

THE PERIOD MATRIX OF THE HYPERELLIPTIC CURVE $w^2 = z^{2g+1} - 1$

By

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Abstract. A geometric algorithm is introduced for finding a symplectic basis of the first integral homology group of a compact Riemann surface, which is a p -cyclic covering of \mathbf{CP}^1 branched over 3 points. The algorithm yields a previously unknown symplectic basis of the hyperelliptic curve defined by the affine equation $w^2 = z^{2g+1} - 1$ for genus $g \geq 2$. We then explicitly obtain the period matrix of this curve, its entries being elements of the $(2g + 1)$ -st cyclotomic field. In the proof, the details of our algorithm play no significant role.

1. Introduction

Let X be a compact Riemann surface of genus $g \geq 2$ or smooth projective algebraic curve over \mathbf{C} . The period matrix τ_g of X depends only on the choice of symplectic basis of the first integral homology group $H_1(X; \mathbf{Z})$. It is known that τ_g is symmetric and its imaginary part is positive definite. The Jacobian variety $J(X)$ of X is defined by a complex torus $\mathbf{C}^g / (\mathbf{Z}^g + \tau_g \mathbf{Z}^g)$. Torelli's theorem states that two given Riemann surfaces X and Y are biholomorphic if and only if $J(X)$ and $J(Y)$ are isomorphic as polarized abelian varieties. It implies that τ_g determines the complex structure of X . In general, calculating τ_g is not easy; the difficulty is in finding a symplectic basis of $H_1(X; \mathbf{Z})$. Tretkoff and Tretkoff [24] gave a method to compute τ_g , using *Hurwitz systems*. Andersen, Bene, and Penner [1] showed a way of finding a symplectic basis of $H_1(X; \mathbf{Z})$ using *chord slides* for linear chord diagrams. By combining these two methods, we explicitly write down a geometric algorithm for finding a symplectic basis of the first

integral homology groups of p -cyclic coverings of \mathbf{CP}^1 branched over 3 points for prime number $p \geq 5$. We call it the *chord slide algorithm*. The Frobenius method that Tretkoff and Tretkoff used is steady but not easy to apply to compact Riemann surfaces of generic genus g . We can apply the chord slide algorithm to special surfaces of generic genus g with good linear chord diagrams. Furthermore, we illustrate calculating the period matrix for this kind of Riemann surfaces. We compute the period matrices of the hyperelliptic and Klein quartic curves defined by the affine equations $y^7 = x(1-x)$ and $y^7 = x(1-x)^2$, respectively. These computations are already known; see [24] for the first curve and [8, 12, 17, 18, 19, 22, 26] for the second.

For any odd number $q \geq 5$, let $C_{q,1}$ be the smooth projective curve over \mathbf{C} defined by the affine equation $y^q = x(1-x)$. This is biholomorphic to the hyperelliptic curve defined by the affine equation $w^2 = z^{2q+1} - 1$ of genus $g = (q-1)/2$. We can apply the chord slide algorithm to this curve $C_{q,1}$ and obtain a symplectic basis $\{a_i, b_i\}_{i=1,2,\dots,g}$ of $H_1(C_{q,1}; \mathbf{Z})$ different to the well-known basis in [2]. The advantage of our method is its applicability to other curves, for example, nonhyperelliptic ones.

For generic genus, few examples of period matrices are known. Schindler [20] computed the period matrices of three types of hyperelliptic curves of genus $g \geq 2$. These matrices are the only examples as far as we know. Kamata [13] introduced an algorithm for calculating those of Fermat-type curves. We know of no other algorithms except Kamata's and Tretkoff and Tretkoff's. For explicit computations for cases of low genus, see [7, 9, 15] except for the above hyperelliptic and Klein quartic curves. We remark that Streit [21] studied the period matrices from the viewpoint of representation theory. Tashiro, Yamazaki, Ito, and Higuchi [23] computed the periods on $C_{q,1}$. We obtain the period matrix τ_g of $C_{q,1}$ using the inverse of the Vandermonde matrix. Our original contribution to the computation of τ_g is in finding the inverse matrix Ω_A^{-1} , which is defined in Section 4. Schindler [20] obtained one for the same hyperelliptic curve of genus $g \geq 2$ defined by the affine equation $w_1^2 = z_1(z_1^{2q+1} - 1)$, which is biholomorphic to $C_{q,1}$. This result contains a recurrence relation. However, we have an explicit representation of τ_g . Set $\zeta = \zeta_q = \exp(2\pi\sqrt{-1}/q)$. A symplectic basis $\{A_i, B_i\}_{i=1,2,\dots,g}$ of $H_1(C_{q,1}; \mathbf{Z})$ is defined later. For variables x_1, x_2, \dots, x_n , we denote by $\sigma_i(x_1, x_2, \dots, x_n)$ the symmetric polynomial

$$\sum_{1 \leq j_1 < \dots < j_i \leq n} x_{j_1} \cdots x_{j_i}$$

for $1 \leq i \leq n$ and $\sigma_0(x_1, x_2, \dots, x_n) = 1$.

THEOREM 4.4. *With respect to the symplectic basis $\{A_i, B_i\}_{i=1,2,\dots,g}$, the period matrix τ_g of $C_{g,1}$ is*

$$\tau_g = \left(\sum_{k=1}^g \frac{(-1)^{i+g}}{2g+1} (1 - \zeta^{2kj}) \sigma_{g-i}(\zeta^2, \zeta^4, \dots, \widehat{\zeta^{2j}}, \dots, \zeta^{2g}) \prod_{m=g-k+1}^{2g-k} (1 - \zeta^{2m}) \right)_{i,j},$$

where the ‘hat’ symbol $\widehat{}$ over an element ζ^{2j} indicates that this element is deleted from the sequence $\zeta^2, \zeta^4, \dots, \zeta^{2g}$.

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2. Algorithm

We begin by writing down the geometric algorithm, called the chord slide algorithm, for finding a symplectic basis of the first integral homology groups of the smooth projective algebraic curves defined by the affine equation $y^p = x^l(1-x)^m$. Here $p \geq 5$ is a prime number, l, m are coprime, and $1 \leq l, m, l+m < p-1$. We denote this curve by $X_{p,l,m}$. This is a Riemann surface of genus $g = (p-1)/2$ and can be considered as a p -sheeted cyclic covering of \mathbf{CP}^1 branched over $\{0, 1, \infty\} \subset \mathbf{CP}^1$. In particular, we simply write the curve $X_{p,1,m} = C_{p,m}$ for the case $l = 1$. Throughout this section, we work with the Klein quartic $C_{7,2}$ as an example. Moreover, we detail an algorithm for calculating the period matrix of $X_{p,l,m}$ using holomorphic 1-forms of Bennama and Carbonne [6]. It is known that up to isomorphism there are only two curves $X_{7,l,m}$. They are $C_{7,1}$ and $C_{7,2}$; see [10, §1.3.2] for example. We next calculate their period matrices.

