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SPACELIKE CONSTANT MEAN CURVATURE AND MAXIMAL SURFACES IN 3-DIMENSIONAL DE SITTER SPACE VIA IWASAWA SPLITTING

By

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Abstract. We study the construction of spacelike constant mean curvature (CMC) surfaces with mean curvature $0 \leq H < 1$ in 3-dimensional de Sitter space $\mathbf{S}^{2,1}$, by using Iwasawa splitting. We also study their singularities and create some criteria for them.

1. Introduction

We can construct CMC $H = 0$ (minimal) surfaces in the 3-dimensional Euclidean space $\mathbf{R}^3$ by using a famous integral formula involving a pair of holomorphic functions satisfying certain conditions, called the *Weierstrass representation*, and many examples of minimal surfaces have been constructed with it. Dorfmeister, Pedit and Wu provided a generalization of the Weierstrass representation formula ([5]), called the DPW method, for constructing CMC $H \neq 0$ surfaces in $\mathbf{R}^3$, using holomorphic data satisfying certain conditions and a matrix loop splitting called the Iwasawa splitting. In other works ([4], [14], etc), Dorfmeister, Inoguchi, Kobayashi, Kilian, Rossman and Schmitt also constructed new examples of CMC surfaces in the 3-dimensional sphere $\mathbf{S}^3$ and hyperbolic space $\mathbf{H}^3$ (non-Euclidean positive definite spaceforms) via the DPW method. In [3], Brander, Rossman and Schmitt constructed spacelike CMC surfaces in the 3-dimensional Minkowski space $\mathbf{R}^{2,1}$. In [6] and [7], Fujioka and Inoguchi studied spacelike and timelike harmonic inverse mean curvature surfaces in $\mathbf{R}^{2,1}$, de Sitter 3-space $\mathbf{S}^{2,1}$ and anti-de Sitter 3-space $\mathbf{H}^{2,1}$, including spacelike CMC surfaces, by using Lax systems. In our previous work [12], we studied spacelike CMC surfaces in $\mathbf{R}^{2,1}$, $\mathbf{S}^{2,1}$ and $\mathbf{H}^{2,1}$, and created criteria for singularities on these

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surfaces, by using the frame-change method, called the s-spectral deformation. However, in [12] we omitted the case of spacelike CMC H surfaces with $0 \le H < 1$ in $\mathbf{S}^{2,1}$ because the Lax pair and Iwasawa splitting are quite different from the case of $H > 1$. On the other hand, in the appendix of [4] (arXiv version), spacelike CMC H surfaces with $0 \le H < 1$ in the de Sitter space $\mathbf{S}^{2,1}$ with no umbilics are considered by using the normal vector of the parallel surfaces of CMC H surfaces with $0 \le H < 1$ in $\mathbf{H}^3$. See Proposition 2.2 in the present paper.

In the present paper, rather, to allow for umbilics as one of the reasons, we give a DPW method for spacelike CMC H surfaces with $0 \le H < 1$ in $\mathbf{S}^{2,1}$ by using Iwasawa splitting. (See Theorems 3.1, 4.1 and 4.2.) We also study the singularities of these surfaces, and specify types of singularities, focusing on the asymptotic behavior of Iwasawa splitting and using the s-spectral deformation. (See Theorems 5.1, 6.3.) In particular we look at singularities of Smyth-type surfaces with $0 \le H < 1$ in $\mathbf{S}^{2,1}$. The data for Smyth-type surfaces with $0 \le H < 1$ in $\mathbf{H}^3$ was given by Dorfmeister, Inoguchi and Kobayashi in [4]—however, in that $\mathbf{H}^3$ case the Iwasawa splitting is more easily extendable beyond the Iwasawa core, and so singularities do not appear on these surfaces. Here instead we go to the $\mathbf{S}^{2,1}$ case to examine singularities.

This paper has seven sections. Section 2 explains Lax pairs and the immersion formula as in [6]. In Section 3, we prove the existence and uniqueness of $\Lambda SL_2(\mathbf{C})_\tau$-Iwasawa splitting. In Section 4, we apply the DPW method for spacelike CMC H surfaces with $0 \le H < 1$ in $\mathbf{S}^{2,1}$, and in Section 5 we consider the behavior of the frames and surfaces when approaching the $\Lambda SL_2(\mathbf{C})_\tau$-Iwasawa small cell $\mathscr{P}$. In Section 6, we introduce criteria of singularities on these surfaces, including cuspidal edges, swallowtails and cuspidal cross caps, and in the last Section 7 we introduced Smyth-type surfaces with umbilics and singularities. Here, we show this Smyth-type surfaces have three types of singularities, i.e. cuspidal edges, swallowtails and cuspidal cross caps (See Theorem 7.2 and Figure 1 below).

2. The Lax Pair in $\mathbf{S}^{2,1}$

2.1. The 3-Dimensional De Sitter Space. Let $\mathbf{R}^{3,1}$ be the Cartesian 4-space with metric $\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \rangle := x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4$. We define the 3-dimensional de Sitter space as the hyperquadric $\mathbf{S}^{2,1} := \{x \mid \langle x, x \rangle = 1\} \subset \mathbf{R}^{3,1}$.

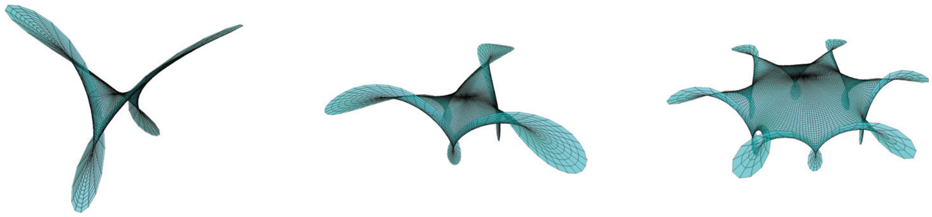

Figure 1: Maximal 2-legged, 3-legged and 6-legged Smyth-type surfaces (left to right).

Let Σ be a simply-connected domain in $\mathbf{C}$ with the usual complex coordinate $w = x + iy$. Let $f : \Sigma \to \mathbf{S}^{2,1}$ be a conformally immersed spacelike surface. Since f is conformal,

$$\langle f_w, f_w \rangle = \langle f_{\bar{w}}, f_{\bar{w}} \rangle = 0, \quad \langle f_w, f_{\bar{w}} \rangle = 2e^{2u}$$

for some function $u : \Sigma \to \mathbf{R}$. For the unit normal vector field N of f satisfying $\langle N, N \rangle = -1$, $\langle f_w, N \rangle = \langle f_{\bar{w}}, N \rangle = 0$, we define the mean curvature H and Hopf differential $\mathscr{A}\, dw^2$ as follows:

$$H := \frac{1}{2e^{2u}} \langle f_{w\bar{w}}, N \rangle, \quad \mathscr{A} := \langle f_{ww}, N \rangle.$$

The Gauss-Codazzi equations are of the following form in the CMC cases:

$$2u_{w\bar{w}} - 2e^{2u}(H^2 - 1) + \frac{1}{2}\mathscr{A}\bar{\mathscr{A}}e^{-2u} = 0, \quad \mathscr{A}_{\bar{w}} = 0. \tag{2.1}$$

The Codazzi equation in (2.1) is equivalent to the Hopf differential $\mathscr{A}$ being holomorphic, and (2.1) is invariant under the deformation $\mathscr{A} \mapsto \mu^{-2}\mathscr{A}$ for $\mu \in \mathbf{S}^1$. When $f(x, y)$ is a spacelike CMC in $\mathbf{S}^{2,1}$, the spectral parameter $\mu \in \mathbf{S}^1$ allows us to create a 1-parameter family of CMC surfaces $f^\mu = f(x, y, \mu)$ associated to $f(x, y)$.

2.2. The 2×2 Matrix Model of $\mathbf{S}^{2,1}$. We identify $\mathbf{R}^{3,1}$ with the space $\{X \in M_{2\times 2} \mid X = \bar{X}^t\}$ of all 2 by 2 Hermitian matrices as follows:

$$\mathbf{R}^{3,1} \ni x = (x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_4 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_4 - x_3 \end{pmatrix}. \tag{2.2}$$

The metric becomes, under this identification, $\langle X, Y \rangle = -\frac{1}{2}\,\mathrm{trace}(X\sigma_2 Y^t \sigma_2)$. In particular, $\langle X, X \rangle = -\det(X)$, and we can identify $\mathbf{S}^{2,1}$ with

$$\{X \in M_{2\times 2} \mid X = \bar{X}^t, \det(X) = -1\} = \{F\sigma_3 \overline{F^t} \mid F \in \mathrm{SL}_2(\mathbf{C})\}.$$

Let f be a conformal spacelike CMC surface in $\mathbf{S}^{2,1}$ with associated family f^{μ}, and let the identity matrix and Pauli matrices be as follows:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\{I, \sigma_1, \sigma_2, \sigma_3\}$ is an orthogonal basis for $\mathbf{R}^{3,1}$. We can define

$$f^{\mu} := \hat{F}\sigma_3\overline{\hat{F}^t}, \quad e_1 := \frac{f_x^{\mu}}{|f_x^{\mu}|} = \frac{f_x^{\mu}}{2e^u} = \hat{F}\sigma_1\overline{\hat{F}^t}, \quad e_2 := \frac{f_y^{\mu}}{|f_y^{\mu}|} = \frac{f_y^{\mu}}{2e^u} = \hat{F}\sigma_2\overline{\hat{F}^t}, \quad N := \hat{F}\overline{\hat{F}^t}$$

for $\hat{F} = \hat{F}(w, \overline{w}, \mu) \in \mathrm{SL}_2(\mathbf{C})$. For this $\hat{F}$, we get the *untwisted* 2×2 *Lax pair* in $\mathbf{S}^{2,1}$ as follows:

$$\text{(2.3)} \quad \hat{F}_w = \hat{F}\hat{S}, \quad \hat{F}_{\overline{w}} = \hat{F}\hat{T}, \quad \text{where } \hat{S} = \frac{1}{2}\begin{pmatrix} -u_w & -\mu^{-2}\mathscr{A}e^{-u} \\ 2(1-H)e^u & u_w \end{pmatrix},$$

$$\hat{T} = \frac{1}{2}\begin{pmatrix} u_{\overline{w}} & -2(1+H)e^u \\ -\mu^2\bar{\mathscr{A}}e^{-u} & -u_{\overline{w}} \end{pmatrix}.$$

