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A METHOD FOR FINDING A MINIMAL POINT OF THE LATTICE IN CUBIC NUMBER FIELDS (II)

By

Kan Kaneko

Abstract. We give a method for finding a minimal point adjacent to 1 of the reduced lattice in cubic number fields using an isotropic vector of the quadratic form and two-dimensional lattice.

1. Introduction

In the previous paper [3] with the same title, we proved six theorems which gave candidates of a minimal point adjacent to 1 in a reduced lattice $\mathcal{R}$.

In this paper we shall improve Theorem 6.1B, Theorem 6.2A and Theorem 6.3A in [3]. We also give such an example that does not seem to occur very frequently in Theorem 6.3B in [3]. We follow the notation and terminology used in the previous paper [3].

In the rest of this introduction, we shall show that ϕ_{10} need not be included in [3, Theorem 6.1B,(3),(ii-a)]. Also, we shall show that ϕ_5 need not be included in [3, Theorem 6.2A,(2),(ii)].

Theorem 6.1B$'$. *Let* $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ *be a reduced lattice of* K *such that* $0 < \lambda < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda, \mu) < 1$, $\omega_2(\lambda, \mu) > 0$, $a > 1$, $2|b| < 1$, $0 < \mu < 1$, $\phi_1 < 1$, $F(\phi_6) < 1$, *where* $a = F(\mu)$, $b = Y_\mu$. *Then*

(1) *If* $F(\phi_2) < 1$, *then the minimal point adjacent to* 1 *is* ϕ_2.

(2) *If* $\phi_2 > 1$, $F(\phi_2) > 1$, *then the minimal point adjacent to* 1 *is* ϕ_6.

(3) *If* $\phi_2 < 1$:

(i) *if* $b < 0$, *then the minimal point adjacent to* 1 *is* ϕ_6;

(ii) *if* $b > 0$, *then the minimal point adjacent to* 1 *is* ϕ_6 *or* ϕ_9.

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Proof. (3) (ii) We assume that $b > 0$, $2\lambda + \mu < 1$ and $\theta_g = \phi_{10} = 3\lambda + 2\mu$. Since $Y_\lambda < -1/2$ and $0 < Y_\mu < 1/2$, we have $Y_{3\lambda+2\mu} = 3Y_\lambda + 2Y_\mu < -3/2 + 1 = -1/2$. From this and $-1 < Y_{3\lambda+2\mu}$, we have $0 < Y_{1+3\lambda+2\mu} < 1/2$. Hence, $F(1 + 3\lambda + 2\mu) = Y_{1+3\lambda+2\mu}^2 + Z_{1+3\lambda+2\mu}^2 < Y_{3\lambda+2\mu}^2 + Z_{3\lambda+2\mu}^2 = F(3\lambda + 2\mu) < 1$. Since $F(1 + 3\lambda + 2\mu) < 1$ and $F(\phi_6) = F(-\phi_6) = F(-1 - \lambda) < 1$, by Remark 1.1 bellow, we have $F\left(\frac{1}{2}(-1 - \lambda) + \frac{1}{2}(1 + 3\lambda + 2\mu)\right) = F(\lambda + \mu) < 1$. Therefore, since $0 < \lambda + \mu < 1$ and $\mathscr{R}$ is a reduced lattice, the assumption such that $b > 0$, $2\lambda + \mu < 1$ and $\theta_g = \phi_{10}$ leads to a contradiction. Hence, if $b > 0$, $2\lambda + \mu < 1$, then $\theta_g \neq \phi_{10}$. $\square$

Remark 1.1. If $F(\alpha) < 1$ and $F(\beta) < 1$, then $F(t\alpha + (1 - t)\beta) < 1$, where $\alpha, \beta \in K$, $0 \leq t \leq 1$ $(t \in \mathbf{Q})$.

Theorem 6.2A′. *Let* $\mathscr{R} = \langle 1, \lambda, \mu \rangle$ *be a reduced lattice of* K *such that* $0 < \lambda < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda, \mu) < 1$, $\omega_2(\lambda, \mu) > 0$, $a > 1$, $2|b| < 1$, $\mu > 1$, $\phi_1 > 1$, *where* $a = F(\mu)$, $b = Y_\mu$. *Then*

(1) *If* $F(\phi_1) < 1$:

(i) *if* $b < 0$, *then the minimal point adjacent to* 1 *is* ϕ_1, ϕ_3 *or* ϕ_4;

(ii) *if* $b > 0$, *then the minimal point adjacent to* 1 *is* ϕ_1 *or* ϕ_7.

(2) *If* $F(\phi_1) > 1$, $F(\phi_6) < 1$, *then the minimal point adjacent to* 1 *is* ϕ_6.

Proof. (2) From [3, Theorem 6.2A,(2),(ii)], suffice it to say that if $b > 0$, then $\theta_g \neq \phi_5$. We assume that $F(\phi_1) > 1$, $F(\phi_6) < 1$. From $F(\phi_1) > 1$, $F(\phi_1 + 1) = F(\phi_6) < 1$, by Lemma 2.1,(2) in Section 2, we have $Y_{\phi_1} < -1/2$. From this and $Y_\mu = b < 1/2$, we have $Y_{\phi_5} = Y_{\phi_1+\mu-1} = Y_{\phi_1} + Y_\mu - 1 < -1/2 + 1/2 - 1 = -1$. Hence, $F(\phi_5) > 1$. Therefore, $\theta_g \neq \phi_5$. $\square$

2. Preliminaries

This section is a preparation for the next section.