2.1. Dessins D’enfants. Let X be a smooth projective algebraic curve over a field k . We assume that k is \mathbf{C} and there exists a covering $\pi : X \rightarrow \mathbf{CP}^1$

branched over $\{0, 1, \infty\} \subset \mathbf{C}P^1$. The inverse image $\pi^{-1}([0, 1])$ in X of the unit interval in $\mathbf{C}P^1$ is called a dessin d'enfants [11]. It is a topological bipartite graph illustrated on the Riemann surface X . Belyi [3] proved that all algebraic curves over $\overline{\mathbf{Q}}$ correspond to dessins d'enfants. The map π is often called the Belyi map; see also [25]. In the rest of this paper, we assume that X is $X_{p,l,m}$ and $\pi : X \ni (x, y) \mapsto x \in \mathbf{C}P^1$ is a p -cyclic covering branched over $\{0, 1, \infty\} \subset \mathbf{C}P^1$. Set the order- p holomorphic automorphism $\sigma(x, y) = (x, \zeta_p y)$. Here, we denote $\zeta_p = \exp(2\pi\sqrt{-1}/p)$. Let $y_0(t)$ be a real analytic function $\sqrt[p]{t^l(1-t)^m}$. A continuous path $I_0 : [0, 1] \rightarrow X$ is defined by the equation $I_0(t) = (t, y_0(t)) \in X$ for $0 \leq t \leq 1$. We immediately obtain $\pi(I_0) = [0, 1] \subset \mathbf{C}P^1$ and the dessin d'enfants $\pi^{-1}([0, 1]) = \bigcup_{i=0}^{p-1} \sigma_*^i(I_0)$. We call $\pi^{-1}(0)$ and $\pi^{-1}(1)$ the white and black vertices, respectively. The dessin $\pi^{-1}([0, 1])$ in X is a bipartite graph. Take a point b_i on $\sigma_*^i(I_0)$ except for endpoints for each i . For the Klein quartic $C_{7,2}$, we draw a dessin d'enfants $\pi^{-1}([0, 1])$; see Figure 1.

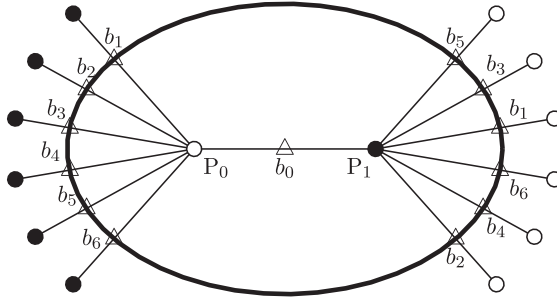


Figure 1: Dessin d'enfants for the Klein quartic $C_{7,2}$

2.2. Intersection Numbers. We now introduce the method of Tretkoff and Tretkoff [24], based on the Hurwitz system, from dessins d'enfants to the intersection numbers of the loops in X . Let V be the set of $2g = p - 1$ labeled points on the unit circle S^1 . A chord diagram on V is a set of oriented simple chords between points of V .

For $i = 1, 2, \dots, p - 1$, let c_i denote the loop $I_0 \cdot \sigma_*^i(I_0)^{-1}$ in X . Here, the product $I_0 \cdot \sigma_*^i(I_0)^{-1}$ signifies that we traverse I_0 first and then $\sigma_*^i(I_0)^{-1}$. It follows that the dessin d'enfants $\pi^{-1}([0, 1])$ equals the union $\bigcup_{i=0}^{p-1} c_i$. We deform the dessin topologically and consider c_i as not loops but chords. We get a chord diagram and compute the intersection numbers $c_i \cdot c_j = 0$ or ± 1 . This satisfies the property in [4, §8.1]. Let $(a_{i,j})$ denote the matrix with (i, j) -th entry $a_{i,j}$. If

the $2g \times 2g$ intersection matrix $(c_i \cdot c_j)$ is regular, then $\{c_i\}_{i=1,2,\dots,2g}$ is a basis of the first integral homology group $H_1(X; \mathbf{Z})$.

For $C_{7,2}$, we obtain the chord diagram in Figure 2 that corresponds to Figure 1. For convenience, the origin and terminal point of c_i is denoted by i and \bar{i} , respectively. The intersection matrix is as follows [24, pp. 482]

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

We can choose from many bases of $H_1(X; \mathbf{Z})$. For example, we [22] can choose the basis $\{\ell_i\}_{i=1,2,\dots,6}$ such that $\ell_i = c_{i-1} \cdot c_i^{-1}$ for $C_{7,2}$.

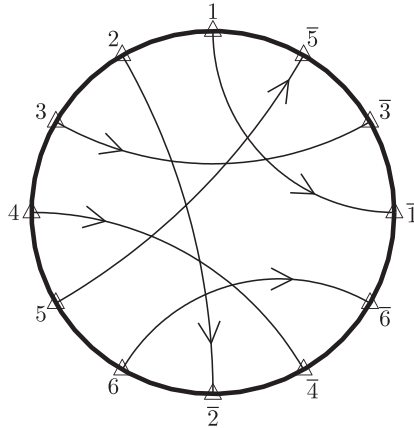


Figure 2: Chord diagram of $C_{7,2}$

We write down the intersection number $c_i \cdot c_j$ for $X_{p,l,m}$. For $1 \leq i \leq p-1$, the integer $i_l \in \{1, 2, \dots, p-1\}$ is uniquely determined such that $i_l l \equiv i$ modulo p . We define i_m similarly. Draw the loops c_i and c_j . The initial point i of c_i is the i_l -th point moving counter-clockwise from the point l and the terminal point \bar{i} is the i_m -th point from the point \bar{m} . We give the associated chord diagram in Figure 3 for the case $j_l - i_l > 0$ and $j_m - i_m > 0$. In this case, we have $c_i \cdot c_j = 1$.

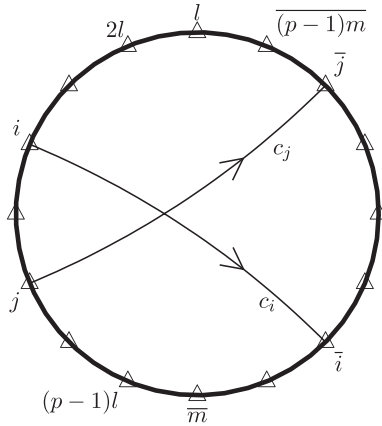


Figure 3: $c_i \cdot c_j = 1$ for $j_l - i_l > 0$ and $j_m - i_m > 0$

REMARK 2.1. We have the intersection number

$$c_i \cdot c_j = \begin{cases} 1 & (j_l - i_l > 0 \text{ and } j_m - i_m > 0), \\ -1 & (j_l - i_l < 0 \text{ and } j_m - i_m < 0), \\ 0 & (\text{otherwise}). \end{cases}$$

This is easily seen to be true.

2.3. Linear Chord Diagrams. We identify chord diagrams with linear chord diagrams. A linear chord diagram in the plane with k chords is defined as an interval $[0, 2k]$, together with k oriented simple arcs in the upper half plane between the integer points $\{1, 2, \dots, 2k\}$.

Cut open a chord diagram from Section 2.2 at a certain point on S^1 . By identifying the $2g$ chords with loops $\{c_1, c_2, \dots, c_{2g}\}$, we have the corresponding linear chord diagram. The end points of c_i 's on S^1 determine the intersection number $c_i \cdot c_j$. We remark that $c_i \cdot c_j$ is independent of the choice of cutting points, but the chord slide method depends on it.