We change the "untwisted" setting to the "twisted" setting by the following transformation (2.4). Let $\tilde{F}$ be defined by

$$\text{(2.4)} \qquad \hat{F} = -\sigma_3(\tilde{F}^{-1})^t \begin{pmatrix} \sqrt{\mu} & 0 \\ 0 & \frac{1}{\sqrt{\mu}} \end{pmatrix}\sigma_3,$$

producing the *twisted* 2×2 *Lax pair* of f^{μ} in $\mathbf{S}^{2,1}$,

$$\text{(2.5)} \quad \tilde{F}_w = \tilde{F}\tilde{S}, \quad \tilde{F}_{\overline{w}} = \tilde{F}\tilde{T}, \quad \text{where } \tilde{S} = \frac{1}{2}\begin{pmatrix} u_w & 2\mu^{-1}(1-H)e^u \\ -\mu^{-1}\mathscr{A}e^{-u} & -u_w \end{pmatrix},$$

$$\tilde{T} = \frac{1}{2}\begin{pmatrix} -u_{\overline{w}} & -\mu\bar{\mathscr{A}}e^{-u} \\ -2\mu(1+H)e^u & u_{\overline{w}} \end{pmatrix}.$$

Now we consider $0 \le H < 1$ case, and we set $H := \tanh(-q)$ for $q \le 0$. We change $\tilde{F}$ to a new frame F, as follows:

$$F = \tilde{F}\begin{pmatrix} e^{q/4} & 0 \\ 0 & e^{-q/4} \end{pmatrix}.$$

We call this F the *extended frame* of spacelike CMC H surfaces with $0 \le H < 1$. Moreover we set

$$\text{(2.6)} \quad \mathscr{H} := -ie^{-q}(1-H) = -ie^{q}(1+H) \in i\mathbf{R}, \quad Q := -i\mathscr{A}, \quad \nu := e^{-q/2}\mu,$$

and we have the following:

$$(2.7)\qquad F_w = FU, \quad F_{\bar{w}} = FV, \quad \text{where } U = \frac{1}{2}\begin{pmatrix} u_w & 2iv^{-1}\mathscr{H}e^u \\ -iv^{-1}Qe^{-u} & -u_w \end{pmatrix},$$

$$V = \frac{1}{2}\begin{pmatrix} -u_{\bar{w}} & iv\bar{Q}e^{-u} \\ -2iv\mathscr{H}e^u & u_{\bar{w}} \end{pmatrix}.$$

We call (2.7) the *extended Lax pair*, and $F = F(w, \bar{w}, v)$ is in the following loop group

$$(2.8)\qquad \Lambda \mathrm{SL}_2(\mathbf{C})_\tau = \{F(\lambda) \in M_{2\times 2} \mid F : \mathbf{S}^1 \xrightarrow{C^\infty} \mathrm{SL}_2(\mathbf{C}), F(-\lambda) = \sigma_3 F(\lambda)\sigma_3,$$
$$\tau(F(\lambda)) = F(\lambda)\},$$

where $\tau(F(\lambda)) := \mathrm{Ad}\left(\begin{pmatrix} e^{(\pi/4)i} & 0 \\ 0 & e^{-(\pi/4)i} \end{pmatrix}\right) \cdot (\overline{F(i\bar{\lambda}^{-1})^t})^{-1}$. For simplicity, we set $R = \begin{pmatrix} e^{-(\pi/4)i} & 0 \\ 0 & e^{(\pi/4)i} \end{pmatrix}$ as in [4] and the symbol * is defined by $F^*(\lambda) := (\overline{F(i\bar{\lambda}^{-1})^t})^{-1}$, and we can rewrite $\tau(F(\lambda)) = \mathrm{Ad}(R^{-1})F^*(\lambda)$.

The following Proposition 2.1 gives us a method for determining spacelike CMC H surfaces with $0 \le H < 1$ in $\mathbf{S}^{2,1}$, from given data u and $\mathscr{A}$.

Proposition 2.1 (The immersion formula for spacelike CMC H surfaces with $0 \le H < 1$ in $\mathbf{S}^{2,1}$). *Let Σ be a simply-connected domain in $\mathbf{C}$. Let u and $\mathscr{A}$ solve* (2.1). *Set $Q = -i\mathscr{A}$ and $\mathscr{H} := -ie^{-q}(1 - \tanh(-q))$ for $q \le 0$. Let $F = F(w, \bar{w}, v) \in \Lambda \mathrm{SL}_2(\mathbf{C})_\tau$ be a solution of the system* (2.7). *Set $F_0 = F|_{v=e^{-q/2}}$. We define the following immersion formulas*

$$(2.9)\qquad f = F_0\begin{pmatrix} e^{-(1/2)q} & 0 \\ 0 & -e^{(1/2)q} \end{pmatrix}\overline{F_0}^t, \quad N = F_0\begin{pmatrix} -e^{-(1/2)q} & 0 \\ 0 & -e^{(1/2)q} \end{pmatrix}\overline{F_0}^t.$$

Then, f is a spacelike CMC $H = \tanh(-q)$ surface in $\mathbf{S}^{2,1}$ with unit normal N.

Proof. Applying a frame change in Theorem 8.5 in [6], we can prove this proposition. We can also prove it by the same argument as in the proof of Proposition 4.1 in [4]. So we omit the proof. □

By the above Proposition 2.1, we can construct all simply-connected spacelike CMC H surfaces with $0 \le H < 1$. However, here we also have the following proposition in Appendix E of [4], and it says that spacelike CMC H surfaces with $0 \le H < 1$ with no umbilics can be constructed as the normal vector of parallel transformation of CMC H surfaces with $0 \leq H < 1$ in the 3-dimensional hyperbolic space $\mathbf{H}^3$. In this proposition, we need the following extended Lax pair for CMC H surfaces with $0 \leq H < 1$ in $\mathbf{H}^3$:

$$(F_{\mathbf{H}^3})_w = F_{\mathbf{H}^3} U_{\mathbf{H}^3}, \quad (F_{\mathbf{H}^3})_{\bar{w}} = F_{\mathbf{H}^3} V_{\mathbf{H}^3}, \quad \text{where}$$

$$U_{\mathbf{H}^3} = \frac{1}{2}\begin{pmatrix} (u_{\mathbf{H}^3})_w & -2v^{-1}\mathscr{H}_{\mathbf{H}^3} e^{u_{\mathbf{H}^3}} \\ v^{-1} Q_{\mathbf{H}^3} e^{-u_{\mathbf{H}^3}} & -(u_{\mathbf{H}^3})_w \end{pmatrix},$$

$$V_{\mathbf{H}^3} = \frac{1}{2}\begin{pmatrix} -(u_{\mathbf{H}^3})_{\bar{w}} & -v\overline{Q_{\mathbf{H}^3}} e^{-u_{\mathbf{H}^3}} \\ 2v\mathscr{H}_{\mathbf{H}^3} e^{u_{\mathbf{H}^3}} & (u_{\mathbf{H}^3})_{\bar{w}} \end{pmatrix},$$

where $u_{\mathbf{H}^3}$ is the metric function for $ds^2 = 4e^{2u_{\mathbf{H}^3}}\,dw d\bar{w}$, and $\mathscr{H}_{\mathbf{H}^3} = ie^{-q}(1 - H_{\mathbf{H}^3})$ for mean curvature $H_{\mathbf{H}^3} = \tanh(-q)$, and $Q_{\mathbf{H}^3} = i\mathscr{A}_{\mathbf{H}^3}$ for Hopf differential $\mathscr{A}_{\mathbf{H}^3}$.

Proposition 2.2 ([4]). *Let Σ be a simply-connected domain in $\mathbf{C}$. Let $F_{\mathbf{H}^3}$ be the extended frame of some CMC $0 \leq H_{\mathbf{H}^3} = \tanh(-q) < 1$ surface $f_{\mathbf{H}^3}(w, \bar{w})$ in $\mathbf{H}^3$ with unit normal $N_{\mathbf{H}^3}$. Set $(F_{\mathbf{H}^3})_0 = (F_{\mathbf{H}^3})|_{v=e^{-q/2}}$ for $q < 0$. Then, by changing parameter such that $z = iw$,*

$$f(z, \bar{z}) = \sinh(-q) f_{\mathbf{H}^3} + \cosh(-q) N_{\mathbf{H}^3} = (F_{\mathbf{H}^3})_0 \begin{pmatrix} -e^{(1/2)q} & 0 \\ 0 & e^{-(1/2)q} \end{pmatrix} \overline{(F_{\mathbf{H}^3})_0}^t \tag{2.10}$$

is a spacelike CMC $H = \tanh(-q)$ surface in $\mathbf{S}^{2,1}$, and $ds^2 = e^{-2u_{\mathbf{H}^3}} \cosh^2(-q)|\mathscr{A}_{\mathbf{H}^3}|^2\, dz d\bar{z}$, $\mathscr{A} = \mathscr{A}_{\mathbf{H}^3}$.

3. Iwasawa Splitting for $\Lambda \mathrm{SL}_2(\mathbf{C})_\tau$

In this section, we introduce the *Birkhoff splitting* and prove the existence and uniqueness of the (*twisted*) $\Lambda \mathrm{SL}_2(\mathbf{C})_\tau$-*Iwasawa splitting*, by using the Birkhoff splitting. First we define the notations for loop groups and algebras.

Definition 3.1.

$$\Lambda \mathrm{SL}_2(\mathbf{C}) = \{\phi : \mathbf{S}^1 \xrightarrow{C^\infty} SL_2(\mathbf{C}) \mid \phi(-\lambda) = \sigma_3 \phi(\lambda) \sigma_3\},$$

$$\Lambda sl_2(\mathbf{C}) = \{A : \mathbf{S}^1 \xrightarrow{C^\infty} sl_2(\mathbf{C}) \mid A(-\lambda) = \sigma_3 A(\lambda) \sigma_3\},$$

$$\Lambda_\triangle^+ \mathrm{SL}_2(\mathbf{C}) = \{B_+(\lambda) \in \Lambda \mathrm{SL}_2(\mathbf{C}) \mid B_+ \textit{ extends holomorphically to } \mathbf{D}.\},$$

$$\Lambda^-_{\triangle}\mathrm{SL}_2(\mathbf{C}) = \{B_-(\lambda) \in \Lambda\mathrm{SL}_2(\mathbf{C}) \mid B_- \textit{ extends holomorphically to } \mathbf{C}\cup\{\infty\}\backslash\bar{\mathbf{D}}.\},$$

$$\Lambda^+_{\mathbf{R}}\mathrm{SL}_2(\mathbf{C}) = \left\{B_+(\lambda) \in \Lambda^+_{\triangle}\mathrm{SL}_2(\mathbf{C}) \mid B_+(0) = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \textit{ for } \rho \in \mathbf{R}, \rho > 0.\right\},$$

where $\mathbf{S}^1 = \{\lambda \in \mathbf{C}\cup\{\infty\} \mid \lambda\bar{\lambda} = 1\}$, $\mathbf{D} = \{\lambda \in \mathbf{C}\cup\{\infty\} \mid \lambda\bar{\lambda} < 1\}$. *We can also define* $\Lambda\mathrm{GL}_2(\mathbf{C})$, *etc., in a similar way.*

Here we introduce the twisted version of the Birkhoff splitting as follows:

Proposition 3.1 ([13]). *For all* $\phi \in \Lambda\mathrm{GL}_2(\mathbf{C})$, *there exist* $\phi_+ \in \Lambda^+\mathrm{GL}_2(\mathbf{C})$, $\phi_- \in \Lambda^-\mathrm{GL}_2(\mathbf{C})$ *and* $a_1, a_2 \in \mathbf{Z}$ *such that*

$$\phi = \phi_- \begin{pmatrix} \lambda^{2a_1} & 0 \\ 0 & \lambda^{2a_2} \end{pmatrix} \phi_+. \tag{3.1}$$

The middle term is uniquely determined by ϕ, *and the big cell* $\mathscr{B}$, *where* $a_1 = a_2 = 0$, *is an open dense subset in* $\Lambda\mathrm{GL}_2(\mathbf{C})$. *When* $\phi \in \mathscr{B}$, *we have a unique splitting such that* $\phi_+ \in \Lambda^+_I\mathrm{GL}_2(\mathbf{C})$, $\phi_- \in \Lambda^-\mathrm{GL}_2(\mathbf{C})$.