Lemma 2.1. (1) *$K \ni 1, \lambda, \mu$ are independent over* $\mathbf{Q} \Rightarrow \omega_2(\lambda, \mu) \notin \mathbf{Q}$.
(2) *Let* $\alpha \in K \backslash \mathbf{Q}$. *If* $F(\alpha) > 1$, $F(1 + \alpha) < 1$, *then* $Y_\alpha < -1/2$.

Proof. (1) Let $K = \mathbf{Q}(\theta)$, $\theta^3 + p\theta + q = 0$ $(p, q \in \mathbf{Q})$ and $\lambda = a_1 + a_2\theta + a_3\theta^2$ $(a_i \in \mathbf{Q})$, $\mu = b_1 + b_2\theta + b_3\theta^2$ $(b_i \in \mathbf{Q})$. Then we have $Y_\lambda = \frac{1}{2}(2a_1 - 2pa_3 -$

$a_2\theta - a_3\theta^2)$, $Y_\mu = \frac{1}{2}(2b_1 - 2pb_3 - b_2\theta - b_3\theta^2)$, $\omega_1(\lambda,\mu) = -\frac{a_2 - a_3\theta}{b_2 - b_3\theta}$. From these and the definition of $\omega_2(\lambda,\mu)$, we obtain the following formula:

$$\omega_2(\lambda,\mu) = \frac{1}{-b_2 + b_3\theta}\left(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + p\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\theta + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\theta^2\right). \tag{2.1}$$

Suppose that $\omega_2(\lambda,\mu) \in \mathbf{Q}$. Then from (2.1), we have

$$-\omega_2 b_2 + \omega_2 b_3\theta = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + p\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\theta + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\theta^2.$$

Since 1, θ, θ^2 are independent over $\mathbf{Q}$, we have $\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = 0$. From this and $\omega_1 = -\frac{a_2 - a_3\theta}{b_2 - b_3\theta}$, we have $\omega_1 \in \mathbf{Q}$. On the other hand, by [3, Proposition 2.2,(3)], $\omega_1(\lambda,\mu) \notin \mathbf{Q}$. Hence, we have reached a contradiction. Therefore, we have $\omega_2(\lambda,\mu) \notin \mathbf{Q}$.

(2) Since $F(1+\alpha) < 1$, we have $-1 < Y_{1+\alpha} < 1$. Suppose that $Y_\alpha > -1/2$. Then $Y_{1+\alpha} = 1 + Y_\alpha > 1/2$. From this, we have $1/4 + Z_{1+\alpha}^2 < Y_{1+\alpha}^2 + Z_{1+\alpha}^2 = F(1+\alpha) < 1$. Hence, $|Z_{1+\alpha}| < \sqrt{3}/2$. Since $Y_\alpha > -1/2$ and $Y_\alpha < 0$, we have $-1/2 < Y_\alpha < 0$. Hence, $F(\alpha) = Y_\alpha^2 + Z_\alpha^2 = Y_\alpha^2 + Z_{1+\alpha}^2 < 1/4 + 3/4 = 1$. Since $F(\alpha) > 1$, we have reached a contradiction. Therefore, we have $Y_\alpha < -1/2$. □

Proposition 2.2. *Let $\mathscr{R} = \langle 1,\lambda,\mu\rangle$ be a reduced lattice of K such that $0 < \omega_1(\lambda,\mu) < 1$, $\omega_2(\lambda,\mu) > 0$, $a > 1$, $2|b| < 1$, $0 < \mu < 1$, $\phi_1 > 1$, where $a = F(\mu)$, $b = Y_\mu$. If $F(\phi_2) > 1$, $F(\phi_6) < 1$:*

(1) *if $F(\phi_1) < 1$, $b < 0$, then the minimal point adjacent to 1 is ϕ_1 or ϕ_3;*

(2) *if $F(\phi_1) < 1$, $b > 0$, then the minimal point adjacent to 1 is ϕ_1,*

(3) *if $F(\phi_1) > 1$, then the minimal point adjacent to 1 is ϕ_6.*

Proof. We assume that $F(\phi_2) > 1$, $F(\phi_6) < 1$.

(a) By [3, Lemma 4.5,(1)], we have $\theta_g \in \{\psi_{i,y};\, y(\neq 0) \in \mathbf{Z}, 1 \le i \le 12\}$.

(b) We shall prove that $y \ge 1$. We note that $[y\omega_i] \le y[\omega_i]$ $(y \le -1)$ and that by [3, Propositon 2.2,(3)] and Lemma 2.1,(1), $[-\omega_i] = -[\omega_i] - 1$. We assume that $y \le -1$. By [3, Remark 4.4,(1)], we have $\psi_{i,y} \le \psi_{12,y}$.

The case $y \le -2$: $\psi_{12,y} = [\omega_2 y] + 2 + y\lambda + ([\omega_1 y] + 1)\mu \le y[\omega_2] + 2 + y\lambda + (y[\omega_1] + 1)\mu = y([\omega_2] + \lambda) + 2 + \mu \le -2([\omega_2] + \lambda) + 2 + \mu < \mu < 1$. The case $y = -1$: $\psi_{12,-1} = [-\omega_2] + 2 - \lambda + ([-\omega_1] + 1)\mu = -[\omega_2] - 1 + 2 - \lambda = -[\omega_2] + 1 - \lambda = -([\omega_2] + \lambda) + 1 < 0$. Therefore, if $y \le -1$, then we have $\psi_{i,y} \neq \theta_g$.