For $C_{7,2}$, choosing point 3 in Figure 2 produces the linear chord diagram in Figure 4.

2.4. Chord Slides. Andersen, Bene, and Penner [1] used chord slides for the linear chord diagrams. They studied chord slides with the Whitehead moves on the dual of the fat graphs embedded in a surface of genus g with one boundary component; see also [4]. We simply use chord slides to compute the intersection numbers and find the matrix T such that $(c_1, c_2, \dots, c_{2g})^t T$ is a symplectic matrix.

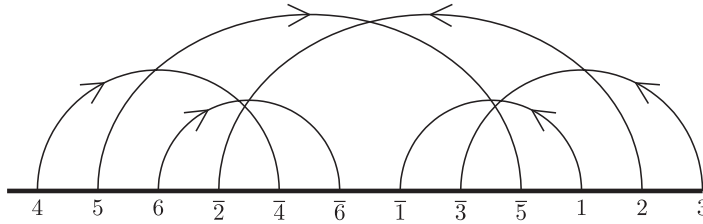


Figure 4: A linear chord diagram of $C_{7,2}$

For the linear chord diagram, we define a *chord slide* of c_i along c_j for the same position by the transformation from $(c_1, c_2, \dots, c_{2g})$ to $(c'_1, c'_2, \dots, c'_{2g})$ such that

$$c'_k = \begin{cases} c_i - c_j & (k = i), \\ c_k & (k \neq i). \end{cases}$$

as homology classes. For the *opposite position*, the c'_i is replaced with $c_i + c_j$. We define the *position* of a chord slide of c_i along c_j in the linear chord diagram. It is the *same position* that c_i initially (ultimately) moves in the direction towards the origin (terminal) point of c_j . The others are *opposite positions*. We remark that the position does not depend on the orientation of the chord; see Figures 5 and 6. Using the origin and terminal points, we simply write the chord slide in Figure 5 and 6 as $c \rightarrow d$ and $\bar{c} \rightarrow \bar{d}$, respectively.

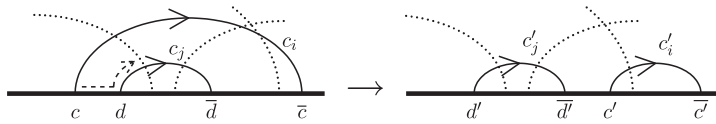


Figure 5: A chord slide of c_i along c_j for the same position

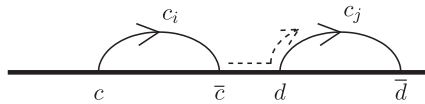


Figure 6: A chord slide for the opposite position

For $i \neq j$, let $M_s(\alpha, \beta)$ and $M_o(\alpha, \beta)$ denote the $2g \times 2g$ matrices for which the (i, j) -th entries are -1 and 1 respectively for $(i, j) = (\alpha, \beta)$ and $\delta_{i,j}$ for $(i, j) \neq (\alpha, \beta)$. Here $\delta_{i,j}$ is Kronecker's delta. After a chord slide of c_i along c_j for the same position, we have

$$(c'_1, c'_2, \dots, c'_{2g}) = (c_1, c_2, \dots, c_{2g})^t M_s(i, j).$$

For the opposite position, ${}^tM_s(i, j)$ may be replaced with ${}^tM_o(i, j)$. Let $s_{i,j}(A)$ and $o_{i,j}(A)$ denote, respectively, matrices $M_s(i, j)A^tM_s(i, j)$ and $M_o(i, j)A^tM_o(i, j)$. By noting the changes in the intersection numbers of the chords $(c_1, c_2, \dots, c_{2g})$, we have

PROPOSITION 2.2. *Consider loops $\{c_k\}_{k=1,2,\dots,2g}$ as chords in the linear chord diagram and A its intersection matrix. If we slide c_i along c_j for the same position, the intersection matrix of $\{c'_k\}_{k=1,2,\dots,2g}$ is $s_{i,j}(A)$. For the opposite position, it is $o_{i,j}(A)$.*

Using this proposition, we have only to deform the intersection matrix into a $2g \times 2g$ symplectic matrix $\begin{pmatrix} O & I_g \\ -I_g & O \end{pmatrix}$. Here I_g is the identity matrix of size g .

For the Klein quartic $C_{7,2}$, the chord slide algorithm yields the matrices

$$s_{6,4} \circ o_{6,2} \circ o_{5,2} \circ o_{5,1} \circ s_{2,1}(A)$$

and

$$M = M_s(6, 4)M_o(6, 2)M_o(5, 2)M_s(5, 3)M_o(5, 1)M_s(2, 1),$$

or explicitly

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 \end{pmatrix},$$

respectively. We denote $(c_1, c_2, \dots, c_6)^t M$ by $(c'_1, c'_2, \dots, c'_6)$ such that $c'_3 = c_5$, $c'_4 = c_3$, and $c'_5 = c_4$. The resulting matrix is denoted by $T_{7,2}$. We have the following matrices $T_{7,2}$ and $T_{7,2}A^tT_{7,2}$,

$$T_{7,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}, \quad T_{7,2}A^tT_{7,2} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

from which we then obtain a symplectic basis $(a_1, a_2, a_3, b_1, b_2, b_3) = (c_1, c_2, \dots, c_6)^t T_{7,2}$ of $H_1(C_{7,2}; \mathbf{Z})$. The matrix $T_{7,2}$ is different from that given in [24].

2.5. Period Matrices. We introduce a method for calculating the period matrix of X using holomorphic 1-forms of Bennama and Carbonne [6]. We compute those of $C_{7,1}$ and $C_{7,2}$.

Let $H^{1,0}(X)$ be the space of holomorphic 1-forms on X . The floor function is denoted by $[\cdot]$. For $n = 1, 2, \dots, p - 1 (= 2g)$, we define α_l and α_m by $\left[\frac{nl}{p}\right]$ and $\left[\frac{nm}{p}\right]$, respectively. Set $d_n = \left[\frac{n(l+m)}{p}\right] - \alpha_l - \alpha_m - 1$. Bennama and Carbonne [6] derived a basis for $H^{1,0}(X)$

$$\omega_{n,d} = \frac{x^{\alpha_l}(1-x)^{\alpha_m}x^d}{y^n} \quad \text{with } 0 \leq d \leq d_n \text{ and } 1 \leq n \leq p - 1.$$

Take a symplectic basis $\{a_i, b_i\}_{i=1,2,\dots,g}$ of $H_1(X; \mathbf{Z})$, i.e., their intersection numbers are $a_i \cdot b_j = \delta_{i,j}$ and $a_i \cdot a_j = b_i \cdot b_j = 0$. We define two $g \times g$ matrices Ω_A and Ω_B by $(\int_{a_j} \omega_i)$ and $(\int_{b_j} \omega_i)$. It is known that the period matrix with respect to $\{a_i, b_i\}_{i=1,2,\dots,g}$ is obtained by $\Omega_A^{-1}\Omega_B$; see [16], for example.

