*}, L)$ are expressed as*

(i) $a(T_{4,2}, \mathscr{L}_{48}) = 48\, a(T_{3,2}, \mathscr{L}_{48}) - a(T_{4,4}, \mathscr{L}_{48})$,

(ii) $a(T_{4,5}, \mathscr{L}_{48}) = 136\, a(T_{3,2}, \mathscr{L}_{48}) - a(T_{4,4}, \mathscr{L}_{48})$,

(iii) $a(T_{4,6}, \mathscr{L}_{48}) = 183\, a(T_{3,2}, \mathscr{L}_{48}) + 2\, a(T_{4,4}, \mathscr{L}_{48})$,

(iv) $a(T_{4,7}, \mathscr{L}_{48}) = 688\ a(T_{3,2}, \mathscr{L}_{48}) - 3\ a(T_{4,4}, \mathscr{L}_{48})$,

(v) $a(T_{4,8}, \mathscr{L}_{48}) = 832\ a(T_{3,2}, \mathscr{L}_{48}) + 3\ a(T_{4,4}, \mathscr{L}_{48})$,

(vi) $a(T_{4,9}, \mathscr{L}_{48}) = 1152\ a(T_{3,2}, \mathscr{L}_{48}) - 4\ a(T_{4,4}, \mathscr{L}_{48})$,

(vii) $a(T_{4,10}, \mathscr{L}_{48}) = 1824\ a(T_{3,2}, \mathscr{L}_{48}) + a(T_{4,4}, \mathscr{L}_{48})$,

(viii) $a(T_{4,12}, \mathscr{L}_{48}) = 5280\ a(T_{3,2}, \mathscr{L}_{48}) - 5\ a(T_{4,4}, \mathscr{L}_{48})$,

(ix) $a(T_{4,13}, \mathscr{L}_{48}) = 6912\ a(T_{3,2}, \mathscr{L}_{48}) + 9\ a(T_{4,4}, \mathscr{L}_{48})$,

(x) $a(T_{4,15}, \mathscr{L}_{48}) = 8112\ a(T_{3,2}, \mathscr{L}_{48}) - 2\ a(T_{4,4}, \mathscr{L}_{48})$,

(xi) $a(T_{4,14}, \mathscr{L}_{48}) = 8334\ a(T_{3,2}, \mathscr{L}_{48}) - 9\ a(T_{4,4}, \mathscr{L}_{48})$,

(xii) $a(T_{4,16}, \mathscr{L}_{48}) = 9432\ a(T_{3,2}, \mathscr{L}_{48}) + 3\ a(T_{4,4}, \mathscr{L}_{48})$,

(xiii) $a(T_{4,19}, \mathscr{L}_{48}) = 34752\ a(T_{3,2}, \mathscr{L}_{48}) - 7\ a(T_{4,4}, \mathscr{L}_{48})$,

(xiv) $a(T_{4,20}, \mathscr{L}_{48}) = 46464\ a(T_{3,2}, \mathscr{L}_{48}) + 11\ a(T_{4,4}, \mathscr{L}_{48})$,

(xv) $a(T_{4,22}, \mathscr{L}_{48}) = 60672\ a(T_{3,2}, \mathscr{L}_{48}) + 2\ a(T_{4,4}, \mathscr{L}_{48})$,

(xvi) $a(T_{4,24}, \mathscr{L}_{48}) = 94362\ a(T_{3,2}, \mathscr{L}_{48}) - 2\ a(T_{4,4}, \mathscr{L}_{48})$,

(xvii) $a(T_{4,23}, \mathscr{L}_{48}) = 93936\ a(T_{3,2}, \mathscr{L}_{48}) + 14\ a(T_{4,4}, \mathscr{L}_{48})$,

(xviii) $a(T_{4,25}, \mathscr{L}_{48}) = 112896\ a(T_{3,2}, \mathscr{L}_{48}) + 4\ a(T_{4,4}, \mathscr{L}_{48})$,

(xix) $a(T_{4,26}, \mathscr{L}_{48}) = 192960\ a(T_{3,2}, \mathscr{L}_{48}) - 10\ a(T_{4,4}, \mathscr{L}_{48})$,