(c) We shall prove that $y = 1$ or 2. Since $\phi_1 = \psi_{4,1} = [\omega_2] + \lambda > 1$, for $y \geq 3$ we have $\psi_{i,y} = [\omega_2 y] + j + y\lambda + ([\omega_1 y] + k)\mu \geq y[\omega_2] + j + y\lambda + (y[\omega_1] + k)\mu \geq 2([\omega_2] + \lambda + [\omega_1]\mu) + [\omega_2] + j + \lambda + ([\omega_1] + k)\mu = 2([\omega_2] + \lambda + [\omega_1]\mu) + [\omega_2] + 1 + \lambda + [\omega_1]\mu + j - 1 + k\mu = 2\psi_{4,1} + \psi_{8,1} + j - 1 + k\mu > \psi_{8,1}$, where $-1 \leq j,k \leq 2$, $(j,k) \neq (2,-1), (2,2), (-1,-1), (-1,2)$. Therefore, if $y \geq 3$, then we have $\psi_{i,y} \neq \theta_g$ $(1 \leq i \leq 12)$.

(d) We shall prove that $y \neq 2$.

(i) The case $b < 0$: By [3, Lemma 4.5,(3),(i)], we have $\theta_g \in \{\psi_{1,y}, \psi_{3,y}, \psi_{4,y}, \psi_{5,y}, \psi_{8,y}, \psi_{9,y}, \psi_{10,y}, \psi_{12,y}\}$. $\psi_{1,2} = [2\omega_2] - 1 + 2\lambda + [2\omega_1]\mu \geq 2[\omega_2] - 1 + 2\lambda = ([\omega_2] + \lambda) - 1 + [\omega_2] + \lambda > \phi_1$. The case $F(\phi_1) < 1$; By [3, Remark 4.4,(1)], we have $\psi_{i,2} > \phi_1$ $(i = 1,3,4,5,8,9,10,12)$. Hence, $\psi_{i,2} \neq \theta_g$ $(i = 1,3,4,5,8,9,10,12)$. The case $F(\phi_1) > 1$; By [3, Lemma 4.5,(10),(12)], we have $\psi_{1,2} = [2\omega_2] - 1 + 2\lambda + [2\omega_1]\mu = 2[\omega_2] + 2\lambda + [2\omega_1]\mu = ([\omega_2] + \lambda) + [\omega_2] + \lambda + [2\omega_1]\mu > \phi_6$. Hence, by [3, Remark 4.4,(1)], we have $\psi_{i,2} > \phi_6$ $(i = 1,3,4,5,8,9,10,12)$. Therefore, $\psi_{i,2} \neq \theta_g$ $(i = 1,3,4,5,8,9,10,12)$.

(ii) The case $b > 0$: By [3, Lemma 4.5,(3),(ii)], we have $\theta_g \in \{\psi_{2,y}, \psi_{4,y}, \psi_{5,y}, \psi_{6,y}, \psi_{7,y}, \psi_{8,y}, \psi_{9,y}, \psi_{11,y}\}$. We have $\psi_{4,2} = [2\omega_2] + 2\lambda + [2\omega_1]\mu \geq 2[\omega_2] + 2\lambda + [2\omega_1]\mu = [\omega_2] + ([\omega_2] + \lambda) + \lambda + [2\omega_1]\mu > \psi_{8,1} = \phi_6$. From this and [3, Remark 4.4,(1)], we have $\psi_{i,2} > \phi_6$ $(i = 4,5,6,7,8,9,11)$. By [3, Lemma 4.5,(12)], for $\psi_{2,2} = [2\omega_2] - 1 + 2\lambda + ([2\omega_1] + 1)\mu$, there are four cases:

1) $\psi_{2,2} = 2[\omega_2] + 2\lambda + 2\mu = [\omega_2] + ([\omega_2] + \lambda) + \lambda + 2\mu > \phi_6$.

2) $\psi_{2,2} = 2[\omega_2] + 2\lambda + \mu = [\omega_2] + ([\omega_2] + \lambda) + \lambda + \mu > \phi_6$.

3) $\psi_{2,2} = 2[\omega_2] - 1 + 2\lambda + 2\mu = ([\omega_2] + \lambda) - 1 + [\omega_2] + \lambda + 2\mu > \phi_1$.

4) $\psi_{2,2} = 2[\omega_2] - 1 + 2\lambda + \mu = ([\omega_2] + \lambda) - 1 + [\omega_2] + \lambda + \mu > \phi_1$.

The case $F(\phi_1) < 1$; we have $\psi_{2,2} \neq \theta_g$.

The case $F(\phi_1) > 1$; Since $F([\omega_2] + \lambda) > 1$, $F([\omega_2] + 1 + \lambda) < 1$, by Lemma 2.1,(2), we have $Y_{[\omega_2]+\lambda} < -1/2$. From this we have $Y_{2[\omega_2]-1+2\lambda+2\mu} = 2Y_{[\omega_2]+\lambda} - 1 + 2Y_\mu < -1 - 1 + 1 = -1$. Hence, we have $F(2[\omega_2] - 1 + 2\lambda + 2\mu) > 1$. Similarly, from $Y_{2[\omega_2]-1+2\lambda+\mu} = 2Y_{[\omega_2]+\lambda} - 1 + Y_\mu < -1 - 1 + 1/2 < -3/2$, we have $F(2[\omega_2] - 1 + 2\lambda + \mu) > 1$. Hence, we have $\psi_{2,2} \neq \theta_g$. By (i), (ii), we conclude that $y \neq 2$.

(e) We shall prove (1), (2) and (3).