We illustrate with curves $C_{7,1}$ and $C_{7,2}$. For $k = 1, 2$, we define a basis $\eta_1^k, \eta_2^k, \eta_3^k$ of $H^{1,0}(C_{7,k})$ as follows:

	η_1^k	η_2^k	η_3^k
$C_{7,1}$	$\frac{dx}{y^6}$	$\frac{dx}{y^5}$	$\frac{dx}{y^4}$
$C_{7,2}$	$\frac{(1-x) dx}{y^6}$	$\frac{(1-x) dx}{y^5}$	$\frac{dx}{y^3}$

Let $B(u, v)$ denote the beta function $\int_0^1 t^{u-1}(1-t)^{v-1} dt$ for $u, v > 0$. Put $(h_1, h_2, h_3, h_4) = (1/7, 2/7, 4/7, 1/7)$. From

$$\int_{I_0} \eta_i^k = \begin{cases} B(i/7, i/7) & (k = 1) \\ B(h_i, h_{i+1}) & (k = 2) \end{cases}$$

the holomorphic 1-forms ω_i^1 and ω_i^2 are denoted by $\eta_i^1/B(i/7, i/7)$ and $\eta_i^2/B(h_i, h_{i+1})$, respectively. The equation

$$\int_{c_j} \omega_i^k = \int_{I_0} \omega_i^k - \int_{\sigma_i^j I_0} \omega_i^k = \int_{I_0} \omega_i^k - \int_{I_0} (\sigma^*)^j \omega_i^k$$

yields

LEMMA 2.3.

$$\int_{c_j} \omega_i^k = \begin{cases} 1 - \zeta_7^{ij} & (k = 1), \\ 1 - \zeta_7^{7h_i j} & (k = 2). \end{cases}$$

REMARK 2.4. These integrals depend only on the cohomology class of ω_i^k and the homology class of c_j .

For $k = 1, 2$, let A_k and B_k be the 3×3 matrices $(\int_{a_j} \omega_i^k)$ and $(\int_{b_j} \omega_i^k)$, respectively. We construct the 3×6 matrices

$$(A_k, B_k) = \left(\int_{c_j} \omega_i^k \right)^t T_{7,k}.$$

Here, for the case $p = 7$, we denote $T_{7,1} = T$, the matrix (3.1) in Remark 3.6. We obtain the period matrix for $C_{7,k}$.

PROPOSITION 2.5. *Let $\tau_{7,k}$ be the period matrix $A_k^{-1}B_k$ of $C_{7,k}$. Then we have*

$$\tau_{7,1} = \begin{pmatrix} 4 + \zeta + 2\zeta^2 + 2\zeta^3 + \zeta^4 + 2\zeta^5 & -1 - 2\zeta - 2\zeta^3 - 2\zeta^5 & -1 + \zeta - \zeta^2 \\ -1 - 2\zeta - 2\zeta^3 - 2\zeta^5 & -1 + \zeta - \zeta^2 - \zeta^4 - \zeta^5 & 1 + \zeta^3 + \zeta^5 \\ -1 + \zeta - \zeta^2 & 1 + \zeta^3 + \zeta^5 & 1 + \zeta^2 \end{pmatrix}$$

and

$$\tau_{7,2} = \frac{1}{4} \begin{pmatrix} 6 + 3\zeta & 4 + 2\zeta & -2 - \zeta \\ 4 + 2\zeta & 4 + 4\zeta & -2\zeta \\ -2 - \zeta & -2\zeta & 2 + 3\zeta \end{pmatrix},$$

where $\zeta = \zeta_7$ and $\xi = \zeta + \zeta^2 + \zeta^4 = (-1 + \sqrt{-7})/2$.

PROOF. By definition of A_k and B_k , we have matrices

$$A_1 = \begin{pmatrix} 1 - \zeta & 1 - \zeta + \zeta^2 - \zeta^3 & 1 - \zeta + \zeta^2 - \zeta^3 + \zeta^4 - \zeta^5 \\ 1 - \zeta^2 & 1 - \zeta^2 + \zeta^4 - \zeta^6 & 1 - \zeta^2 + \zeta^4 - \zeta^6 + \zeta^8 - \zeta^{10} \\ 1 - \zeta^3 & 1 - \zeta^3 + \zeta^6 - \zeta^9 & 1 - \zeta^3 + \zeta^6 - \zeta^9 + \zeta^{12} - \zeta^{15} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 - \zeta^2 & 1 - \zeta + \zeta^2 - \zeta^4 & 1 - \zeta + \zeta^2 - \zeta^3 + \zeta^4 - \zeta^6 \\ 1 - \zeta^4 & 1 - \zeta^2 + \zeta^4 - \zeta^8 & 1 - \zeta^2 + \zeta^4 - \zeta^6 + \zeta^8 - \zeta^{12} \\ 1 - \zeta^6 & 1 - \zeta^3 + \zeta^6 - \zeta^{12} & 1 - \zeta^3 + \zeta^6 - \zeta^9 + \zeta^{12} - \zeta^{18} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 - \zeta & \zeta - \zeta^2 & 1 - \zeta^2 + \zeta^3 - \zeta^5 \\ 1 - \zeta^2 & \zeta^2 - \zeta^4 & 1 - \zeta^4 + \zeta^6 - \zeta^{10} \\ 1 - \zeta^4 & \zeta^4 - \zeta^8 & 1 - \zeta^8 + \zeta^{12} - \zeta^{20} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 1 - \zeta^3 & 1 - \zeta^4 & \zeta - \zeta^2 + \zeta^4 - \zeta^6 \\ 1 - \zeta^6 & 1 - \zeta^8 & \zeta^2 - \zeta^4 + \zeta^8 - \zeta^{12} \\ 1 - \zeta^{12} & 1 - \zeta^{16} & \zeta^4 - \zeta^8 + \zeta^{16} - \zeta^{24} \end{pmatrix}.$$

The determinants of A_1 and A_2 are $-7(\zeta^4 + \zeta^5)$ and $7(1 - \zeta)$, respectively. Using the adjoint matrices of A_1 and A_2 , we obtain their inverse matrices

$$A_1^{-1} = -\frac{1 + \zeta^2 + \zeta^3 + \zeta^5}{7} \begin{pmatrix} -\zeta + 3\zeta^2 + 4\zeta^3 + \zeta^4 & -1 + 3\zeta - \zeta^2 + 2\zeta^3 - \zeta^4 - 2\zeta^5 & -3 - 2\zeta - 2\zeta^2 - \zeta^3 - 4\zeta^4 - 2\zeta^5 \\ 1 + 4\zeta + 3\zeta^2 - \zeta^3 & 1 - \zeta + 3\zeta^3 + 4\zeta^5 & -\zeta + 2\zeta^2 - 2\zeta^4 + \zeta^5 \\ -1 - 3\zeta - 4\zeta^2 - 2\zeta^3 - 2\zeta^4 - 2\zeta^5 & 1 + 2\zeta^2 - 2\zeta^3 - \zeta^5 & 2 + \zeta + 2\zeta^3 + 3\zeta^4 - \zeta^5 \end{pmatrix},$$

$$A_2^{-1} = \frac{2 + \zeta}{28} \begin{pmatrix} 3 + \zeta - \zeta^3 - 3\zeta^4 & 4 - 2\zeta + 2\zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 & 3 - 3\zeta^2 + \zeta^4 - \zeta^5 \\ -\zeta - \zeta^2 - 5\zeta^3 - 4\zeta^4 - 3\zeta^5 & 5 + \zeta + 4\zeta^2 + 2\zeta^3 + 4\zeta^4 + 5\zeta^5 & 3 + 2\zeta - \zeta^2 + 3\zeta^3 + 2\zeta^4 - 2\zeta^5 \\ -2\zeta - \zeta^2 + \zeta^3 + 2\zeta^4 & -1 + \zeta - 3\zeta^2 - \zeta^3 - 2\zeta^4 - \zeta^5 & -\zeta + 2\zeta^2 - 2\zeta^4 + \zeta^5 \end{pmatrix}.$$