We also introduce the specialized version of the Birkhoff splitting for $\Lambda\mathrm{SL}_2(\mathbf{C})$, as in [3].

Proposition 3.2 (the specialized version of the Birkhoff splitting [3]). *For all* $\phi \in \Lambda\mathrm{SL}_2(\mathbf{C})$, *there exist* $B_+ \in \Lambda^+_{\triangle}\mathrm{SL}_2(\mathbf{C})$, $B_- \in \Lambda^-_{\triangle}\mathrm{SL}_2(\mathbf{C})$ *and* $k \in \mathbf{Z}$ *such that*

$$\phi = B_- M B_+, \tag{3.2}$$

where either

$$M = \begin{pmatrix} \lambda^{2k} & 0 \\ 0 & \lambda^{-2k} \end{pmatrix}, \quad \text{or} \quad M = \begin{pmatrix} 0 & \lambda^{2k+1} \\ -\lambda^{-2k-1} & 0 \end{pmatrix}. \tag{3.3}$$

Remark 3.1 ([3]).

(1) The factor M is uniquely determined by ϕ. For $\phi \in \mathscr{B}$, where $k = 0$, there is a unique splitting $\phi = B_- B_+$ with $B_- \in \Lambda^-_{\triangle}\mathrm{SL}_2(\mathbf{C})$, $B_+ \in \Lambda^+_I\mathrm{SL}_2(\mathbf{C})$, with $M = I$.

(2) For the $\Lambda\mathrm{SL}_2(\mathbf{C})_\tau$-Iwasawa splitting, we define

$$\mathscr{P} = \left\{\phi \in \Lambda\mathrm{SL}_2(\mathbf{C}) \mid M \text{ of the Birkhoff splitting of } \phi \text{ is } \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}\right\},$$

and we call this $\mathscr{P}$ the *Birkhoff first small cell.*

In order to prove the $\Lambda SL_2(\mathbf{C})_\tau$-Iwasawa splitting, we need some definitions and lemmas.

Definition 3.2 (operator τ). *For $\phi(\lambda) \in \Lambda SL_2(\mathbf{C})$, we define an operator τ by*

$$\tau(\phi(\lambda)) := R^{-1}(\overline{\phi(i\bar{\lambda}^{-1})^t})^{-1}R \quad \textit{for } R = \begin{pmatrix} e^{-(\pi/4)i} & 0 \\ 0 & e^{(\pi/4)i} \end{pmatrix}.$$

Remark 3.2. For $\phi \in \Lambda SL_2(\mathbf{C})$, the following two statements hold:
- $\tau(\tau(\phi(\lambda))) = \sigma_3\phi(-\lambda)\sigma_3 = \phi(\lambda)$.
- $\tau(\phi(\lambda)) = \phi(\lambda)$ for all $\lambda \in \mathbf{S}^1 \Leftrightarrow \phi(\lambda) \in \Lambda SL_2(\mathbf{C})_\tau$.

Thus, τ is an automorphism on $\Lambda SL_2(\mathbf{C})_\tau$.

Lemma 3.1. *Let $\psi \in \Lambda SL_2(\mathbf{C})$ such that $(\tau(\psi))^{-1} = \psi$. If $\psi \in \mathscr{B} \cup \mathscr{P}$, then we have*

$$\psi = \tau(B_+)^{-1} \cdot (\pm I) \cdot B_+ \quad \textit{or} \quad \psi = \tau(B_+)^{-1} \cdot \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} \cdot B_+ \quad \textit{for } B_+ \in \Lambda_\triangle^+ SL_2(\mathbf{C}).$$

However, if $\psi \notin \mathscr{B} \cup \mathscr{P}$, then we do not have

$$\psi = \tau(B_+)^{-1} \cdot \begin{pmatrix} \lambda^{2k} & 0 \\ 0 & \lambda^{-2k} \end{pmatrix} \cdot B_+ \quad \textit{or} \quad \psi = \tau(B_+)^{-1} \cdot \begin{pmatrix} 0 & \lambda^{2k+1} \\ -\lambda^{-2k-1} & 0 \end{pmatrix} \cdot B_+$$

for $B_+ \in \Lambda_\triangle^+ SL_2(\mathbf{C})$.

Proof. (i) Noting the result in Proposition 3.2, suppose

$$\psi = B_- \begin{pmatrix} \lambda^{2k} & 0 \\ 0 & \lambda^{-2k} \end{pmatrix} B_+$$

for some $B_+ \in \Lambda_\triangle^+ SL_2(\mathbf{C})$ and $B_- \in \Lambda_\triangle^- SL_2(\mathbf{C})$ and some $k \in \mathbf{Z}$. By $\tau(\psi)^{-1} = \psi$, we have

$$\delta \cdot \tau(B_+ \cdot \tau(B_-))^{-1} \cdot \begin{pmatrix} -\lambda^{-2k} & 0 \\ 0 & -\lambda^{2k} \end{pmatrix} = \begin{pmatrix} \lambda^{2k} & 0 \\ 0 & \lambda^{-2k} \end{pmatrix} \cdot B_+ \cdot \tau(B_-)$$

for $\delta = 1$ (resp. -1) if $k \in 2\mathbf{Z} + 1$ (resp. $k \in 2\mathbf{Z}$). Setting $\mathbf{B}(\lambda) = B_+ \cdot \tau(B_-)$, this equals

$$\delta \cdot \tau(\mathbf{B}(\lambda))^{-1} \cdot \begin{pmatrix} -\lambda^{-2k} & 0 \\ 0 & -\lambda^{2k} \end{pmatrix} = \begin{pmatrix} \lambda^{2k} & 0 \\ 0 & \lambda^{-2k} \end{pmatrix} \cdot \mathbf{B}(\lambda), \tag{3.4}$$

Now we note that $\mathbf{B}(\lambda) \in \Lambda_{\triangle}^{+}\mathrm{SL}_2(\mathbf{C})$, and we let $\mathbf{B}(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}$. Then, (3.4) equals

$$\delta \cdot \begin{pmatrix} -\lambda^{-2k}\overline{a^*(\lambda)} & -i\lambda^{2k}\overline{c^*(\lambda)} \\ i\lambda^{-2k}\overline{b^*(\lambda)} & -\lambda^{2k}\overline{d^*(\lambda)} \end{pmatrix} = \begin{pmatrix} \lambda^{2k}a(\lambda) & \lambda^{2k}b(\lambda) \\ \lambda^{-2k}c(\lambda) & \lambda^{-2k}d(\lambda) \end{pmatrix}, \tag{3.5}$$

where $a^*(\lambda) = a(i\bar{\lambda}^{-1})$.

If $k > 0$, then the upper-left component of (3.5) implies $-\delta\overline{a^*(\lambda)} = \lambda^{4k}a(\lambda)$, and this a has the power series expansion $\sum_{j=0}^{\infty} a_j\lambda^j$, where the a_j do not depend on λ since $\mathbf{B}(\lambda) \in \Lambda_{\triangle}^{+}\mathrm{SL}_2(\mathbf{C})$. By these conditons, we know $a \equiv 0$. Similarly, the upper-right and lower-left component of (3.5) imply that

$$b, c : \text{complex constants} \quad \text{and} \quad \delta ib = c.$$

However, by the twisted property, we have $b \equiv c \equiv 0$, and this contradicts $\det(\mathbf{B}(\lambda)) = 1$. Similarly, if $k < 0$, then we get the contradiction. Thus, we conclude $k = 0$.

By $k = 0$, we have $\delta = -1$, $b \equiv c \equiv 0$, $a \in \mathbf{R}\backslash\{0\}$: constant, $d = a^{-1}$. Here we notice that $\sqrt{a}$ is well-defined, and we get the following:

$$\tau\left(\begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix}\right)^{-1} = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix} \quad \text{if } a > 0 \tag{3.6}$$

$$\tau\left(\begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix}\right)^{-1} = \begin{pmatrix} -\sqrt{a} & 0 \\ 0 & -\sqrt{a}^{-1} \end{pmatrix} \quad \text{if } a < 0. \tag{3.7}$$

Thus, we separately consider these cases to finish the case (i).

(i)-(1) If $a > 0$, then

$$\mathbf{B} = B_+ \cdot \tau(B_-) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{for } B_\pm \in \Lambda_{\triangle}^{\pm}\mathrm{SL}_2(\mathbf{C}).$$

This implies that $\begin{pmatrix} \sqrt{a}^{-1} & 0 \\ 0 & \sqrt{a} \end{pmatrix} B_+ = \tau\left(B_-\begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix}\right)^{-1}$.

Now let $\beta_+ := \begin{pmatrix} \sqrt{a}^{-1} & 0 \\ 0 & \sqrt{a} \end{pmatrix} B_+ \in \Lambda_{\triangle}^{+}\mathrm{SL}_2(\mathbf{C})$ and $\beta_- := B_-\begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix} \in \Lambda_{\triangle}^{-}\mathrm{SL}_2(\mathbf{C})$, then

$$\psi = B_- B_+ = \beta_- \beta_+ = \tau(\beta_+)^{-1} \cdot \beta_+.$$

(i)-(2) If $a < 0$, similarly we have

$$\mathbf{B} = B_+ \cdot \tau(B_-) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{for } B_\pm \in \Lambda_\triangle^\pm \mathrm{SL}_2(\mathbf{C}).$$

Thus this implies that $\begin{pmatrix} \sqrt{a}^{-1} & 0 \\ 0 & \sqrt{a} \end{pmatrix} B_+ = \tau\left(B_- \begin{pmatrix} -\sqrt{a} & 0 \\ 0 & -\sqrt{a}^{-1} \end{pmatrix}\right)^{-1}$. Now let $\beta_+ := \begin{pmatrix} \sqrt{a}^{-1} & 0 \\ 0 & \sqrt{a} \end{pmatrix} B_+ \in \Lambda_\triangle^+ \mathrm{SL}_2(\mathbf{C})$ and $\beta_- := B_- \begin{pmatrix} -\sqrt{a} & 0 \\ 0 & -\sqrt{a}^{-1} \end{pmatrix} \in \Lambda_\triangle^+ \mathrm{SL}_2(\mathbf{C})$, then

$$\psi = B_- B_+ = \beta_-(-I)\beta_+ = \tau(\beta_+)^{-1}(-I)\beta_+. \tag{3.8}$$

This completes the proof of the case (i).