(i) The case $b < 0$: From (d), $\theta_g \in \{\psi_{1,1}, \psi_{3,1}, \psi_{4,1}, \psi_{5,1}, \psi_{8,1}, \psi_{9,1}, \psi_{10,1}, \psi_{12,1}\}$. By [3, Remark 4.4,(1)], $\phi_6 = \psi_{8,1} < \psi_{9,1} < \psi_{10,1} < \psi_{12,1}$, so $\theta_g \in \{\psi_{1,1}, \psi_{3,1}, \psi_{4,1}, \psi_{5,1}, \psi_{8,1}\}$. From $F(\psi_{8,1}) < 1$, we have $F(\psi_{1,1}) > 1$. Therefore, we have $\theta_g \in \{\psi_{3,1}, \psi_{4,1}, \psi_{8,1}\}$.

(1) If $F(\phi_1) < 1$, then we have $\theta_g = \phi_1$ or ϕ_3.

(3) We assume that $F(\phi_1) > 1$. By [3, Lemma 4.5,(4)], $F(\psi_{3,1}) > F(\psi_{4,1}) > 1$. Hence, we have $\theta_g = \phi_6$.

(ii) The case $b > 0$: From (d), $\theta_g \in \{\psi_{2,1}, \psi_{4,1}, \psi_{5,1}, \psi_{6,1}, \psi_{7,1}, \psi_{8,1}, \psi_{9,1}, \psi_{11,1}\}$. By [3, Remark 4.4,(1)], $\phi_6 = \psi_{8,1} < \psi_{9,1} < \psi_{11,1}$, so $\theta_g \in \{\psi_{2,1}, \psi_{4,1}, \psi_{5,1}, \psi_{6,1}, \psi_{7,1}, \psi_{8,1}\}$. By [3, Lemma 4.5,(9)], $F(\psi_{2,1}) > 1$. Also, by [3, Lemma 4.5,(5)], $F(\psi_{7,1}) > 1$. Therefore, we have $\theta_g \in \{\psi_{4,1}, \psi_{6,1}, \psi_{8,1}\}$. By [3, Lemma 4.2,(1)], we have $F(\psi_{6,1}) - F(\psi_{5,1}) = a(c_1+2)^2 + 2bc_2(c_1+2) + c_2^2 - a(c_1+1)^2 - 2bc_2(c_1+1) - c_2^2 = 2c_1a + 3a + 2bc_2 = 2a(c_1+1) + a\left(1 + \frac{2b}{a}c_2\right) > 0$, where $c_1 = [\omega_1] - \omega_1, c_2 = [\omega_2] - \omega_2$. Hence, $F(\psi_{6,1}) > F(\psi_{5,1})$. From this and $F(\psi_{5,1}) > 1$, we have $F(\psi_{6,1}) > 1$. Therefore, $\theta_g \in \{\psi_{4,1}, \psi_{8,1}\}$. From this we have (2) $F(\phi_1) < 1 \Rightarrow \theta_g = \phi_1$ and (3) $F(\phi_1) > 1 \Rightarrow \theta_g = \phi_6$. □

Corollary 2.3. *Let* $\mathscr{R} = \langle 1, \lambda, \mu \rangle$ *be a reduced lattice of* K *such that* $0 < \omega_1(\lambda, \mu) < 1$, $\omega_2(\lambda, \mu) > 0$, $a > 1$, $0 < b < 1/2$, $0 < \mu < 1$, $\phi_1 > 1$, *where* $a = F(\mu)$, $b = Y_\mu$. *If* $F(\phi_2) > 1$, $F(\phi_6) < 1$, *then the minimal point adjacent to* 1 *is* ϕ_1 *or* ϕ_6.

Remark 2.4. From the proof in [4, Theorem 2.1] and Proposition 2.2,(3), we can see that Theorem 6.1A in [3] does not require the assumption $0 < X_\mu < X_\lambda$, $0 < \lambda < 1$.

The following two lemmas are used to prove Lemma 3.1 in Section 3.

Lemma 2.5 ([5, Chapter 4, Section 2, p. 51]). *Let* $\mathscr{R}$ *be a reduced lattice with the normalized basis* $\{1, N, M\}$. *If* $\theta_g^\tau = (N+M)^\tau$, *then* $F(M_{(3)}) > 1$.

Lemma 2.6 ([6, Lemma 4.3]). *Let* $\mathscr{R}$ *be a reduced lattice. For* $\alpha \in \mathscr{R}$ *such that* $F(\alpha_{(3)}) < 1$, *we define* $\alpha_* := \begin{cases} \alpha_{(1)} & \text{if } F(\alpha_{(1)}) < 1 \\ \alpha_{(2)} & \text{if } F(\alpha_{(1)}) > 1. \end{cases}$ *Let* $\alpha, \beta \in \mathscr{R}$ *such that* $X_\alpha > 0$, $|Z_\alpha| < \sqrt{3}/2$, $F(\beta) < 1$. *If* $X_\alpha < X_\beta$, $Z_\alpha Z_\beta > 0$, *then* $\alpha_* < \beta$.

3. Improved form of the Theorem 6.3A in [3]

In this section we shall improve Theorem 6.3A,(1),(ii-a) and Theorem 6.3A,(2) in [3]. If we improve Theorem 6.3A,(2) in [3], we can further reduce the maximum number of candidates $\varphi \in \mathscr{R}$ such that we must check whether $F(\varphi) < 1$ or not from at most four to at most three (see Remark 4.4).

To improve Theorem 6.3A,(1),(ii-a), we need the following lemma.