It suffices to calculate $A_1^{-1}B_1$ and $A_2^{-1}B_2$. □

3. Two Symplectic Bases of One Family of Hyperelliptic Curves

For *odd* integer $q \geq 5$, let $C_{q,1}$ be a plane algebraic curve defined by the affine equation $y^q = x(1-x)$. We obtain two symplectic bases for the first integral homology group $H_1(C_{q,1}; \mathbf{Z})$. By substituting $y = \sqrt[q]{\frac{1}{4}}z$ and $x = \frac{\sqrt{-1}w + 1}{2}$ into the above equation, we have $w^2 = z^q - 1$, corresponding to a hyperelliptic curve of genus $g = (q-1)/2$. Set the order- q holomorphic automorphism $\sigma(x, y) = (x, \zeta y)$ with $\zeta = \zeta_q = \exp(2\pi\sqrt{-1}/q)$. In terms of parameters z and w , we define a loop $\gamma_k : [0, 1] \rightarrow C_{q,1}$, $k = 0, 1, \dots, 2g$, by

$$\gamma_k(t) = \begin{cases} (\zeta^k \cdot 2t, \sqrt{-1}\sqrt{1 - (2t)^p}) & (0 \leq t \leq 1/2), \\ (\zeta^k(2-2t), -\sqrt{-1}\sqrt{1 - (2-2t)^p}) & (1/2 \leq t \leq 1). \end{cases}$$

We define the path $I_0 : [0, 1] \rightarrow C_{q,1}$ in a similar manner as in Section 2.1. It is easy to prove

LEMMA 3.1. *For $k = 0, 1, \dots, 2g$, the two paths $\sigma_*^k(I_0)$ and γ_k are homotopic with relative endpoints.*

We recall the following well-known fact (see [2], for example).

PROPOSITION 3.2. *For $i = 1, 2, \dots, g$, we denote $A_i = \gamma_{2i-1} \cdot \gamma_{2i}^{-1}$ and $B_i = \gamma_{2i-1} \cdot \gamma_{2i-2}^{-1} \cdot \dots \cdot \gamma_1 \cdot \gamma_0^{-1}$. We then have $\{A_i, B_i\}_{i=1,2,\dots,g}$ is a symplectic basis of $H_1(C_{q,1}; \mathbf{Z})$.*

We call this basis a natural type. Indeed, this proposition immediately follows from a two-sheeted covering $C_{q,1} \ni (z, w) \rightarrow z \in \mathbf{CP}^1$ branched over the $2g + 2$ points $\{1, \zeta, \zeta^2, \dots, \zeta^{2g}, \infty\} \subset \mathbf{CP}^1$. We find another symplectic basis of $H_1(C_{q,1}; \mathbf{Z})$. Although in general q is not prime, the chord slide algorithm can be similarly applied to $C_{q,1}$. We recall $c_i = I_0 \cdot \sigma_*^i(I_0)^{-1}$. Remark 2.1 gives us the intersection numbers of c_i 's

$$c_i \cdot c_j = \begin{cases} 1 & (i < j), \\ 0 & (i = j), \\ -1 & (i > j). \end{cases}$$

For $k = 1, 2, \dots, g - 1$ and $i = 1, 2, \dots, 2g$, let $h_{i,k}$ be the composition $o_{i,2k-1} \circ s_{i,2k}$. Moreover, we define the move f_k by

$$h_{2g,k} \circ h_{2g-1,k} \circ \dots \circ h_{2k+2,k} \circ h_{2k+1,k}.$$

LEMMA 3.3. *Let $A = (c_i \cdot c_j)$ be the intersection matrix of c_i 's. Then, the $2g \times 2g$ matrix*

$$f_{g-1} \circ f_{g-2} \circ \dots \circ f_2 \circ f_1(A)$$

is equal to the $2g \times 2g$ matrix

$$J_g = \begin{pmatrix} J & & & O \\ & J & & \\ & & \ddots & \\ O & & & J \end{pmatrix},$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

PROOF. The intersection matrices of c_i 's in Figures 7, 8, 9, and 10 represent A , $h_{3,1}(A)$, $f_1(A)$, and $f_{g-1} \circ f_{g-2} \circ \dots \circ f_2 \circ f_1(A)$ respectively. We illustrate each move f_i , in particular f_1 . We place emphasis on the endpoint series in the linear chord diagrams. The endpoint series in Figure 7 is

$$1, 2, 3, \dots, 2g - 1, 2g, \bar{1}, \bar{2}, \bar{3}, \dots, \overline{2g - 1}, \overline{2g}.$$

In this figure, we first take chord slides $\bar{3} \rightarrow \bar{2}$ and $\bar{3} \rightarrow 1$. Figure 8 is obtained and the intersection matrix becomes $h_{3,1}(A)$. Similarly, we take chord slides

$$\bar{4} \rightarrow \bar{2}, \bar{4} \rightarrow 1, \bar{5} \rightarrow \bar{2}, \bar{5} \rightarrow 1, \dots, \overline{2g} \rightarrow \bar{2}, \overline{2g} \rightarrow 1.$$

We obtain Figure 9 and the intersection matrix becomes $f_1(A)$. The endpoints series in this figure is

$$1, 2, 3, \dots, 2g - 1, 2g, \bar{3}, \bar{4}, \bar{5}, \dots, \overline{2g - 1}, \overline{2g}, \bar{1}, \bar{2}.$$

For each k -th step, $k = 2, 3, \dots, g - 1$, we take chord slides

$$\begin{aligned} \overline{2k + 1} &\rightarrow \overline{2k}, \overline{2k + 1} \rightarrow 2k - 1, \overline{2k + 2} \rightarrow \overline{2k}, \overline{2k + 2} \\ &\rightarrow 2k - 1, \dots, \overline{2g} \rightarrow \overline{2k}, \overline{2g} \rightarrow 2k - 1. \end{aligned}$$

This step corresponds to f_k . Finally, we have Figure 10 and the intersection matrix becomes $f_{g-1} \circ f_{g-2} \circ \dots \circ f_2 \circ f_1(A)$. The endpoint series in this figure is

$$1, 2, 3, \dots, 2g - 1, 2g, \overline{2g - 1}, \overline{2g}, \overline{2g - 3}, \overline{2g - 2}, \dots, \bar{3}, \bar{4}, \bar{1}, \bar{2}.$$