(xx) $a(T_{4,28}, \mathscr{L}_{48}) = 199872\ a(T_{3,2}, \mathscr{L}_{48}) - 12\ a(T_{4,4}, \mathscr{L}_{48})$,

(xxi) $a(T_{4,30}, \mathscr{L}_{48}) = 335352\ a(T_{3,2}, \mathscr{L}_{48}) + 8\ a(T_{4,4}, \mathscr{L}_{48})$,

(xxii) $a(T_{4,31}, \mathscr{L}_{48}) = 455712\ a(T_{3,2}, \mathscr{L}_{48}) - 2\ a(T_{4,4}, \mathscr{L}_{48})$,

(xxiii) $a(T_{4,32}, \mathscr{L}_{48}) = 503577\ a(T_{3,2}, \mathscr{L}_{48}) + 18\ a(T_{4,4}, \mathscr{L}_{48})$,

(xxiv) $a(T_{4,34}, \mathscr{L}_{48}) = 977472\ a(T_{3,2}, \mathscr{L}_{48}) - 22\ a(T_{4,4}, \mathscr{L}_{48})$,

(xxv) $a(T_{4,35}, \mathscr{L}_{48}) = 1439112\ a(T_{3,2}, \mathscr{L}_{48}) - 2\ a(T_{4,4}, \mathscr{L}_{48})$,

(xxvi) $a(T_{4,36}, \mathscr{L}_{48}) = 2172672\ a(T_{3,2}, \mathscr{L}_{48}) + 3\ a(T_{4,4}, \mathscr{L}_{48})$,

(xxvii) $a(T_{4,37}, \mathscr{L}_{48}) = 2386992\ a(T_{3,2}, \mathscr{L}_{48}) - 7\ a(T_{4,4}, \mathscr{L}_{48})$,

(xxviii) $a(T_{4,38}, \mathscr{L}_{48}) = 2788672\ a(T_{3,2}, \mathscr{L}_{48}) + 13\ a(T_{4,4}, \mathscr{L}_{48})$,

(xxix) $a(T_{4,39}, \mathscr{L}_{48}) = 6344672\ a(T_{3,2}, \mathscr{L}_{48}) - 12\ a(T_{4,4}, \mathscr{L}_{48})$.

REMARK 6. There are four overlapping cases in Proposition 5.3 and Proposition 5.4. Those are $a(T_{4,4}, \mathscr{L}_{48}), a(T_{4,5}, \mathscr{L}_{48}), a(T_{4,7}, \mathscr{L}_{48})$ and $a(T_{4,8}, \mathscr{L}_{48})$. Therefore these overlapping cases should be consistent. First we have $a(T_{4,4}, \mathscr{L}_{48}) = -552\, a(T_{3,1}, \mathscr{L}_{48}) + 3\, a(T_{4,3}, \mathscr{L}_{48})$. Next we have $a(T_{4,5}, \mathscr{L}_{48}) = 1929\, a(T_{3,1}, \mathscr{L}_{48}) - 3\, a(T_{4,3}, \mathscr{L}_{48}) = 136\, a(T_{3,2}, \mathscr{L}_{48}) - a(T_{4,4}, \mathscr{L}_{48})$. From these we must have $1929\, a(T_{3,1}, \mathscr{L}_{48}) = 136\, a(T_{3,2}, \mathscr{L}_{48}) + 552\, a(T_{3,1}, \mathscr{L}_{48})$. This equality is verified to be true by using the values $a(T_{3,1}, \mathscr{L}_{48}), a(T_{3,2}, \mathscr{L}_{48})$ given in Section 4.3.1. Other overlapping cases are also verified to be true. By using $a(T_{4,4}, \mathscr{L}_{48}) = -552\, a(T_{3,1}, \mathscr{L}_{48}) + 3\, a(T_{4,3}, \mathscr{L}_{48})$ all $a(T_{4,j}, \mathscr{L}_{48})$ $1 \le j \le 39$ can be expressed as linear functions of $a(T_{4,4}, \mathscr{L}_{48})$ solely. This reflects Table 5 below.