(ii) Again noting the result in Proposition 3.2, suppose

$$\psi = B_- \begin{pmatrix} 0 & \lambda^{2k+1} \\ -\lambda^{-2k-1} & 0 \end{pmatrix} B_+$$

for some $B_+ \in \Lambda_\triangle^+ \mathrm{SL}_2(\mathbf{C})$ and $B_- \in \Lambda_\triangle^- \mathrm{SL}_2(\mathbf{C})$. By $\tau(\psi)^{-1} = \psi$, we have

$$\delta \cdot \begin{pmatrix} i\lambda^{-2k-1}\overline{c^*(\lambda)} & -\lambda^{2k+1}\overline{a^*(\lambda)} \\ \lambda^{-2k-1}\overline{d^*(\lambda)} & i\lambda^{2k+1}\overline{b^*(\lambda)} \end{pmatrix} = \begin{pmatrix} \lambda^{2k+1}c(\lambda) & \lambda^{2k+1}d(\lambda) \\ -\lambda^{-2k-1}a(\lambda) & -\lambda^{-2k-1}b(\lambda) \end{pmatrix} \tag{3.9}$$

for $\mathbf{B}(\lambda) = B_+ \cdot \tau(B_-) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}$. As in the case (i), by the form of the power series expansion of $b(\lambda)$ and $c(\lambda)$, we conclude

$$c \equiv 0, \quad b(\lambda) = b_1\lambda^1 + b_3\lambda^3 + \cdots + b_{4k+1}\lambda^{4k+1}, \tag{3.10}$$

$$-\delta i\lambda^{4k+2}\overline{b^*(\lambda)} = b(\lambda) \quad \text{if } k \geq 0,$$

$$b \equiv 0, \quad c(\lambda) = c_1\lambda^1 + c_3\lambda^3 + \cdots + c_{-4k-3}\lambda^{-4k-3}, \tag{3.11}$$

$$\delta i\lambda^{-4k-2}\overline{c^*(\lambda)} = c(\lambda) \quad \text{if } k < 0.$$

Similarly, by the off-diagonal components of (3.9), we have

$$\delta = -1 (\Leftrightarrow k \in 2\mathbf{Z}), \quad a \in \mathbf{S}^1 : \text{complex constant}, \quad d = a^{-1}. \tag{3.12}$$

Finally, we need to change B_+ and B_- to $\beta_+ = YB_+ \in \Lambda_\triangle^+ \mathrm{SL}_2(\mathbf{C})$ and $\beta_- = B_- X^{-1} \in \Lambda_\triangle^- \mathrm{SL}_2(\mathbf{C})$ by using $X \in \Lambda_\triangle^- \mathrm{SL}_2(\mathbf{C})$ and $Y \in \Lambda_\triangle^+ \mathrm{SL}_2(\mathbf{C})$. To complete the requirement, these matrices X and Y should satisfy the following condition:

$$\mathbf{B} = Y^{-1}\tau(X) \tag{3.13}$$

and

$$X^{-1}\begin{pmatrix} \lambda^{2l} & 0 \\ 0 & \lambda^{-2l} \end{pmatrix} Y = \begin{pmatrix} 0 & \lambda^{2k+1} \\ -\lambda^{-2k-1} & 0 \end{pmatrix} \quad \text{or} \tag{3.14}$$

$$X^{-1}\begin{pmatrix} 0 & \lambda^{2l+1} \\ -\lambda^{-2l-1} & 0 \end{pmatrix} Y = \begin{pmatrix} 0 & \lambda^{2k+1} \\ -\lambda^{-2k-1} & 0 \end{pmatrix}$$

for some $l \in \mathbf{Z}$. By direct computation with (3.10), (3.11), (3.12) and (3.13), we know that the diagonal case in (3.14) does not occur for any k, and that the off-diagonal case in (3.14) occurs only for $k = 0$. For $k = 0$, as in (i), we set $\beta_+ := \begin{pmatrix} \sqrt{a}^{-1} & -\frac{1}{2}b\sqrt{a}\lambda \\ 0 & \sqrt{a} \end{pmatrix} B_+ \in \Lambda_{\triangle}^{+}\mathrm{SL}_2(\mathbf{C})$ and $\beta_- := B_- \begin{pmatrix} \sqrt{a}^{-1} & 0 \\ -\frac{1}{2}b\sqrt{a}\lambda^{-1} & \sqrt{a} \end{pmatrix} \in \Lambda_{\triangle}^{-}\mathrm{SL}_2(\mathbf{C})$, then

$$\psi = B_-\begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} B_+ = \beta_-\begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}\beta_+ = \tau(\beta_+)^{-1}\begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}\beta_+.$$

This completes the proof of Lemma 3.1. □

Here we define the $\Lambda\mathrm{SL}_2(\mathbf{C})_\tau$-*Iwasawa big cell* $\mathscr{B}_\tau$ and $\Lambda\mathrm{SL}_2(\mathbf{C})_\tau$-*Iwasawa first small cell* $\mathscr{P}_\tau$.

Definition 3.3.

$$\mathscr{B}_\tau = \{\phi \in \Lambda\mathrm{SL}_2(\mathbf{C}) \mid \tau(\phi)^{-1}\phi \in \mathscr{B}\}, \quad \mathscr{P}_\tau = \{\phi \in \Lambda\mathrm{SL}_2(\mathbf{C}) \mid \tau(\phi)^{-1}\phi \in \mathscr{P}\}.$$

We introduce one of the main theorems here, and this Theorem 3.1 plays an important role in the next section about the DPW method, in order to construct the solution of the extended Lax pair which was introduced in the previous section. We will also study the asymptotic behavior of F near $\mathscr{P}_\tau$ related to singularities of surfaces in the two latter sections. In Theorem 3.1, we use the map

$$\Psi\left(\begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}\right) = \begin{pmatrix} a(\lambda^2) & \lambda b(\lambda^2) \\ \lambda^{-1}c(\lambda^2) & d(\lambda^2) \end{pmatrix}. \tag{3.15}$$

Theorem 3.1 ($\Lambda\mathrm{SL}_2(\mathbf{C})_\tau$-Iwasawa splitting).

(1) *For all* $\phi \in \mathscr{B}_\tau$, *there exist* $F \in \Lambda\mathrm{SL}_2(\mathbf{C})_\tau \cup \Psi(i\sigma_2) \cdot \Lambda\mathrm{SL}_2(\mathbf{C})_\tau$ *and* $B \in \Lambda_{\triangle}^{+}\mathrm{SL}_2(\mathbf{C})$ *such that*

$$\phi = FB. \tag{3.16}$$

We can choose $B \in \Lambda_{\mathbf{R}}^{+}\mathrm{SL}_2(\mathbf{C})$, *and then* F *and* B *are uniquely determined. We call this unique splitting "normalized."*

(2) *For all* $\phi \in \mathscr{P}_\tau$, *there exist* $F \in \Lambda\mathrm{SL}_2(\mathbf{C})_\tau \cup \Psi(i\sigma_2)\cdot\Lambda\mathrm{SL}_2(\mathbf{C})_\tau$ *and* $B \in \Lambda_{\mathbf{R}}^{+}\mathrm{SL}_2(\mathbf{C})$ *such that*

$$\phi = FCB, \quad C = \begin{pmatrix} \frac{1}{2} & \lambda \\ -\frac{1}{2}\lambda^{-1} & 1 \end{pmatrix}. \tag{3.17}$$

Proof. Take any $\phi \in \Lambda\mathrm{SL}_2(\mathbf{C})$, and set $\psi = \tau(\phi)^{-1}\phi$. Then, we have $\tau(\psi)^{-1} = \psi$, thus we can apply Lemma 3.1 for this ψ. This implies that

$$\psi = \tau(B_+)^{-1}\tau(W)^{-1}WB_+ = \tau(WB_+)^{-1}WB_+$$

for $W = I$ or $W = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}$ or $W = \begin{pmatrix} \frac{1}{2} & \lambda \\ -\frac{1}{2}\lambda^{-1} & 1 \end{pmatrix}$, and for some $B_+ \in \Lambda_{\triangle}^{+}\mathrm{SL}_2(\mathbf{C})$. The first two cases occur when $\psi \in \mathscr{B}$ (*i.e.* $\phi \in \mathscr{B}_\tau$), and the third case occurs when $\psi \in \mathscr{P}$ (i.e. $\phi \in \mathscr{P}_\tau$).

First we show that $\hat{F} := \tau(\phi)\tau(WB_+)^{-1} \in \Lambda\mathrm{SL}_2(\mathbf{C})_\tau$. The twisted property is automatically satisfied by definition, and we need to check $\tau(\hat{F}) = \hat{F}$. However, it is also clear because $\tau(\hat{F}) = \hat{F}$ is equivalent to $\phi = \tau(\phi)\tau(WB_+)^{-1}WB_+$.

Next we will consider cases:

(i) The case $W = I$, $W \in \Lambda\mathrm{SL}_2(\mathbf{C})_\tau$. We set $F := \hat{F}W \in \Lambda\mathrm{SL}_2(\mathbf{C})_\tau$ and $B := B_+ \in \Lambda_{\triangle}^{+}\mathrm{SL}_2(\mathbf{C})$, and this is the required splitting. Moreover, about normalization, we set

$$B|_{\lambda=0} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

for $\alpha \in \mathbf{C}\backslash\{0\}$. Splitting to

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} & 0 \\ 0 & \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} \begin{pmatrix} \sqrt{\alpha\bar{\alpha}} & 0 \\ 0 & \frac{1}{\sqrt{\alpha\bar{\alpha}}} \end{pmatrix},$$

we notice that $\begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} & 0 \\ 0 & \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbf{C})_\tau$, and that $\sqrt{\alpha\bar{\alpha}} \in \mathbf{R}_{>0}$. Hence, we can split to $\phi = F'B'$ for $F' := F\begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} & 0 \\ 0 & \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbf{C})_\tau$ and $B' := \begin{pmatrix} \sqrt{\frac{\bar{\alpha}}{\alpha}} & 0 \\ 0 & \sqrt{\frac{\alpha}{\bar{\alpha}}} \end{pmatrix} B \in \Lambda_{\mathbf{R}}^{+}\mathrm{SL}_2(\mathbf{C})$. So the uniqueness of this splitting follows from uniqueness of the Birkhoff splitting on $\mathscr{B}$.