Clearly, the intersection matrix $f_{g-1} \circ f_{g-2} \circ \dots \circ f_2 \circ f_1(A)$ is equal to J_g from Figure 10. \square

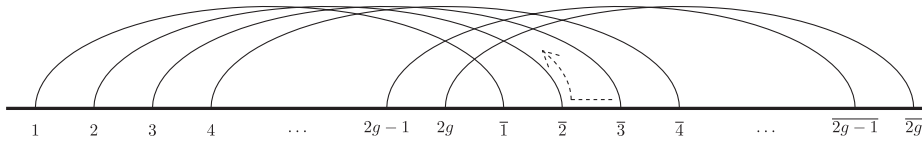


Figure 7: The initial linear chord diagram

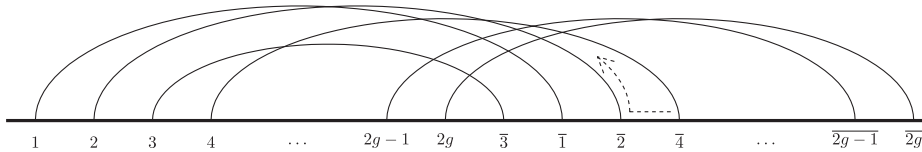


Figure 8: After chord slides $\bar{3} \rightarrow \bar{2}$ and $\bar{3} \rightarrow \bar{1}$ in Figure 7

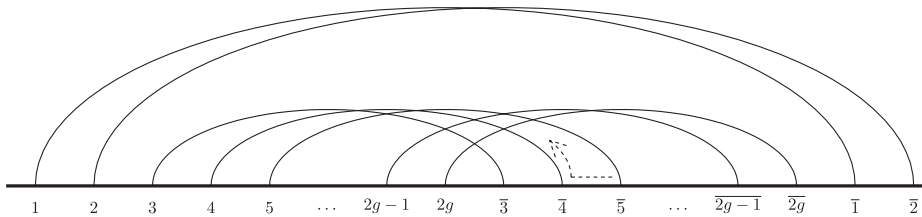


Figure 9: After the first step in Figure 7

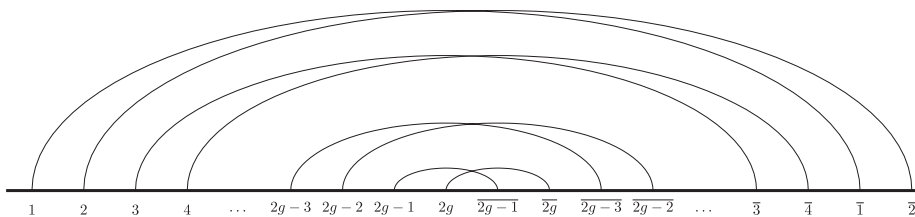


Figure 10: The final linear chord diagram

For $g = 3$, we give explicit forms for the intersection matrices A , $f_1(A)$, and $f_2 \circ f_1(A)$ corresponding to Figure 7, 9, and 10, respectively.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix}, \quad \text{and}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

REMARK 3.4. We obtain the matrix T' such that $f_{g-1} \circ f_{g-2} \circ \cdots \circ f_2 \circ f_1(A) = T' A' T'$

$$T' = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & 1 & & 0 & 0 \\ 1 & -1 & 1 & -1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & \\ & & & & & 1 & 0 \\ 1 & -1 & 1 & -1 & & 0 & 1 \end{pmatrix}.$$

Let $c'_1, c'_2, \dots, c'_{2g}$ be a basis of $H_1(C_{g,1}; \mathbf{Z})$ obtained by Lemma 3.3. Interchanging this basis gives us a symplectic basis

$$(a_1, \dots, a_g, b_1, \dots, b_g) = (c'_1, c'_3, \dots, c'_{2g-1}, c'_2, c'_4, \dots, c'_{2g}).$$

We interchange the rows of the matrix $f_{g-1} \circ f_{g-2} \circ \cdots \circ f_2 \circ f_1(A)$ in the following way. All odd rows move to rows $1, 2, \dots, g$ -th and all even rows to rows $g+1, g+2, \dots, 2g$. The resulting matrix is denoted by T .

THEOREM 3.5. *Set $(a_1, \dots, a_g, b_1, \dots, b_g) = (c_1, c_2, \dots, c_{2g})^t T$. Then, $\{a_i, b_i\}_{i=1, 2, \dots, g}$ is a symplectic basis of $H_1(C_{g,1}; \mathbf{Z})$.*

REMARK 3.6. The matrix T takes the explicit form

$$T = \begin{pmatrix} 1 & 0 & & \cdots & & & & & & 0 \\ 1 & -1 & 1 & 0 & & \cdots & & & & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 & & \cdots & & 0 \\ & & & \vdots & & & & & & \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & -1 & 1 & 0 \\ 0 & 1 & 0 & & \cdots & & & & & & 0 \\ 1 & -1 & 0 & 1 & 0 & & & \cdots & & & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & 0 & & \cdots & & 0 \\ & & & \vdots & & & & & & & \\ 1 & -1 & 1 & -1 & 1 & -1 & & \cdots & -1 & 0 & 1 \end{pmatrix}.$$

For $g = 3$, the matrix $T = T_{7,1}$ is given by

$$(3.1) \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 \end{pmatrix}.$$

We prove that the two symplectic bases $\{A_i, B_i\}_{i=1, 2, \dots, g}$ and $\{a_i, b_i\}_{i=1, 2, \dots, g}$ are different. From Lemma 3.1, we have matrix K

$$K = \begin{pmatrix} -1 & 1 & 0 & & \cdots & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & 0 \\ & & & \vdots & & & \\ 0 & \cdots & & 0 & -1 & 1 \\ -1 & 0 & & \cdots & & 0 \\ -1 & 1 & -1 & 0 & \cdots & 0 \\ & & & \vdots & & \\ -1 & 1 & -1 & 1 & \cdots & -1 & 0 \end{pmatrix}$$

such that $(A_1, \dots, A_g, B_1, \dots, B_g) = (c_1, c_2, \dots, c_{2g})^t K$. By comparing T and K , we have

PROPOSITION 3.7. *The two symplectic bases $\{A_i, B_i\}_{i=1,2,\dots,g}$ and $\{a_i, b_i\}_{i=1,2,\dots,g}$ are not equal up to ordering.*

4. Period Matrices for Hyperelliptic Curves

Let τ_g and τ_g^{CS} denote the period matrices of $C_{q,1}$ with respect to symplectic bases $\{A_i, B_i\}_{i=1,2,\dots,g}$ and $\{a_i, b_i\}_{i=1,2,\dots,g}$ of $H_1(C_{q,1}; \mathbf{Z})$. We compute τ_g of $C_{q,1}$. Moreover, we obtain the relations among τ_g , τ_g^{CS} , and Schindler's period matrix [20]. Set $\eta_i = \frac{dx}{y^{q-i}}$ for $i = 1, 2, \dots, g$. Bennama [5] proved that $\{\eta_i\}_{i=1,2,\dots,g}$ is a basis of $H^{1,0}(C_{q,1})$. From Section 2.5, we have for the period of η_i along c_j

$$\int_{c_j} \eta_i = (1 - \zeta^{ij})B(i/q, i/q).$$

For simplicity, we denote $\omega_i = \eta_i/B(i/q, i/q)$. We recall the two $g \times g$ matrices Ω_A and Ω_B which are $(\int_{A_j} \omega_i)$ and $(\int_{B_j} \omega_i)$, respectively. We form the $g \times 2g$ matrix,

$$(\Omega_A, \Omega_B) = \left(\int_{c_j} \omega_i \right)^t K.$$

We have the periods of the matrices Ω_A and Ω_B , for which Tashiro, Yamazaki, Ito, and Higuchi [23] obtained the same result.