6. Some Mathematical Statements

Salvati Manni [15] Theorem 3 showed

THEOREM 6.1 (Salvati Manni). *Let us assume $N = 32, 48$ (the dimensions of the lattices); then about the theta series associated to extremal lattices we can say*

(i) *it is unique in degree* 3,

(ii) *in degree* 4 *their difference is, up to a multiplicative constant (possibly* 0), *equal to a power of Schottky's polynomial J.*

We normalize that the Fourier coefficient of J^3 at the index $T_{4,3}$ is 1.

THEOREM 6.2. *Let $\Theta_4(Z, \mathscr{L}_{48}) = \sum_T a(T, \mathscr{L}_{48}) e^{2\pi i \sigma(TZ)}$ be the Fourier expansion of Siegel theta series of degree* 4 *for any even unimodular extremal* 48*-dimensional lattice $\mathscr{L}_{48}$. Then we have*

(1) *$\Theta_4(Z, \mathscr{L}_{48})$ is uniquely determined by the value of the Fourier coefficient at the index $T_{4,3}$ which is equivalent to $(T_{3,1}, \{3,3,3\}, 3)$.*

(2) *The series defined by*

$$P\Theta_4(Z) = \Theta_4(Z, \mathscr{L}_{48}) - a(T_{4,3}, \mathscr{L}_{48}) J^3$$

is a Siegel modular form of degree 4 *and weight* 16, *and independent of the choice of extremal lattice $\mathscr{L}_{48}$.*

PROOF. (1) follows from (2). The latter part of (2) can be obtained from Salvati Manni's Theorem. Indeed, that Theorem says the equality

$$\Theta_4(Z, \mathscr{L}_{48}^{(1)}) - \Theta_4(Z, \mathscr{L}_{48}^{(2)}) = cJ^3$$

holds for certain constant c. We compare the both sides of the above equation at the index $T_{4,3}$ in their Fourier expansions. The left hand equals $(a(T_{4,3}, \mathscr{L}_{48}^{(1)}) - a(T_{4,3}, \mathscr{L}_{48}^{(2)}))$ whereas the right hand equals c. This implies that

$$\Theta_4(Z, \mathscr{L}_{48}^{(1)}) - \Theta_4(Z, \mathscr{L}_{48}^{(2)}) = (a(T_{4,3}, \mathscr{L}_{48}^{(1)}) - a(T_{4,3}, \mathscr{L}_{48}^{(2)}))J^3,$$

or

$$\Theta_4(Z, \mathscr{L}_{48}^{(1)}) - a(T_{4,3}, \mathscr{L}_{48}^{(1)})J^3 = \Theta_4(Z, \mathscr{L}_{48}^{(2)}) - a(T_{4,3}, \mathscr{L}_{48}^{(2)})J^3.$$

for any two 48-dimensional even unimodular extremal lattices $\mathscr{L}_{48}^{(1)}$, $\mathscr{L}_{48}^{(2)}$. □

We call $P\Theta_4(Z)$ a pan theta series in 48 dimension. As to the range of $a(T_{4,3}, \mathscr{L}_{48})$ we have simple bounds.

Theorem 6.3. *Let $\mathscr{L}_{48}$ be any even unimodular* 48 *dimensional extremal lattice, and $a(T_{4,3}, \mathscr{L}_{48})$ be the Fourier coefficients at the index $T_{4,3}$. Then we have the inequalities*

$$184\, a(T_{3,1}, \mathscr{L}_{48}) \le a(T_{4,3}, \mathscr{L}_{48}) \le 278\, a(T_{3,1}, \mathscr{L}_{48}).$$

Proof. These inequalities are derived from the non-negativities of $a(T_{4,1}, \mathscr{L}_{48})$ and $a(T_{4,4}, \mathscr{L}_{48})$ in Proposition 5.3. □

Table 5. Fourier coefficients of the cube of Schottky modular form J and the 48 dimensional pan theta series