Lemma 3.1. *Let* $\mathscr{R} = \langle 1, \lambda, \mu \rangle$ *be a reduced lattice of* K *such that* $0 < \lambda < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda, \mu) < 1$, $\omega_2(\lambda, \mu) > 0$, $a > 1$, $2|b| < 1$, $\mu < 0$, $\phi_1 > 1$, *where*

$a = F(\mu)$, $b = Y_\mu$. Then if $F(\phi_1) < 1$, $[\omega_2] = 1$, $\lambda + \mu < 0$, then $\theta_g \neq 1 + \phi_9 = 1 + 2\lambda + \mu$.

PROOF. We assume that $F(\phi_1) < 1$, $[\omega_2] = 1$, $\lambda + \mu < 0$ and $\theta_g = 1 + \phi_9 = 1 + 2\lambda + \mu$. We take a normalized basis $\{1, N, M\}$ of $\mathscr{R}$ and fix it. $1 + 2\lambda + \mu$ appears only in the following two cases of the proof of [3, Theorem 6.3A,(1)]:

1) (1-2) in [3, Table 1] i.e., $\psi_{1,2}$ $(\omega_1 > 1/2)$,

2) (1-3) in [3, Table 1] i.e., $\psi_{1,d+1}$ $(d = 1)$.

We note that by [3, Theorem 3.6], $\lambda^\tau = N^\tau$, $(N - M)^\tau$ or M^τ. Moreover, by [3, Theorem 3.3], we see that $\lambda^\tau = (N - M)^\tau \Rightarrow \mu^\tau = -dN^\tau + (d+1)M^\tau$ and that $\lambda^\tau = M^\tau \Rightarrow \mu^\tau = N^\tau - dM^\tau$. In the case (1-2) in [3, Table 1], we have only one case that $\lambda^\tau = N^\tau$, $\mu^\tau = M^\tau$. In the case (1-3) in [3, Table 1], we have two cases, that is, $\lambda^\tau = (N - M)^\tau$ and $\lambda^\tau = M^\tau$. Hence, only the following three cases are possible:

(i) The case $\lambda^\tau = N^\tau$, $\theta_g^\tau = (2N + M)^\tau$, $\omega_1(\lambda, \mu) > 1/2$ which corresponds to (1-2) in [3, Table 1],

(ii) The case $\lambda^\tau = M^\tau$, $\theta_g^\tau = (N + M)^\tau$, $d(\lambda, \mu) = 1$ which corresponds to (1-3) in [3, Table 1],

(iii) The case $\lambda^\tau = (N - M)^\tau$, $\theta_g^\tau = N^\tau$, $d(\lambda, \mu) = 1$ which corresponds to (1-3) in [3, Table 1].

(i) The case $\lambda^\tau = N^\tau$, $\theta_g^\tau = (2N + M)^\tau$, $\omega_1(\lambda, \mu) > 1/2$: From $\omega_1 = |Z_N|/|Z_M| > 1/2$, we have $2|Z_N| > |Z_M|$. From this, we have $Z_{1+\lambda} Z_{1+2\lambda+\mu} = Z_N Z_{2N+M} > 0$. So, since $|Z_{1+\lambda}| = |Z_N| < \sqrt{3}/2$, $F(1 + 2\lambda + \mu) < 1$, $0 < X_{1+\lambda} < X_{1+2\lambda+\mu}$ and $Z_{1+\lambda} Z_{1+2\lambda+\mu} > 0$, Lemma 2.6 leads to $(1 + \lambda)_* < 1 + 2\lambda + \mu$. Since $F(\lambda) > 1$, $F(1 + \lambda) < 1$, we see $1 + \lambda = (1 + \lambda)_*$. Hence, $1 + \lambda < 1 + 2\lambda + \mu$. Therefore, this case is impossible.

(ii) The case $\lambda^\tau = M^\tau$, $\theta_g^\tau = (N + M)^\tau$, $d(\lambda, \mu) = 1$: By Lemma 2.5, this case is impossible.

(iii) The case $\lambda^\tau = (N - M)^\tau$, $\theta_g^\tau = N^\tau$, $d(\lambda, \mu) = 1$:

(a) The case $|Z_\lambda| < \sqrt{3}/2$; Since $0 < X_{1+\lambda} < X_{1+2\lambda+\mu}$, $Z_{1+\lambda} Z_{1+2\lambda+\mu} = Z_{N-M} Z_N > 0$, $|Z_{1+\lambda}| = |Z_\lambda| < \sqrt{3}/2$, Lemma 2.6 leads to $1 + \lambda = (1 + \lambda)_* < 1 + 2\lambda + \mu$. Therefore, $\theta_g \neq 1 + 2\lambda + \mu$.

(b) The case $|Z_\lambda| > \sqrt{3}/2$; Since $|Z_{1+\lambda}| = |Z_\lambda| > \sqrt{3}/2$, $F(1 + \lambda) < 1$, we have $|Y_{1+\lambda}| < 1/2$. If $-1/2 < Y_{1+\lambda} < 0$, then $Y_{1+2\lambda+\mu} < -1$, so $F(1 + 2\lambda + \mu) > 1$. Hence, we conclude that

$$0 < Y_{1+\lambda} < 1/2. \tag{3.1}$$

Since $0 < \lambda < 1$, $-1/2 < \mu < 0$, we see $1/2 < 1 + \lambda + \mu < 1$. Hence, as $\mathscr{R}$ is a reduced lattice, we have

$$F(1 + \lambda + \mu) > 1. \tag{3.2}$$

Since $1^\tau + \lambda^\tau = N^\tau - M^\tau$, $1^\tau + 2\lambda^\tau + \mu^\tau = N^\tau$, we see