PROPOSITION 4.1. *We have*

$$\Omega_A = -\text{diag}(-1 + \zeta^i) \text{diag}(\zeta^i)(\zeta^{2i(j-1)}) \quad \text{and}$$

$$\Omega_B = \text{diag}(-1 + \zeta^i) \left(\sum_{k=0}^{j-1} \zeta^{2ik} \right),$$

where $\text{diag}(a_i)$ is the diagonal matrix with (i, i) -th entry a_i .

PROOF. From the definition of Ω_A and Ω_B , we obtain $\Omega_A = (\zeta^{i(2j-1)} - \zeta^{2ij})$ and $\Omega_B = \left(\sum_{k=0}^{2j-1} (-1)^{k+1} \zeta^{ik} \right)$. We have only to compute the (i, j) -th entries of Ω_A and Ω_B

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$$\zeta^{i(2j-1)} - \zeta^{2ij} = -(-1 + \zeta^i)\zeta^i\zeta^{2i(j-1)},$$

$$\sum_{k=0}^{2j-1} (-1)^{k+1}\zeta^{ik} = -\frac{1 - \zeta^{2ij}}{1 + \zeta^i} = (-1 + \zeta^i)\frac{1 - \zeta^{2ij}}{1 - \zeta^{2i}} = (-1 + \zeta^i)\sum_{k=0}^{j-1}\zeta^{2ik}. \quad \square$$

REMARK 4.2. The matrix $(\zeta^{2i(j-1)})$ is a Vandermonde matrix.

To compute the matrix τ_g from $\Omega_A^{-1}\Omega_B$, we introduce a lemma. For variables x_1, x_2, \dots, x_n , we denote by $\sigma_i(x_1, x_2, \dots, x_n)$ the symmetric polynomial

$$\sum_{1 \leq j_1 < \dots < j_i \leq n} x_{j_1} \cdots x_{j_i}$$

for $1 \leq i \leq n$ and $\sigma_0(x_1, x_2, \dots, x_n) = 1$. Knuth [14, Exercise 40 in § 1.2.3] derived the inverse matrix of a Vandermonde matrix.

LEMMA 4.3. *Let a_1, a_2, \dots, a_n be distinct complex constants. We denote the Vandermonde matrix of size n by $V_n = (a_i^{j-1})$. Its inverse matrix is then*

$$V_n^{-1} = \left((-1)^{i-1} \frac{\sigma_{n-i}(a_1, \dots, \widehat{a_j}, \dots, a_n)}{\prod_{m=1, m \neq j}^n (a_m - a_j)} \right).$$

The ‘hat’ symbol is as defined earlier.

The above proposition and lemma give

THEOREM 4.4. *The period matrix τ_g of $C_{g,1}$ with respect to the symplectic basis $\{A_i, B_i\}_{i=1,2,\dots,g}$ is expressible as*

$$\tau_g = \left(\sum_{k=1}^g \frac{(-1)^{i+g}}{2g+1} (1 - \zeta^{2kj}) \sigma_{g-i}(\zeta^2, \zeta^4, \dots, \widehat{\zeta^{2j}}, \dots, \zeta^{2g}) \prod_{m=g-k+1}^{2g-k} (1 - \zeta^{2m}) \right).$$

PROOF. We compute $\Omega_A^{-1}\Omega_B$ as follows:

$$\begin{aligned} \Omega_A^{-1}\Omega_B &= -(\zeta^{2i(j-1)})^{-1} \text{diag}(\zeta^{-i}) \left(\sum_{k=0}^{j-1} \zeta^{2ik} \right) \\ &= -(\zeta^{2i(j-1)})^{-1} \left(\zeta^{-i} \frac{1 - \zeta^{2ij}}{1 - \zeta^{2i}} \right). \end{aligned}$$

From the equation

$$(\zeta^{2i(j-1)})^{-1} = \left((-1)^{i-1} \frac{\sigma_{g-i}(\zeta^2, \zeta^4, \dots, \widehat{\zeta^{2j}}, \dots, \zeta^{2g})}{\prod_{m=1, m \neq j}^n (\zeta^{2m} - \zeta^{2k})} \right),$$

we have for the (i, j) -th entry of τ_g

$$\sum_{k=1}^g (-1)^i \frac{\sigma_{g-i}(\zeta^2, \zeta^4, \dots, \widehat{\zeta^{2j}}, \dots, \zeta^{2g})}{\prod_{m=1, m \neq k}^g (\zeta^{2m} - \zeta^{2k})} \frac{1 - \zeta^{2kj}}{\zeta^k (1 - \zeta^{2k})}.$$

Moreover, we obtain

$$\begin{aligned} \zeta^k (1 - \zeta^{2k}) \prod_{m=1, m \neq k}^g (\zeta^{2m} - \zeta^{2k}) &= \zeta^{k+2k(g-1)+2k} (\zeta^{-2k} - 1) \prod_{m=1, m \neq k}^g (\zeta^{2(m-k)} - 1) \\ &= (-1)^g \prod_{m=-k, m \neq 0}^{g-k} (1 - \zeta^{2m}) \\ &= (-1)^g \frac{2g+1}{\prod_{m=g-k+1}^{2g-k} (1 - \zeta^{2m})}, \end{aligned}$$

for each $k = 1, 2, \dots, g$. The last equality follows from

$$\prod_{m=-k, m \neq 0}^{g-k} (1 - \zeta^{2m}) \prod_{m=g-k+1}^{2g-k} (1 - \zeta^{2m}) = \prod_{l=1}^{2g} (1 - \zeta^l) = 2g + 1.$$

This establishes the result. \square

Setting $\zeta = \zeta_7$, we calculate τ_3 of $C_{7,1}$ to be

$$\begin{pmatrix} -\zeta^5 & -1 - \zeta^2 - \zeta^4 - \zeta^5 & 1 + \zeta + \zeta^3 + \zeta^5 \\ -1 - \zeta^2 - \zeta^4 - \zeta^5 & 1 + \zeta + 2\zeta^3 - \zeta^4 + \zeta^5 & 2 + \zeta^2 + \zeta^3 + \zeta^5 \\ 1 + \zeta + \zeta^3 + \zeta^5 & 2 + \zeta^2 + \zeta^3 + \zeta^5 & 1 + \zeta^2 \end{pmatrix}.$$

In general, the period matrix depends only on the choice of the symplectic basis and the complex structure of the compact Riemann surface. Two period matrices τ_g and τ'_g are obtained from the same compact Riemann surface if and only if there exists a symplectic matrix $\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathrm{Sp}(2g, \mathbf{Z})$ such that $\tau'_g = (P + \tau_g R)^{-1} (Q + \tau_g S)$. Here P , Q , R , and S are $g \times g$ \mathbf{Z} -coefficient matrices.