(ii) The case $W = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}$, $\hat{F}W \in \Psi(i\sigma_2)\cdot\Lambda\mathrm{SL}_2(\mathbf{C})_\tau$. We can prove the existence and normalization of splitting in the same way as in (i).

(iii) The case $W = \begin{pmatrix} \frac{1}{2} & \lambda \\ -\frac{1}{2}\lambda^{-1} & 1 \end{pmatrix}$, we set $F := \hat{F}$, $C := W$ and $B := B_+$, and this is the required splitting. Moreover, about normalization, we set $B|_{\lambda=0} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ for $\alpha \in \mathbf{C}\backslash\{0\}$. Splitting to the same form as in (i), we have

$$C\begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} & 0 \\ 0 & \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} + \sqrt{\frac{\bar{\alpha}}{\alpha}} & \left(\sqrt{\frac{\bar{\alpha}}{\alpha}} - \sqrt{\frac{\alpha}{\bar{\alpha}}}\right)\lambda \\ \left(\sqrt{\frac{\bar{\alpha}}{\alpha}} - \sqrt{\frac{\alpha}{\bar{\alpha}}}\right)\lambda^{-1} & \sqrt{\frac{\alpha}{\bar{\alpha}}} + \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} C,$$

and we notice that $\frac{1}{2}\begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} + \sqrt{\frac{\bar{\alpha}}{\alpha}} & \left(\sqrt{\frac{\bar{\alpha}}{\alpha}} - \sqrt{\frac{\alpha}{\bar{\alpha}}}\right)\lambda \\ \left(\sqrt{\frac{\bar{\alpha}}{\alpha}} - \sqrt{\frac{\alpha}{\bar{\alpha}}}\right)\lambda^{-1} & \sqrt{\frac{\alpha}{\bar{\alpha}}} + \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbf{C})_\tau$. Hence we can split to $\phi = F'CB'$ for $F' := F\frac{1}{2}\begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} + \sqrt{\frac{\bar{\alpha}}{\alpha}} & \left(\sqrt{\frac{\bar{\alpha}}{\alpha}} - \sqrt{\frac{\alpha}{\bar{\alpha}}}\right)\lambda \\ \left(\sqrt{\frac{\bar{\alpha}}{\alpha}} - \sqrt{\frac{\alpha}{\bar{\alpha}}}\right)\lambda^{-1} & \sqrt{\frac{\alpha}{\bar{\alpha}}} + \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbf{C})_\tau$ and $B' := \begin{pmatrix} \sqrt{\frac{\bar{\alpha}}{\alpha}} & 0 \\ 0 & \sqrt{\frac{\alpha}{\bar{\alpha}}} \end{pmatrix} B \in \Lambda^+_{\mathbf{R}}\mathrm{SL}_2(\mathbf{C})$. □

Remark 3.3. $\mathscr{B}_\tau$ becomes an open dense subset of $\Lambda\mathrm{SL}_2(\mathbf{C})$ because we can use the same argument as in the proof of Theorem 1.2 (4) in [3].

4. The DPW Method for Spacelike CMC H Surfaces with $0 \le H < 1$ in $\mathbf{S}^{2,1}$

4.1. Holomorphic Potential. Here we will show that all simply-connected spacelike CMC H surfaces with $0 \le H < 1$ in $\mathbf{S}^{2,1}$ are given by a holomorphic potential, defined in Definition 4.1.

Definition 4.1 (holomorphic potential [3], [5]). *Let Σ be a simply-connected domain, $z \in \Sigma$ and $\lambda \in \mathbf{C}$. A* holomorphic potential *ξ is of the form*

$$\xi := A\,dz, \quad A = A(z,\lambda) = \sum_{j=-1}^{\infty} A_j(z)\lambda^j, \tag{4.1}$$

where each $A_j(z)$ is a 2×2 matrix that is independent of λ, is holomorphic in $z \in \Sigma$, is traceless, is a diagonal (resp. off-diagonal) matrix when j is even (resp. odd), and the upper-right entry of $A_{-1}(z)$ is never zero.

4.2. The Inverse Problem of the DPW Method. We will show that given a holomorphic potential ξ, then we get a conformal spacelike CMC H surface

f with $0 \le H < 1$ in $\mathbf{S}^{2,1}$. However, in Theorem 4.1, we will see that finding spacelike CMC H surfaces with $0 \le H < 1$ is equivalent to finding the solution of the extended Lax pair of the form (2.7), and then the surface is found by using the immersion formula (2.9). So to prove that the DPW method finds all spacelike CMC H surfaces with $0 \le H < 1$, we want to prove that the DPW method produces all integrable Lax pairs of the form (2.7) and all their solutions F.

Theorem 4.1 (The inverse problem of the DPW method). *Let* $\xi := A(\lambda)\,dz = \sum_{j=-1}^{\infty} A_j(z)\lambda^j\,dz$ *be a holomorphic potential over a simply-connected domain* Σ *in* $\mathbf{C}$ *including the origin, and let* $\phi : \Sigma \to \Lambda\mathrm{SL}_2(\mathbf{C})$ *be a solution of*

$$d\phi = \phi\xi \quad \text{and} \quad \phi(z)|_{z=0} = I. \tag{4.2}$$

Define the open set $\Sigma^o := \phi^{-1}(\mathscr{B}_\tau) \subset \Sigma$, *and take the unique* $\Lambda\mathrm{SL}_2(\mathbf{C})_\tau$*-Iwasawa splitting on* Σ^o:

$$\phi = FB, \tag{4.3}$$

where

$$F \in \Lambda\mathrm{SU}_2(\mathbf{C})_\tau, \quad B \in \Lambda^+_{\mathbf{R}}\mathrm{SL}_2(\mathbf{C}). \tag{4.4}$$

Then, after a change of coordinates and notations, F *satisfies the extended Lax pair in* (2.7).

We can prove Theorem 4.1 in the same way as in [3], [4]. We omit the proof.

4.3. The Ordinary Problem of the DPW Method. We will consider the converse of Theorem 4.1, in the same way as in [4].

Theorem 4.2 (The ordinary problem of the DPW method). *Let* $f : \Sigma \to \mathbf{S}^{2,1}$ *be a spacelike CMC* H *surfaces with* $0 \le H < 1$ *for simply-connected domain* Σ, *and let* F *be the extended frame of* f *satisfying* (2.7). *Then, there exist* ϕ *such that* $\phi = FB$ *for some* $B \in \Lambda^+_{\triangle}\mathrm{SL}_2(\mathbf{C})$ *and holomorphic potential* ξ *such that* $\xi = \phi^{-1}\,d\phi$.

We can also prove Theorem 4.2 in the same way as in [3], [4]. Again we omit the proof.

5. Behavior of the Frame and Surface When Approaching $\mathscr{P}_\tau$

We have introduced $\Lambda\mathrm{SL}_2(\mathbf{C})_\tau$-Iwasawa splitting and defined the $\Lambda\mathrm{SL}_2(\mathbf{C})_\tau$-Iwasawa big cell $\mathscr{B}_\tau$ and small cell $\mathscr{P}_\tau$. In the previous section, we also showed that on $\Sigma^0 = \phi^{-1}(\mathscr{B}_\tau)$ the surface f is immeresed since the metric function e^u is positive definite. However, here we will show that on $\phi^{-1}(\mathscr{P}_\tau)$ f is not immersed, since the metric function e^u is approaching zero there.

First, we show the following lemma.

Lemma 5.1. *Let* $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^\infty a_j\lambda^j & \sum_{j=1}^\infty b_j\lambda^j \\ \sum_{j=1}^\infty c_j\lambda^j & \sum_{j=0}^\infty d_j\lambda^j \end{pmatrix} \in \Lambda_\triangle^+\mathrm{SL}_2(\mathbf{C})$, *and let* $C = \begin{pmatrix} \frac{1}{2} & \lambda \\ -\frac{1}{2}\lambda^{-1} & 1 \end{pmatrix}$. *Then, there exist three factorizations*:

(1) *If* $|2a_0 + b_1| > |d_0|$, *then*

$$BC^{-1} = K_1\check{B}, \quad K_1 := \begin{pmatrix} u & v\lambda \\ \bar{v}\lambda^{-1} & \bar{u} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbf{C})_\tau, \quad \check{B} \in \Lambda_\triangle^+\mathrm{SL}_2(\mathbf{C}),$$

where u *and* v *are constant in* λ, *and determined by* a, b, c, d.

(2) *If* $|2a_0 + b_1| < |d_0|$, *then*

$$BC^{-1} = K_2\check{B}, \quad K_2 := \begin{pmatrix} u & v\lambda \\ -\bar{v}\lambda^{-1} & -\bar{u} \end{pmatrix} \in \Psi(i\sigma_2)\Lambda\mathrm{SL}_2(\mathbf{C})_\tau, \quad \check{B} \in \Lambda_\triangle^+\mathrm{SL}_2(\mathbf{C}),$$

where u *and* v *are constant in* λ, *and determined by* a, b, c, d.

(3) *If* $|2a_0 + b_1| = |d_0|$, *then*

$$BC^{-1} = K_3C\check{B}, \quad K_3 := \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\alpha}^{-1} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbf{C})_\tau, \quad \check{B} \in \Lambda_\triangle^+\mathrm{SL}_2(\mathbf{C}),$$

where $\alpha := -\frac{2a_0+b_1}{d_0}$.

First two cases imply $BC^{-1} \in \mathscr{B}_\tau$, *and third one implies that* $BC^{-1} \in \mathscr{P}_\tau$.