$$M^\tau = \lambda^\tau + \mu^\tau. \tag{3.3}$$

From (3.1), we have $-1 < Y_\lambda < -1/2$ and $-1/2 < Y_{1+\lambda+\mu}$. Hence, we see

$$Y_\lambda < Y_{1+\lambda+\mu}. \tag{3.4}$$

Since M^τ is adjacent to $(N - M)^\tau$, we have

$$|Z_M| < |Z_{N-M}|. \tag{3.5}$$

If $|Y_{1+\lambda+\mu}| < |Y_{1+\lambda}|$, then by $|Z_{1+\lambda+\mu}| = |Z_M| < |Z_{N-M}| = |Z_{1+\lambda}|$, we obtain $F(1 + \lambda + \mu) = Z^2_{1+\lambda+\mu} + Y^2_{1+\lambda+\mu} < Z^2_{1+\lambda} + Y^2_{1+\lambda} = F(1 + \lambda) < 1$. From this, by (3.2), we conclude that

$$|Y_{1+\lambda+\mu}| > |Y_{1+\lambda}|. \tag{3.6}$$

If $Y_{1+\lambda+\mu} > 0$, then we have $|Y_{1+\lambda+\mu}| < |Y_{1+\lambda}|$. From this, by (3.6), we conclude that

$$Y_{1+\lambda+\mu} < 0. \tag{3.7}$$

By (3.6), (3.7) and (3.1), we see $-Y_{1+\lambda+\mu} > 1 + Y_\lambda$, so $Y_{1+2\lambda+\mu} < -1$. From this, $F(1 + 2\lambda + \mu) > 1$. Hence, $\theta_g \neq 1 + 2\lambda + \mu$.

By (a), (b), this case is impossible. Therefore, by (i), (ii), (iii), the assumption leads to a contradiction. □

Theorem 6.3A$'$. *Let $\mathscr{R} = \langle 1, \lambda, \mu \rangle$ be a reduced lattice of K such that $0 < \lambda < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda, \mu) < 1$, $\omega_2(\lambda, \mu) > 0$, $a > 1$, $2|b| < 1$, $\mu < 0$, $\phi_1 > 1$, where $a = F(\mu)$, $b = Y_\mu$. Then*

(1) *If $F(\phi_1) < 1$, then the minimal point adjacent to 1 is ϕ_1, ϕ_2 or ϕ_4.*

(2) *If $\phi_2 > 1$:*
  (i) *if $F(\phi_1) > 1$, $F(\phi_8) < 1$, then the minimal point adjacent to 1 is ϕ_2 or ϕ_8;*
  (ii) *if $F(\phi_1) > 1$, $F(\phi_8) > 1$, $F(\phi_6) < 1$, then the minimal point adjacent to 1 is ϕ_6.*

(3) *If $\phi_2 < 1$, $F(\phi_1) > 1$, $F(\phi_6) < 1$, then the minimal point adjacent to 1 is ϕ_6 or ϕ_8.*

Proof. (1) is followed by [3, Theorem 6.3A,(1)] and Lemma 3.1.

(2) We assume that $\phi_2(\lambda,\mu) = [\omega_2(\lambda,\mu)] + \lambda + \mu > 1$.

(i) We assume that $F(\phi_1) > 1$, $F(\phi_8) < 1$. We put $\lambda^+ := \lambda + \mu$, $\mu^- := -\mu$.

(a) Since $\omega_1(\lambda^+,\mu^-) = -Z_{\lambda+\mu}/Z_{-\mu} = -(Z_\lambda + Z_\mu)/(-Z_\mu) = 1 - \omega_1(\lambda,\mu)$, we have $0 < \omega_1(\lambda^+,\mu^-) = 1 - \omega_1(\lambda,\mu) < 1$.

(b) Since $\omega_2(\lambda^+,\mu^-) = -Y_{\lambda^+} - \omega_1(\lambda^+,\mu^-)Y_{\mu^-} = -Y_\lambda - Y_\mu + \omega_1(\lambda^+,\mu^-)Y_\mu = -Y_\lambda - Y_\mu + (1-\omega_1(\lambda,\mu))Y_\mu = -Y_\lambda - \omega_1(\lambda,\mu)Y_\mu = \omega_2(\lambda,\mu)$, we have $\omega_2(\lambda^+,\mu^-) = \omega_2(\lambda,\mu) > 0$.

(c) $a(\mu^-) = F(\mu^-) = F(-\mu) = F(\mu) = a(\mu) > 1$.

(d) $b(\mu^-) = Y_{\mu^-} = -Y_\mu = -b(\mu)$. From this and $-1/2 < b(\mu) < 0$, we have $0 < b(\mu^-) < 1/2$. Also from $-1/2 < \mu < 0$, we have $0 < \mu^- < 1/2 < 1$.

(e) Since $\phi_2(\lambda^+,\mu^-) = [\omega_2(\lambda^+,\mu^-)] + \lambda^+ + \mu^- = [\omega_2(\lambda,\mu)] + \lambda = \phi_1(\lambda,\mu)$, we have $F(\phi_2(\lambda^+,\mu^-)) = F(\phi_1(\lambda,\mu)) > 1$. Also, we have $\phi_1(\lambda^+,\mu^-) = [\omega_2(\lambda^+,\mu^-)] + \lambda^+ = [\omega_2(\lambda,\mu)] + \lambda + \mu = \phi_2(\lambda,\mu)$.