D	reduced form	J^3	48 dim pan *Theta*
*324	$T_{4,1} = (3,3,3,3,0,0,0,3,3,3)$	-1	889166157594624000
400	$T_{4,2} = (3,3,3,3,2,2,2,-3,-3,1)$	-3	3319980113608704000
*405	$T_{4,3} = (3,3,3,3,3,0,0,3,0,3)$	1	0
405	$T_{4,4} = (3,3,3,3,1,0,0,3,3,3)$	3	-1765538557526016000
432	$T_{4,5} = (3,3,3,3,3,1,0,1,3,3)$	-3	6169789633093632000
448	$T_{4,6} = (3,3,3,3,2,2,-2,1,1,3)$	6	2395231317513216000
468	$T_{4,7} = (3,3,3,3,1,1,0,3,3,3)$	-9	27576944643096576000
477	$T_{4,8} = (3,3,3,3,2,1,0,0,3,3)$	9	21647037966188544000
480	$T_{4,9} = (3,3,3,3,2,0,0,2,-2,-3)$	-12	44368751576088576000
493	$T_{4,10} = (3,3,3,3,3,2,0,1,2,3)$	3	57303240573616128000
504	$T_{4,11} = (3,3,3,3,3,2,0,3,0,0)$	-18	112150079675891712000
512	$(3,3,3,3,2,2,-2,-2,-2,2)$	24	
517	$T_{4,12} = (3,3,3,3,3,2,-1,0,2,1)$	-15	179816263956725760000
525	$T_{4,13} = (3,3,3,3,3,2,-1,2,-1,1)$	27	207949737058172928000
528	$T_{4,14} = (3,3,3,3,3,2,-1,2,2,2)$	-6	266231700093026304000
528	$T_{4,15} = (3,3,3,3,3,2,-1,-1,2,-1)$	-27	285779762192590848000
529	$(3,3,3,3,3,2,0,2,2,3)$	-42	

Table 5 (continued)