Proof. By direct computation, we have the following results for the three cases:

(1) $K_1 = \begin{pmatrix} u & v\lambda \\ \bar{v}\lambda^{-1} & \bar{u} \end{pmatrix}$,

$$\check{B} = \begin{pmatrix} a\bar{u} - cv\lambda + \frac{b}{2}\bar{u}\lambda^{-1} - \frac{d}{2}v & -a\bar{u}\lambda + cv\lambda^2 + \frac{b}{2}\bar{u} - \frac{d}{2}v\lambda \\ -a\bar{v}\lambda^{-1} + cu - \frac{b}{2}\bar{v}\lambda^{-2} + \frac{d}{2}u\lambda^{-1} & a\bar{v} - cu\lambda - \frac{b}{2}\bar{v}\lambda^{-1} + \frac{d}{2}u \end{pmatrix},$$

where u and v are the solutions of the following equations:

$$|u|^2 - |v|^2 = 1, \quad 2(2a_0 + b_1)\bar{v} - d_0u = 0.$$

(2) $K_2 = \begin{pmatrix} u & v\lambda \\ -\bar{v}\lambda^{-1} & -\bar{u} \end{pmatrix}$,

$$\check{B} = \begin{pmatrix} -a\bar{u} - cv\lambda - \frac{b}{2}\bar{u}\lambda^{-1} - \frac{d}{2}v & a\bar{u}\lambda + cv\lambda^2 - \frac{b}{2}\bar{u} - \frac{d}{2}v\lambda \\ a\bar{v}\lambda^{-1} + cu + \frac{b}{2}\bar{v}\lambda^{-2} + \frac{d}{2}u\lambda^{-1} & -a\bar{v} - cu\lambda + \frac{b}{2}\bar{v}\lambda^{-1} + \frac{d}{2}u \end{pmatrix},$$

where u and v are the solutions of the following equations:

$$-|u|^2 + |v|^2 = 1, \quad 2(2a_0 + b_1)\bar{v} + d_0 u = 0.$$

(3) $K_3 = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\alpha}^{-1} \end{pmatrix}$,

$$\check{B} = \begin{pmatrix} \sqrt{2\alpha}e^{-i\theta} & 0 \\ 0 & \sqrt{2\alpha}e^{i\theta} \end{pmatrix}$$

$$\times \begin{pmatrix} a\bar{u} - cv\lambda + \frac{b}{2}\bar{u}\lambda^{-1} - \frac{d}{2}v & -a\bar{u}\lambda + cv\lambda^2 + \frac{b}{2}\bar{u} - \frac{d}{2}v\lambda \\ a\bar{v}\lambda^{-1} + cu + \frac{b}{2}\bar{v}\lambda^{-2} + \frac{d}{2}u\lambda^{-1} & -a\bar{v} - cu\lambda + \frac{b}{2}\bar{v}\lambda^{-1} + \frac{d}{2}u \end{pmatrix},$$

where $u = \frac{\alpha}{\sqrt{2}}e^{-i\theta}$ and $v = \frac{1}{\sqrt{2}}e^{i\theta}$. □

This is one of our main theorems:

Theorem 5.1. *Let ϕ_n be a sequence in $\mathscr{B}_\tau$, with $\lim_{n\to\infty}\phi_n = \phi_0 \in \mathscr{P}_\tau$. Let $\phi_n = F_n B_n$ be a $\Lambda\mathrm{SL}_2(\mathbf{C})_\tau$-Iwasawa splitting. Then,*

(1) Writing F_n as

$$F_n = \begin{pmatrix} x_n & y_n \\ \pm i\overline{y_n^*} & \pm\overline{x_n^*} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbf{C})_\tau \cup \Psi(i\sigma_2)\cdot\Lambda\mathrm{SL}_2(\mathbf{C})_\tau \quad \text{for } \overline{x_n^*} := \overline{x_n(i\overline{\lambda^{-1}})},$$

we have $\lim_{n\to\infty}|x_n| = \lim_{n\to\infty}|y_n| = \infty$ for all λ.

(2) Writing the constant term of B_n with respect to λ as

$$B_n|_{\lambda=0} = \begin{pmatrix} \rho_n & 0 \\ 0 & \rho_n^{-1} \end{pmatrix},$$

we have $\lim_{n\to\infty}|\rho_n| = 0$. This implies that f has singularities on $\mathscr{P}_\tau$, since the metric function u is defined as $u = 2\log(\rho)$ (i.e. the metric $ds^2 = 4e^{2u}\,dwd\overline{w} \to 0$ if $n \to \infty$).

Proof. (1) By $\Lambda\mathrm{SL}_2(\mathbf{C})_\tau$-Iwasawa splitting in Theorem 3.1, we have $\phi_0 = F_0 C B_0$. Expressing ϕ_n as

$$\phi_n = \hat{\phi}_n C B_0, \quad \hat{\phi}_n := \phi_n B_0^{-1} C^{-1},$$

we have $\lim_{n\to\infty}\hat{\phi}_n = F_0$. So, for sufficiently large n, we have $\hat{\phi}_n \in \mathscr{B}_\tau$. Thus, for these n, $\hat{\phi}_n$ is $\Lambda\mathrm{SL}_2(\mathbf{C})_\tau$-Iwasawa split into $\hat{\phi}_n = \hat{F}_n\hat{B}_n$, and $\lim_{n\to\infty}\hat{F}_n = F_0$, $\lim_{n\to\infty}\hat{B}_n = I$. Applying Lemma 5.1 with λ replacing $-\lambda$, we have

$$\phi_n = \hat{F}_n\hat{B}_n C B_0 = \hat{F}_n X_n \check{B}_n B_0,$$

where $X_n = K_1(-\lambda)$ or $K_2(-\lambda)$. Thus, $X_n = \begin{pmatrix} u_n & v_n\lambda \\ \pm\overline{v_n}\lambda^{-1} & \pm\overline{u_n} \end{pmatrix}$ for u_n, v_n: constant in λ. By the computaion in the proof of Lemma 5.1, we also have

$$\frac{|u_n|}{|v_n|} = \frac{|2\hat{a}_{0,n} + \hat{b}_{1,n}|}{|\hat{d}_{0,n}|},$$

where $\hat{a}_{0,n}$, $\hat{b}_{1,n}$, $\hat{d}_{0,n}$ are determined by the components of $\hat{B}_n = \begin{pmatrix} \sum_{j=0}^{\infty} \hat{a}_{j,n}\lambda^j & \sum_{j=1}^{\infty} \hat{b}_{j,n}\lambda^j \\ \sum_{j=1}^{\infty} \hat{c}_{j,n}\lambda^j & \sum_{j=0}^{\infty} \hat{d}_{j,n}\lambda^j \end{pmatrix}$. So

$$\lim_{n\to\infty} \hat{b}_{1,n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \hat{a}_{0,n} = \lim_{n\to\infty} \hat{d}_{0,n} = 1 \tag{5.1}$$

because $\lim_{n\to\infty} \hat{B}_n = I$. By (5.1) and $|u_n|^2 - |v_n|^2 = \pm 1$, we get $\lim_{n\to\infty}|u_n| = \lim_{n\to\infty}|v_n| = \lim_{n\to\infty}\|X_n\| = \infty$ for some suitable matrix norm $\|\cdot\|$. Now the uniqueness of the $\Lambda\mathrm{SL}_2(\mathbf{C})_\tau$-Iwasawa splitting says that

$$F_n = \hat{F}_n X_n D_n$$

for some diagonal matrix D_n which is constant in λ. Then we have

$$\|X_n\| = \|\hat{F}^{-1}F_n\| \leq \|\hat{F}^{-1}\| \cdot \|F_n\|,$$

so $\lim_{n\to\infty}\|\hat{F}^{-1}\| \cdot \|F_n\| = \lim_{n\to\infty}\|F_n\| = \infty$, since $\lim_{n\to\infty}\|\hat{F}^{-1}\| = \|F_0\|$ is finite. Because $|x_n|^2 - |y_n|^2 = \pm 1$, we get the conclusion $\lim_{n\to\infty}|x_n| = \lim_{n\to\infty}|y_n| = \infty$.

(2) In the same way as in the proof of (1), we get $\lim_{n\to\infty}|\rho_n^{-1}| = \infty$, by using (5.1). $\square$

6. Criteria for Singularities of Spacelike CMC H Surfaces with $0 \leq H < 1$ in $\mathbf{S}^{2,1}$

In this section, we study singularities of spacelike CMC H surfaces f with $0 \leq H < 1$ in $\mathbf{S}^{2,1}$, similarly to our previous work [12] of spacelike CMC H surfaces with $H > 1$ in $\mathbf{S}^{2,1}$. We will use the frame-changing method, called the s-spectral deformation, in order to specify the types of singularities. Here we introduce criteria for cuspidal edges, swallowtails and cuspidal cross caps on spacelike CMC H surfaces with $0 \leq H < 1$ in $\mathbf{S}^{2,1}$.

Let $f : \Sigma \to \mathbf{S}^{2,1}$ be a spacelike CMC immersion of a simply-connected domain $\Sigma \subset \mathbf{C}$, with the metric $ds^2 = 4g^2\, dwd\overline{w} = 4e^{2u}\, dwd\overline{w}$ and unit normal vector N. First, we consider the frame $\mathfrak{F}$ such that

$$f = \mathfrak{F}\sigma_3\overline{\mathfrak{F}}^t,$$

$$(6.1)\qquad \mathfrak{F}_w = \mathfrak{F}\boldsymbol{A}, \quad \mathfrak{F}_{\overline{w}} = \mathfrak{F}\boldsymbol{B}, \quad \text{where } \boldsymbol{A} = \frac{1}{2}\begin{pmatrix} u_w & 2e^u(1-H) \\ -e^{-u}\mathscr{A} & -u_w \end{pmatrix},$$

$$\boldsymbol{B} = \frac{1}{2}\begin{pmatrix} -u_{\overline{w}} & -e^{-u}\overline{\mathscr{A}} \\ -2e^u(1+H) & u_{\overline{w}} \end{pmatrix}.$$

This $\mathfrak{F}$ is related to $\tilde{F}$ in (2.5) such that $\mathfrak{F} = \tilde{F}|_{\mu=1}$. For this $\mathfrak{F}$, the compatibility condition implies the Gauss and Codazzi equations (2.1). We define *s-spectral deformations* as follows:

Definition 6.1. *The* s-spectral deformation *of the spacelike CMC surface f in $\mathbf{S}^{2,1}$ is the deformation defined by $(1+H) \to is(1+H)$, $(1-H) \to is^{-1}(1-H)$ in Equations* (6.1) *for the parameter $s > 0$.*

The s-spectral deformation maps spacelike CMC surfaces to other spacelike CMC surfaces conformally, as follows:

Theorem 6.1. *For all $s \in \mathbf{R}_{>0}$, the s-spectral deformation deforms a surface f in $\mathbf{S}^{2,1}$, with mean curvature H, metric $ds^2 = 4e^{2u}\,dwd\overline{w}$ and Hopf differential $\mathscr{A}$, into a surface f^s with mean curvature $H^s = \frac{s(1+H)-s^{-1}(1-H)}{s(1+H)+s^{-1}(1-H)}$, metric $4e^{2u^s}\,dwd\overline{w} = 4(k^s)^2e^{2u}\,dwd\overline{w}$ and Hopf differential $\mathscr{A}^s = -ik^s\mathscr{A}$ for $k^s = \frac{s(1+H)+s^{-1}(1-H)}{2}$.*

Proof. Checking the Gauss-Weingarten equations for f^s, we get the conclusion. $\square$

Lemma 6.1. $(f^s)^{-1/s} = f$.

Proof. Direct computation implies $(H^s)^{-1/s} = H$, $(k^s)^{-1/s} = -\frac{1}{k^s}$, $(\mathscr{A}^s)^{-1/s} = \mathscr{A}$, $(u^s)^{-1/s} = u$. $\square$

We define the s-spectral Lax pair.