(f) Since $\phi_6(\lambda^+,\mu^-) = [\omega_2(\lambda^+,\mu^-)] + 1 + \lambda^+ = [\omega_2(\lambda,\mu)] + 1 + \lambda + \mu = \phi_8(\lambda,\mu)$, we have $F(\phi_6(\lambda^+,\mu^-)) = F(\phi_8(\lambda,\mu)) < 1$. With (a) to (f), Corollary 2.3 for $\mathscr{R} = \langle 1,\lambda,\mu\rangle = \langle 1,\lambda^+,\mu^-\rangle$ leads to $\theta_g = \phi_1(\lambda^+,\mu^-)$ or $\phi_6(\lambda^+,\mu^-)$. Hence, we have $\theta_g = \phi_2(\lambda,\mu)$ or $\phi_8(\lambda,\mu)$.

(ii) We assume that $F(\phi_1) > 1$, $F(\phi_8) > 1$, $F(\phi_6) < 1$. By [3, Lemma 4.2,(1)], we have $F(\phi_8) - F(\phi_2) = F(\psi_{9,1}) - F(\psi_{5,1}) = a(c_1+1)^2 + 2b(c_1+1)(c_2+1) + (c_2+1)^2 - a(c_1+1)^2 - 2b(c_1+1)c_2 - c_2^2 = 2b(c_1+1) + 2c_2 + 1$, where $c_1 = [\omega_1] - \omega_1$, $c_2 = [\omega_2] - \omega_2$. By [3, Lemma 4.5,(10)], we have $c_2 < -1/2$. From this and $b < 0$, we have $F(\phi_8) - F(\phi_2) = 2b(c_1+1) + 2c_2 + 1 < 0$. Therefore, we have $F(\phi_2) > F(\phi_8)$. From this and $F(\phi_8) > 1$, we have $\theta_g \neq \phi_8, \phi_2$. Therefore, by [3, Theorem 6.3A,(2)], we have $\theta_g = \phi_6$.

(3) We assume that $\phi_2 < 1$, $F(\phi_1) > 1$, $F(\phi_6) < 1$. By [3, Theorem 6.3A,(2)], we have $\theta_g = \phi_6$ or ϕ_8. □

4. Revised Main Theorems

In this section, we shall summarize main theorems in [3, Section 6]. We also give an example such that $\theta_g = \phi_6 + \phi_9 = 1 + 3\lambda + \mu$.

For the simplicity, we denote the following conditions by (#):

(#) $0 < \lambda < 1$, $0 < X_\mu < X_\lambda$, $0 < \omega_1(\lambda,\mu) < 1$, $\omega_2(\lambda,\mu) > 0$, $a > 1$, $2|b| < 1$, where $a = F(\mu)$, $b = Y_\mu$.

Theorem 4.1A. *Let $\mathscr{R} = \langle 1,\lambda,\mu\rangle$ be a reduced lattice of K such that* (#), $0 < \mu < 1$, $\phi_1 > 1$. *Then*

(1) *If* $F(\phi_1) < 1$:
(i) *if* $b < 0$, *then the minimal point adjacent to* 1 *is* ϕ_1, ϕ_3 *or* ϕ_4;
(ii) *if* $b > 0$, *then the minimal point adjacent to* 1 *is* ϕ_1 *or* ϕ_5.
(2) *If* $F(\phi_1) > 1$, $F(\phi_2) < 1$:
(i) *if* $b < 0$, *then the minimal point adjacent to* 1 *is* ϕ_2;
(ii) *if* $b > 0$, *then the minimal point adjacent to* 1 *is* ϕ_2 *or* ϕ_5.
(3) *If* $F(\phi_1) > 1$, $F(\phi_2) > 1$, $F(\phi_6) < 1$, *then the minimal point adjacent to* 1 *is* ϕ_6.

Theorem 4.2A. *Let* $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ *be a reduced lattice of* K *such that* (#), $\mu > 1$, $\phi_1 > 1$. *Then*
(1) *If* $F(\phi_1) < 1$:
(i) *if* $b < 0$, *then the minimal point adjacent to* 1 *is* ϕ_1, ϕ_3 *or* ϕ_4;
(ii) *if* $b > 0$, *then the minimal point adjacent to* 1 *is* ϕ_1 *or* ϕ_7.
(2) *If* $F(\phi_1) > 1$, $F(\phi_6) < 1$, *then the minimal point adjacent to* 1 *is* ϕ_6.

Theorem 4.3A. *Let* $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ *be a reduced lattice of* K *such that* (#), $\mu < 0$, $\phi_1 > 1$. *Then*
(1) *If* $F(\phi_1) < 1$, *then the minimal point adjacent to* 1 *is* ϕ_1, ϕ_2 *or* ϕ_4.
(2) *If* $\phi_2 > 1$:
(i) *if* $F(\phi_1) > 1$, $F(\phi_8) < 1$, *then the minimal point adjacent to* 1 *is* ϕ_2 *or* ϕ_8;
(ii) *if* $F(\phi_1) > 1$, $F(\phi_8) > 1$, $F(\phi_6) < 1$, *then the minimal point adjacent to* 1 *is* ϕ_6.
(3) *If* $\phi_2 < 1$, $F(\phi_1) > 1$, $F(\phi_6) < 1$, *then the minimal point adjacent to* 1 *is* ϕ_6 *or* ϕ_8.