D	reduced form	J^3	48 dim pan *Theta*
532	$T_{4,16} = (3,3,3,3,3,1,-1,-1,2,1)$	9	300151150097670144000
540	$T_{4,17} = (3,3,3,3,2,0,0,0,3,3)$	−30	450826432536969216000
544	$(3,3,3,3,2,2,-1,0,1,3)$	12	
549	$T_{4,18} = (3,3,3,3,3,2,0,3,0,1)$	36	573646506103996416000
565	$T_{4,19} = (3,3,3,3,2,2,-1,3,3,1)$	−21	1137774456506548224000
573	$T_{4,20} = (3,3,3,3,2,2,-1,3,3,0)$	33	1485278502155255808000
576	$(3,3,3,3,3,1,-1,2,1,3)$	−72	
576	$T_{4,21} = (3,3,3,3,3,1,-1,3,0,1)$	−54	1747105951172714496000
576	$(3,3,3,3,3,1,-1,-1,1,2)$	−36	
580	$T_{4,22} = (3,3,3,3,3,2,-1,0,-2,2)$	6	1961283049773465600000
585	$(3,3,3,3,3,2,0,0,0,3)$	−6	
588	$(3,3,3,3,2,2,0,1,-2,-2)$	−48	
589	$(3,3,3,3,3,2,0,1,1,3)$	60	
592	$T_{4,23} = (3,3,3,3,3,2,0,-1,2,0)$	42	3017324585448456192000
592	$T_{4,24} = (3,3,3,3,3,2,-1,1,2,1)$	−6	3059368871179106304000
597	$T_{4,25} = (3,3,3,3,2,2,1,3,3,0)$	12	3648984385676378112000
605	$(3,3,3,3,2,2,-1,-1,2,1)$	33	
608	$(3,3,3,3,2,2,-2,-1,2,-1)$	−12	
609	$(3,3,3,3,1,0,0,2,-2,3)$	6	
612	$T_{4,26} = (3,3,3,3,3,2,-1,0,0,2)$	−30	6266510441027665920000
612	$T_{4,27} = (3,3,3,3,1,1,0,1,3,3)$	−99	6344942572382109696000
613	$T_{4,28} = (3,3,3,3,2,1,0,3,3,2)$	−36	6493881102218625024000
621	$(3,3,3,3,3,1,-1,2,1,-2)$	−45	
621	$(3,3,3,3,3,2,0,1,0,3)$	−126	
621	$T_{4,29} = (3,3,3,3,1,0,0,3,0,3)$	111	8467522921894772736000
628	$T_{4,30} = (3,3,3,3,3,2,-1,0,-1,2)$	24	10845981623111491584000
628	$(3,3,3,3,2,2,1,0,3,2)$	54	
636	$(3,3,3,3,2,2,-1,0,3,0)$	18	
637	$T_{4,31} = (3,3,3,3,1,1,-1,2,-3,0)$	−6	14761399210564091904000
640	$(3,3,3,3,2,2,0,2,2,3)$	144	
640	$T_{4,32} = (3,3,3,3,1,1,1,3,3,2)$	54	16276158128619777024000
640	$(3,3,3,3,3,1,0,0,1,3)$	34	
640	$(3,3,3,3,2,2,2,0,2,-2)$	−24	
*648	$T_{4,33} = (3,3,3,3,0,0,0,3,3,0)$	−164	21534811123966672896000
649	$(3,3,3,3,3,2,0,0,2,1)$	−54	
660	$T_{4,34} = (3,3,3,3,3,2,-1,-1,-1,1)$	−66	31693489696333430784000
661	$(3,3,3,3,3,1,-1,0,2,1)$	12	
664	$(3,3,3,3,2,1,0,0,2,3)$	−78	
665	$(3,3,3,3,2,2,1,-1,2,2)$	130	
672	$T_{4,35} = (3,3,3,3,1,1,-1,3,0,3)$	−6	46608020590808162304000
672	$(3,3,3,3,3,2,0,2,0,2)$	48	
672	$(3,3,3,3,2,2,2,1,-2,-2)$	−64	
672	$(3,3,3,3,2,1,0,2,-2,-1)$	24	
676	$(3,3,3,3,3,1,0,1,1,3)$	−375	
684	$(3,3,3,3,2,2,0,3,0,0)$	60	
685	$T_{4,36} = (3,3,3,3,3,2,-1,0,1,-1)$	9	70354945978854211584000
685	$(3,3,3,3,2,1,0,1,-1,-3)$	42	
688	$T_{4,37} = (3,3,3,3,3,2,-1,0,1,0)$	−21	77313182912338673664000

Table 5 (continued)

D	reduced form	J^3	48 dim pan *Theta*
688	$(3,3,3,3,3,1,-1,0,-2,2)$	-228	
693	$(3,3,3,3,1,1,-1,3,2,2)$	18	
693	$(3,3,3,3,3,1,-1,-1,-2,2)$	207	
693	$(3,3,3,3,3,1,0,0,0,3)$	267	
693	$T_{4,38}=(3,3,3,3,1,1,0,3,3,0)$	39	9028595722967344742400O
697	$(3,3,3,3,3,2,0,2,2,1)$	-146	
700	$(3,3,3,3,2,0,0,0,3,2)$	86	
704	$(3,3,3,3,2,2,-2,1,1,1)$	24	
705	$(3,3,3,3,2,2,-1,1,-2,0)$	-98	
705	$(3,3,3,3,2,2,1,3,0,0)$	-96	
709	$(3,3,3,3,2,1,0,1,3,2)$	-197	
712	$(3,3,3,3,3,1,0,0,2,-2)$	-288	
713	$(3,3,3,3,2,2,-1,1,2,-2)$	54	
720	$(3,3,3,3,2,2,2,-1,2,-1)$	-54	
720	$(3,3,3,3,2,0,0,0,2,3)$	228	
720	$T_{4,39}=(3,3,3,3,3,2,-1,0,0,0)$	-36	20548830764007073382400O
720	$(3,3,3,3,2,2,2,3,1,1)$	468	
720	$(3,3,3,3,2,2,0,0,2,2)$	16	
720	$(3,3,3,3,1,1,1,-3,-2,1)$	198	

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