Definition 6.2 (s-spectral Lax pair). *We define $\mathfrak{F}^s$ as a solution of the following system*:

$$\mathfrak{F}^s_w = \mathfrak{F}^s \boldsymbol{A}^s, \quad \mathfrak{F}^s_{\bar{w}} = \mathfrak{F}^s \boldsymbol{B}^s, \quad \textit{where } \boldsymbol{A}^s := \frac{1}{2}\begin{pmatrix} u_w & 2e^u i s^{-1}(1-H) \\ -e^{-u}\mathscr{A} & -u_w \end{pmatrix},$$

$$\boldsymbol{B}^s := \frac{1}{2}\begin{pmatrix} -u_{\bar{w}} & -e^{-u}\bar{\mathscr{A}} \\ -2e^u i s(1+H) & u_{\bar{w}} \end{pmatrix}.$$

Further, we define the form $\Omega^s := (\mathfrak{F}^s)^{-1}\, d\mathfrak{F}^s$.

Theorem 6.2. *For f given by the frame $\mathfrak{F}$, and mean curvature $0 \le H < 1$, there exists a member of the s-spectral deformation, for the special value $s = s_0 := \sqrt{\frac{1-H}{H+1}}$, that generates a frame $\tilde{\mathfrak{F}} = \mathfrak{F}^{s_0} \in \mathrm{SU}_{1,1}$ (defined in p. 3 of* [12], *etc.*).

Proof. It is easy to see that choosing $s = s_0 := \sqrt{\frac{1-H}{H+1}}$ gives the only deformation that makes the Maurer-Cartan form become an $\mathrm{su}_{1,1}$-valued form. $\square$

As s approaches s_0, the mean curvature goes to infinity, and $\tilde{f} := \tilde{\mathfrak{F}}\sigma_3\bar{\tilde{\mathfrak{F}}}^t$ degenerates to a point, but there still exists a map $\tilde{\mathfrak{F}}$ from Σ to $\mathrm{SU}_{1,1}$ such that $\tilde{\mathfrak{F}}^{-1}\, d\tilde{\mathfrak{F}} = \tilde{\Omega}$ defined by the following (6.2).

Definition 6.3. *We call $\tilde{\mathfrak{F}} : \Sigma \to \mathrm{SU}_{1,1}$ the adjusted frame of $\mathfrak{F}$ and the form $\tilde{\Omega} = \tilde{\mathfrak{F}}^{-1}\, d\tilde{\mathfrak{F}}$ the adjusted Maurer-Cartan form, where*

$$\tilde{\Omega} = \frac{1}{2}\begin{pmatrix} u_w & -2ie^u\sqrt{1-H^2} \\ -e^{-u}\mathscr{A} & -u_w \end{pmatrix} dw + \frac{1}{2}\begin{pmatrix} -u_{\bar{w}} & -e^{-u}\bar{\mathscr{A}} \\ -2ie^u\sqrt{1-H^2} & u_{\bar{w}} \end{pmatrix} d\bar{w} \tag{6.2}$$
$$=: \tilde{\boldsymbol{A}}\, dw + \tilde{\boldsymbol{B}}\, d\bar{w}.$$

Remark 6.1. Defining $G := \mathfrak{F} \cdot \tilde{\mathfrak{F}}^{-1}$, we have $G\sigma_3\bar{G}^t = \mathfrak{F}\sigma_3\bar{\mathfrak{F}}^t = f$.

By the above remark, we can use a new frame G instead of $\mathfrak{F}$, to construct the criteria for singularities of f.

We denote $\tilde{\mathfrak{F}} = \tilde{\mathfrak{F}}(w, \bar{w}) = e^{-u/2}\begin{pmatrix} u_1 & u_2 \\ \overline{u_2} & \overline{u_1} \end{pmatrix} \in \mathrm{SU}_{1,1}$, where $g = e^u$ is the metric function of f, and $|u_1|^2 - |u_2|^2 = g$. So we define $h := \frac{\overline{u_2}}{u_1}$ and $\omega := u_1^2$. By Remark 6.1, we have $f = \mathfrak{F}\sigma_3\bar{\mathfrak{F}}^t = G\sigma_3\bar{G}^t$. Setting $\alpha := (1-H) - i\sqrt{1-H^2}$ and $\beta := -(1+H) + i\sqrt{1-H^2}$, we get

$$ds^2 := 4g^2\, dw d\overline{w} = (1 - |h|^2)^2 |\omega|^2\, dw d\overline{w},$$

$$G^{-1}\, dG = \alpha \begin{pmatrix} -h & 1 \\ -h^2 & h \end{pmatrix} \omega\, dw + \beta \begin{pmatrix} \bar{h} & -\bar{h}^2 \\ 1 & -\bar{h} \end{pmatrix} \overline{\omega}\, d\overline{w}.$$

This implies that, wherever ds^2 is finite, f has a singularity if and only if $h \in \mathbf{S}^1$ or $\omega = 0$. However we consider only extended CMC surfaces f defined in the following Definition 6.4.

Definition 6.4 ([16]). *A CMC surface f restricted to the subdomain $\mathscr{D} = \{p \in \Sigma \,|\, ds^2 < \infty\}$ is called an extended CMC surface if ω, resp. $h^2\omega$, is never zero on $\mathscr{D}$ when $|h| < \infty$, resp. $|h| = \infty$.*

Remark 6.2. By this definition, any point $p \in \Sigma$ is singular only when $|h(p)| = 1$. (See [16].)

We have the following criteria for singularities of spacelike extended CMC H surfaces with $0 \leq H < 1$ in $\mathbf{S}^{2,1}$. The proof of Theorem 6.3 is parallel to the proof of Theorem 7.5 in [12]. (See also [9], [16], [17].)

Theorem 6.3. *Let Σ be a simply connected domain, and let $f : \Sigma \to \mathbf{S}^{2,1}$ be a spacelike extended CMC H surface with $0 \leq H < 1$. Then:*

(1) *A point $p \in \Sigma$ is a singular point if and only if $h \in \mathbf{S}^1$.*

(2) *f is a front at a singular point $p \in \Sigma$ if and only if $\mathrm{Re}\big(\frac{\mathscr{A}}{h^2\omega^2}\big)\big|_p \neq 0$. If this is the case, p is a non-degenerate singular point.*

(3) *f has a cuspidal edge at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Re}\left(\frac{\mathscr{A}}{h^2\omega^2}\right)\bigg|_p \neq 0 \quad \textit{and} \quad \mathrm{Im}\left(\frac{\mathscr{A}}{h^2\omega^2}\right)\bigg|_p \neq 0.$$

(4) *f has a swallowtail at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Re}\left(\frac{\mathscr{A}}{h^2\omega^2}\right)\bigg|_p \neq 0, \quad \mathrm{Im}\left(\frac{\mathscr{A}}{h^2\omega^2}\right)\bigg|_p = 0$$

and

$$\mathrm{Re}\left\{\overline{\left(\frac{\mathscr{A}}{h\omega}\right)}\left(\frac{\mathscr{A}_w h\omega - \mathscr{A}(2\mathscr{A} + 2h\omega_w)}{h^3\omega^3}\right)\right\}\bigg|_p \neq \mathrm{Re}\left\{\left(\frac{\mathscr{A}}{h\omega}\right)\left(\frac{-2\mathscr{A}\omega_{\overline{w}}}{h^2\omega^3}\right)\right\}\bigg|_p.$$

(5) *f has a cuspidal cross cap at a singular point $p \in \Sigma$ if and only if*

$$\operatorname{Re}\left(\frac{\mathscr{A}}{h^2\omega^2}\right)\bigg|_p = 0, \quad \operatorname{Im}\left(\frac{\mathscr{A}}{h^2\omega^2}\right)\bigg|_p \neq 0$$

and

$$\operatorname{Im}\left\{\overline{\left(\frac{\mathscr{A}}{h\omega}\right)}\left(\frac{\mathscr{A}_w h\omega - \mathscr{A}(2\mathscr{A} + 2h\omega_w)}{h^3\omega^3}\right)\right\}\bigg|_p \neq \operatorname{Im}\left\{\left(\frac{\mathscr{A}}{h\omega}\right)\left(\frac{-2\mathscr{A}\omega_{\bar{w}}}{h^2\omega^3}\right)\right\}\bigg|_p.$$

Proof. We can use the same argument as in the proof of Theorem 7.5 of [12], which is the case of CMC $H > 1$. Then, we get the exactly the same claim as Theorem 7.5 of [12], for example: "f has a swallowtail at a singular point $p \in \Sigma$ if and only if

$$\operatorname{Re}\left(\frac{h_w}{h^2\omega}\right)\bigg|_p \neq 0, \quad \operatorname{Im}\left(\frac{h_w}{h^2\omega}\right)\bigg|_p = 0 \quad \text{and}$$

$$\operatorname{Re}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\}\bigg|_p \neq \operatorname{Re}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}\bigg|_p."$$

However, we also have $\mathscr{A} = -2h_w\omega$, $\mathscr{A}_{\bar{w}} = 0$ and $h_{\bar{w}}|_p = 0$, since H is constant. Applying these, we get the conclusion. □

7. Example: Smyth-Type Surfaces with $0 \leq H < 1$ in $\mathbf{S}^{2,1}$ and Their Singularities

7.1. Smyth Surfaces. Smyth studied a generalization of Delaunay surfaces in $\mathbf{R}^3$, which are CMC surfaces with rotationally invariant metrics, in [15]. These surfaces are called *Smyth surfaces*, and there are numerous studies about them. For example, Bobenko and Its studied relationships between Smyth surfaces and Painleve III equations in [1]. The DPW method was also applied to Smyth surfaces in Riemannian spaceforms, in [1], [5]. In our previous work of [12], we constructed the analogue of Smyth surfaces in $\mathbf{R}^{2,1}$, $\mathbf{S}^{2,1}$ and $\mathbf{H}^{2,1}$, and specified the types of singularities they have. However, in [12] we omitted the case of $0 \leq H < 1$ in $\mathbf{S}^{2,1}$ because the Iwasawa splitting given here becomes quite different from the $H > 1$ case.

Here we will construct Smyth-type surfaces with $0 \leq H < 1$ in $\mathbf{S}^{2,1}$. In Equation (7.3), we will notice that Smyth-type surfaces with $0 \leq H < 1$ in $\mathbf{S}^{2,1}$ have an umbilic point at the origin, thus Smyth-type surfaces can be constructed by applying only Proposition 2.1, not by Proposition 2.2. Hence, Smyth-type surfaces with $0 \le H < 1$ are good examples of applying the DPW method introduced in this paper. We also identify the types of singularities on Smyth-type surfaces, using the criteria in Section 6.