Theorem 4.1B. *Let* $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ *be a reduced lattice of* K *such that* (#), $0 < \mu < 1$, $\phi_1 < 1$, $F(\phi_6) < 1$. *Then*
(1) *If* $F(\phi_2) < 1$, *then the minimal point adjacent to* 1 *is* ϕ_2.
(2) *If* $\phi_2 > 1$, $F(\phi_2) > 1$, *then the minimal point adjacent to* 1 *is* ϕ_6.
(3) *If* $\phi_2 < 1$:
(i) *if* $b < 0$, *then the minimal point adjacent to* 1 *is* ϕ_6;
(ii) *if* $b > 0$, *then the minimal point adjacent to* 1 *is* ϕ_6 *or* ϕ_9.

Theorem 4.2B. *Let* $\mathcal{R} = \langle 1, \lambda, \mu \rangle$ *be a reduced lattice of* K *such that* (#), $\mu > 1$, $\phi_1 < 1$, $F(\phi_6) < 1$. *Then the minimal point adjacent to* 1 *is* ϕ_6.

THEOREM 4.3B. *Let* $\mathscr{R} = \langle 1, \lambda, \mu \rangle$ *be a reduced lattice of* K *such that* $(\#)$, $\mu < 0$, $\phi_1 < 1$, $F(\phi_6) < 1$. *Then*

(1) *If* $F(\phi_8) < 1$, *then the minimal point adjacent to* 1 *is* ϕ_8.

(2) *If* $F(\phi_8) > 1$:

(i) *if* $\phi_9 < 0$, *then the minimal point adjacent to* 1 *is* ϕ_6 *or* $\phi_6 + \phi_9$;

(ii) *if* $\phi_9 > 0$, *then the minimal point adjacent to* 1 *is* ϕ_6 *or* $1 + \phi_9$.

REMARK 4.4. From these six theorems above, we see that

(i) $\theta_g \in S := \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8, \phi_9, 1 + \phi_9, \phi_6 + \phi_9\}$,

(ii) in each case of the theorems, the maximum number of candidates $\varphi \in S$ such that we must check whether $F(\varphi) < 1$ or not is at most three.

REMARK 4.5. In practical computation, if we take a F-point as λ, then we can change (2) in Theorem 4.1A as follows:

(2)′ If $F(\phi_1) > 1$, $F(\phi_2) < 1$, $F(\phi_6) < 1$,

then the minimal point adjacent to 1 is ϕ_2.

Indeed, by the proof of Theorem 6.2A′, $F(\phi_1) > 1$ and $F(\phi_6) < 1$ imply that $\theta_g \neq \phi_5$.

EXAMPLE 4.6. Let $K = \mathbf{Q}(\theta)$ be a cubic number field defined by $\theta^3 - 51589 = 0$ $(\theta = 37.22651403)$. Then $\mathscr{R}_{988} = \langle 1, (-3553 - 76\theta + 5\theta^2)/9912, (-1352 + 415\theta - 11\theta^2)/9912 \rangle = \langle 1, \lambda, \mu \rangle$. $0 < \lambda < 1$, $\mu < 0$. $0 < X_\mu < X_\lambda$. $\omega_1(\lambda, \mu) = \dfrac{76 + 5\theta}{415 + 11\theta}$. $Y_\lambda = \dfrac{1}{2c}(-7106 + 76\theta - 5\theta^2)$ $(c = 9912)$. $Y_\mu = \dfrac{1}{2c}(-2704 - 415\theta + 11\theta^2)$. $\omega_1 = 0.31793235$. $Y_\lambda = -0.56526693$. $Y_\mu = -0.14674417$. $\omega_2 = 0.61192165$. Hence $[\omega_2] = 0$, $\phi_1 = [\omega_2] + \lambda = \lambda < 1$.

(1) $N_{K/\mathbf{Q}}(x + y\theta + z\theta^2) = x^3 - 3 \times 51589xyz + 51589y^3 + 51589^2z^3$.

(a) By (1), $F(\phi_6) = F([\omega_2] + 1 + \lambda) = F(1 + \lambda) = F\left(\dfrac{1}{c}(6359 - 76\theta + 5\theta^2)\right)$ $= \dfrac{1}{c^2}F(6359 - 76\theta + 5\theta^2) = \dfrac{1}{c^2}\dfrac{N_{K/\mathbf{Q}}(6359 - 76\theta + 5\theta^2)}{6359 - 76\theta + 5\theta^2} = \dfrac{1}{c^2}\dfrac{941151982680}{6359 - 76\theta + 5\theta^2} =$ $0.91591078 < 1$.

(b) By (1), $F(1 + 3\lambda + \mu) = F\left(\dfrac{-2099 + 187\theta + 4\theta^2}{c}\right) = \dfrac{1}{c^2}F(-2099 + 187\theta + 4\theta^2) = \dfrac{1}{c^2}\dfrac{N_{K/\mathbf{Q}}(-2099 + 187\theta + 4\theta^2)}{-2099 + 187\theta + 4\theta^2} = \dfrac{1}{c^2}\dfrac{741426600096}{-2099 + 187\theta + 4\theta^2} = 0.72523368 < 1$.

(c) Since $-8458 + 263\theta - \theta^2 < 0$, $2\lambda + \mu = \dfrac{-8458 + 263\theta - \theta^2}{c} < 0$. From this $\phi_8 = 1 + \lambda + \mu < 1$. So $F(\phi_8) > 1$.

(d) Since $2\lambda + \mu < 0$, we have $1 + 3\lambda + \mu < 1 + \lambda$. Therefore, by Theorem 4.3B,(2),(i), we have $\theta_g = 1 + 3\lambda + \mu = \phi_6 + \phi_9$.

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Institute of Mathematics
University of Tsukuba
1-1-1 Tennohdai, Tsukuba
Ibaraki 305-8573, Japan