7.2. Reflective Symmetry of Smyth-Type Surfaces. Define

$$\xi = \lambda^{-1}\begin{pmatrix} 0 & 1 \\ cw^k & 0 \end{pmatrix} dw, \quad c \in \mathbf{C},\ w \in \Sigma = \mathbf{C}, \tag{7.1}$$

and take a solution ϕ such that $d\phi = \phi\xi$ and $\phi_{w=0} = I$. Now we can assume $c \in \mathbf{R}_{>0}$ using a reparametrization of w and a rigid motion of f, as in [3].

The following Proposition 7.1 is proven in the same way as Theorem 8.2 in [12].

Proposition 7.1. *The surfaces $f : \phi^{-1}(\mathscr{B}_\tau) \to \mathbf{S}^{2,1}$, produced via the DPW method, from ξ in* (7.1), *with $\phi|_{w=0} = I$ and $F_0 = F|_{\lambda = e^{-q/2}}$ for $q < 0$, has reflective symmetry with respect to $k+2$ geodesic planes that meet equiangularly along a geodesic line.*

7.3. The Gauss Equation of Smyth-Type Surfaces. Here we assume the mean curvature is $\mathscr{H} = \frac{i}{2}$ for simplicity, and then we show that the metric of Smyth-type surfaces is rotationally invariant.

Theorem 7.1. *The Gauss equation* (2.1) *for a surface in $\mathbf{S}^{2,1}$ generated by ξ in* (7.1), *with $\phi|_{w=0} = I$, is equivalent to a special case of the Painleve III equations, and the metric function u is rotationally invariant.*

Proof. When $\mathscr{H} = \frac{i}{2}$, the Gauss equation is of the following form:

$$4u_{w\bar{w}} + e^{2u} + |Q|^2 e^{-2u} = 0. \tag{7.2}$$

By the proof of Theorem 4.1, we have

$$Q = -2\mathscr{H}\frac{b_{-1}}{a_{-1}} = -icw^k \quad (i.e.\ \mathscr{A} = cw^k). \tag{7.3}$$

Set $v := u - \frac{1}{2}\log|Q|$, and (7.2) is equivalent to

$$4v_{w\bar{w}} + 2|c| \cdot |w|^k \cosh(2v) = 0. \tag{7.4}$$

Using $v_{w\bar{w}} = \frac{1}{4}\partial_r^2 v + \frac{1}{4r}\partial_r v$ for $r := |w|$, (7.4) becomes

$$\partial_r^2 v + \frac{1}{r}\partial_r v + 2|c| \cdot r^k \cosh(2v) = 0. \tag{7.5}$$

Next we set $x := \frac{1}{1+\frac{k}{2}} r^{1+k/2}\sqrt{|c|}$, and (7.5) is equivalent to

$$\partial_x^2 v + \frac{1}{x}\partial_x v + 2\cosh(2v) = 0. \tag{7.6}$$

(7.6) is a special case of the Painleve III equation that is $y_{xx} = \frac{1}{y}(y_x)^2 - \frac{1}{x}y_x - \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$, for $y = e^v$, $\alpha = \beta = 0$, $\gamma = \delta = -1$. $\square$

Using polar coordinates $w = re^{i\theta}$ and setting $g := e^u$, the Gauss equation and the suitable choice of the initial conditions are as follows:

$$g\left(g_{rr} + \frac{g_r}{r}\right) - g_r^2 + c^2 r^{2k} - 4\mathscr{H}^2 g^4 = 0, \quad g|_{r=0} = 1, \quad g_r|_{r=0} = 0. \tag{7.7}$$

This solution g depends only on r, and some examples of g are seen in Figure 2. The singular set $S(f) := \{(r,\theta) \in \mathbf{C} \mid g(r) = 0\}$ corresponding to this data of g is seen in the right-side of Figure 2. As in Figure 3, Smyth-type surfaces with $\mathscr{H} = \frac{i}{2}$ in $\mathbf{S}^{2,1}$ arrive at the singular set $S(f)$ repeatedly before they diverge to infinity. This phenomenon does not occur in the case of $H > 1$. (See [12].)

7.4. The Types of Singularities on Smyth-Type Surfaces. By numerical calculation, we know that these Smyth surfaces have cuspidal edges, swallowtails and cuspidal cross caps, using the criteria as in Section 6, see Figure 4.

Fact 7.1. *There exist Smyth-type surfaces in* $\mathbf{S}^{2,1}$ *which have cuspidal edges, swallowtails and cuspidal cross caps.* (*See Figures* 4.)

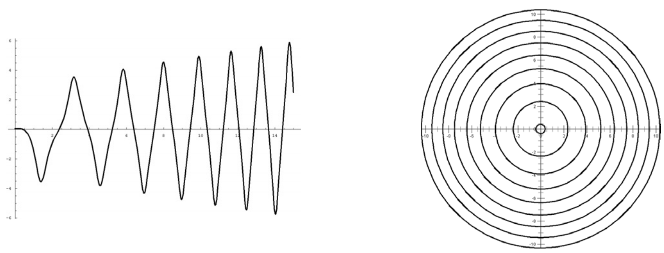

Figure 2: The left image is a solution g of Equation (7.7), and the right image is the corresponding singular set.

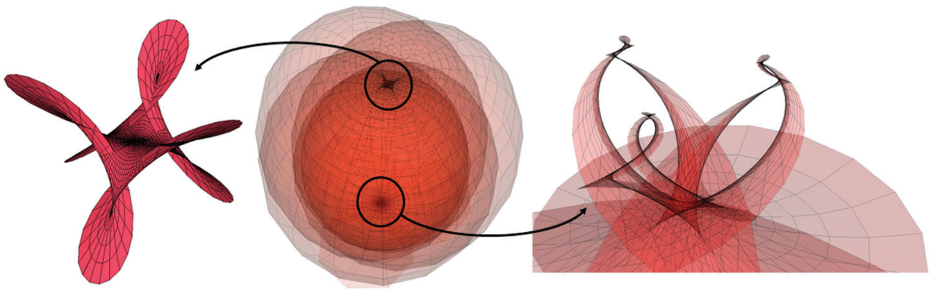

Figure 3: The middle image is a 3-legged Smyth-type surface with $\mathscr{H} = \frac{i}{2}$, and the left image is part of the middle one, from the origin to the first singular set. The right image is a part of the middle one, near second singular set (using the hollow ball model as in [8]).

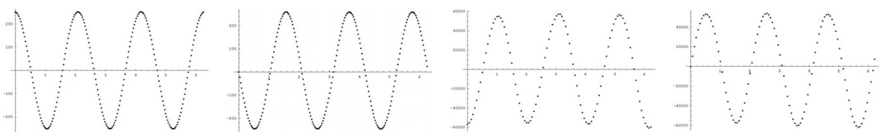

Figure 4: The values of $\mathrm{Re}\left(\frac{\mathscr{A}}{h^2\omega^2}\right)\Big|_p$, $\mathrm{Im}\left(\frac{\mathscr{A}}{h^2\omega^2}\right)\big|_p$, $\mathrm{Re}\left\{\overline{\left(\frac{\mathscr{A}}{h\omega}\right)}\left(\frac{\mathscr{A}_w h\omega - \mathscr{A}(2\mathscr{A}+2h\omega_w)}{h^3\omega^3}\right)\right\}\Big|_p - \mathrm{Re}\left\{\left(\frac{\mathscr{A}}{h\omega}\right)\left(\frac{-2\mathscr{A}\omega_{\bar{w}}}{h^2\omega^3}\right)\right\}\Big|_p$ and $\mathrm{Im}\left\{\overline{\left(\frac{\mathscr{A}}{h\omega}\right)}\left(\frac{\mathscr{A}_w h\omega - \mathscr{A}(2\mathscr{A}+2h\omega_w)}{h^3\omega^3}\right)\right\}\Big|_p - \mathrm{Im}\left\{\left(\frac{\mathscr{A}}{h\omega}\right)\left(\frac{-2\mathscr{A}\omega_{\bar{w}}}{h^2\omega^3}\right)\right\}\Big|_p$ for a 3-legged Smyth-type surface with $\mathscr{H} = \frac{i}{2}$ in $\mathbf{S}^{2,1}$ at (r_0, θ) such that $g(r_0) = 0$ and $0 \le \theta \le 2\pi$ (left to right).

Here we show, for the surfaces in Fact 7.1, that there are at least $2(k+2)$-swallowtails, without relying on numerical calculation, and using only geometric properties. Before doing that, we have a lemma.

Lemma 7.1. *Let $\tilde{\mathfrak{F}} = \tilde{\mathfrak{F}}(w, \overline{w})$ be the solution of the adjusted Lax pair* (6.2) *with $\tilde{\mathfrak{F}}_{w=0} = I$ for the case of a Smyth-type surface. Then $\tilde{\mathfrak{F}}(w) = \sigma_3 \overline{\tilde{\mathfrak{F}}(\overline{w})} \sigma_3$.*

Proof. By direct computation, we have $\tilde{A}(w) = -\tilde{B}(\overline{w})^t$. By this equation and $\tilde{\mathfrak{F}}_{w=0} = I$, we get the conclusion. □

Corollary 7.1.
(1) $h(w) = -\overline{h(\overline{w})}$, $\omega(w) = \overline{\omega(\overline{w})}$.
(2) *At $(r, \theta) = (r_0, 0)$ for $g(r_0) = 0$, we have $h(r_0, 0) = \pm i$, $\omega(r_0, 0) \in \mathbf{R} \backslash \{0\}$, $\omega_w(r_0, 0) = \overline{\omega_{\overline{w}}(r_0, 0)}$.*

Proposition 7.2. *Let $f(w) = f(r, \theta)$ be a $(k+2)$-legged Smyth-type surface in $\mathbf{S}^{2,1}$, and let r_0 satisfy $g(r_0) = 0$. Then f has a swallowtail at $(r_0, 0)$.*

Proof. We will use the criteria of Theorem 6.3, and by the data of the above Corollary 7.1 we can get the conclusion. $\square$

Similarly, we have the same conclusion when $\theta = \frac{\pi}{k+2}$.

Proposition 7.3. *Let $f(w) = f(r, \theta)$ be a $(k+2)$-legged Smyth-type surface in $\mathbf{S}^{2,1}$, and let r_0 satisfy $g(r_0) = 0$. Then f has a swallowtail at $\left(r_0, \frac{\pi}{k+2}\right)$.*

By the above two propositions and the reflective symmetry, we get the following main result:

Theorem 7.2. *If a $(k+2)$-legged Smyth-type surface in $\mathbf{S}^{2,1}$ has singularities, then it has at least $2(k+2)$ swallowtails.*

Remark 7.1. We have checked numerically that there are cuspidal cross caps along the cuspidal edges between each adjacent pair of swallowtails. Thus the surface as in Theorem 7.2 will also have at least $2(k+2)$ cuspidal cross caps.

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