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The symplectic matrix $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ for the two period matrices of $C_{q,1}$ with respect to $\{a_i, b_i\}_{i=1,2,\dots,g}$ and $\{A_i, B_i\}_{i=1,2,\dots,g}$ is given by $({}^tK)^{-1}{}^tT$, which we denote by H . This matrix can be computed as

$$H = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} O & I_g \\ -I_g & -I_g \end{pmatrix} \in \mathrm{Sp}(2g, \mathbf{Z}).$$

Here I_g is the identity matrix of size g . The equation

$$\tau_g^{\mathrm{CS}} = (O + \tau_g \cdot (-I_g))^{-1}(I_g - \tau_g)$$

gives us

PROPOSITION 4.5. *The relation between the two period matrices τ_g and τ_g^{CS} is*

$$\tau_g^{\mathrm{CS}} = -\tau_g^{-1} + I_g.$$

In particular, for $g = 3$, we have $\tau_{7,1} = -\tau_3^{-1} + I_3$. Nevertheless, the period matrix τ_g^{CS} is complicated.

We introduce Schindler's period matrix, denoted by τ_g^S , for the hyperelliptic curve defined by the affine equation $w_1^2 = z_1(z_1^{2g+1} - 1)$, and here denoted $C'_{q,1}$. This curve is biholomorphic to $C_{q,1}$. For $i = 1, 2, \dots, g$, elements t_i of the q -th cyclotomic field $\mathbf{Q}(\zeta)$ are defined as follows:

$$\begin{aligned} t_1 &= (-1)^g \zeta^{g^2}, \\ t_2 &= t_1 \left(1 - \frac{1}{1 + \zeta} \right), \\ t_{i+1} &= \frac{t_1 (1 - \sum_{k=2}^i \zeta^{g-i+k-1} t_k t_{i-k+2})}{1 + \zeta^{-i}} \quad (i = 2, 3, \dots, g-1). \end{aligned}$$

THEOREM 4.6 (Schindler [20]). *The (i, j) -th entry of the period matrix τ_g^S is obtained by*

$$s_{i,j} = 1 - \frac{1}{t_1} \sum_{k=1}^i t_k t_{j-i+k}$$

for $1 \leq i \leq j \leq g$ and $s_{j,i}$ for $g \geq i > j \geq 1$.

If we set $z_1 = 1/z$ and $w_1 = \sqrt{-1}w/z^{g+1}$, we obtain the biholomorphism from $C'_{g,1}$ to $C_{g,1}$. This implies that a symplectic basis for Schindler's period matrix [20] is given by

$$(A_g, A_{g-1}, \dots, A_1, B_g, B_{g-1}, \dots, B_1) = (A_1, A_2, \dots, A_g, B_1, B_2, \dots, B_g) \begin{pmatrix} L_g & O \\ O & L_g \end{pmatrix},$$

using the symplectic basis of natural type. Here the (i, j) -th entry of the $g \times g$ matrix L_g is 1 for $i + j = g + 1$ and 0 otherwise. It immediately follows that $L_g^{-1} = L_g$ and $\begin{pmatrix} L_g & O \\ O & L_g \end{pmatrix} \in \mathrm{Sp}(2g, \mathbf{Z})$. From the equation

$$\tau_g^S = (L_g + \tau_g \cdot O)^{-1} (O + \tau_g L_g),$$

we have

PROPOSITION 4.7. *The relation between the two period matrices τ_g and τ_g^S is*

$$\tau_g^S = L_g \tau_g L_g.$$

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References

- [1] Jørgen Ellegaard Andersen, Alex James Bene, and R. C. Penner, Groupoid extensions of mapping class representations for bordered surfaces., *Topology Appl.* **156** (2009), no. 17, 2713–2725 (English).
- [2] V. I. Arnol'd, Remark on the branching of hyperelliptic integrals as functions of the parameters., *Funct. Anal. Appl.* **2** (1968), 187–189 (English. Russian original).
- [3] G. V. Belyĭ, On Galois extensions of a maximal cyclotomic field., *Math. USSR, Izv.* **14** (1980), 247–256 (English).
- [4] Alex James Bene, A chord diagrammatic presentation of the mapping class group of a once bordered surface., *Geom. Dedicata* **144** (2010), 171–190 (English).

- [5] H. Bennama, Base of differentials of the first kind of the curves $y^q = \prod_{i=0}^p (x - a_i)^{2i}$: Application to Weierstrass points. (Base de différentielles de première espèce des courbes $y^q = \prod_{i=0}^p (x - a_i)^{2i}$: Application aux points de Weierstrass.), *J. Algebra* **203** (1998), no. 1, 261–269 (French).
- [6] H. Bennama and P. Carbonne, Period lattice of curves $y^q = \prod_{j=0}^n (z - a_j)^{2j}$ (q prime). (Réseau des périodes des courbes $y^q = \prod_{j=0}^n (z - a_j)^{2j}$ (q premier).), *Manuscr. Math.* **84** (1994), no. 2, 163–175 (French).
- [7] Kevin Berry and Marvin Tretkoff, The period matrix of Macbeath’s curve of genus seven., *Curves, Jacobians, and abelian varieties, Proc. AMS-IMS-SIAM Jt. Summer Res. Conf. Schottky Probl., Amherst/AM (USA) 1990, Contemp. Math.* **136**, 31–40 (1992), 1992.
- [8] H. W. Braden and T. P. Northover, Klein’s curve., *J. Phys. A* **43** (2010), no. 43, 434009, 17 (English).
- [9] E. Bujalance, A. F. Costa, J. M. Gamboa, and G. Riera, Period matrices of Accola-MacLachlan and Kulkarni surfaces., *Ann. Acad. Sci. Fenn., Math.* **25** (2000), no. 1, 161–177 (English).
- [10] Ernesto Girondo and Gabino González-Diez, Introduction to compact Riemann surfaces and dessins d’enfants., *London Mathematical Society Student Texts 79*. Cambridge: Cambridge University Press. xii, 298 p., 2012 (English).
- [11] Alexandre Grothendieck, Sketch of a programme. (Esquisse d’un programme.), Schneps, Leila (ed.) et al., *Geometric Galois actions. 1. Around Grothendieck’s “Esquisse d’un programme”*. Proceedings of the conference on geometry and arithmetic of moduli spaces, Luminy, France, August 1995. Cambridge: Cambridge University Press. Lond. Math. Soc. Lect. Note Ser. **242**, 5–48; English translation: 243–283 (1997), 1997.
- [12] Yasuo Kamata, A note on Klein curve., *Kumamoto J. Math.* **15** (2002), 7–15 (English).
- [13] Yasuo Kamata, The algorithm to calculate the period matrix of the curve $x^m + y^n = 1$., *Tsukuba J. Math.* **26** (2002), no. 1, 15–37 (English).
- [14] D. E. Knuth, The art of computer programming. Vol. 1: Fundamental algorithms., Addison-Wesley Series in Computer Science and Information Processing. London: Addison-Wesley Publishing Company. XXII, 634 p., 1968 (English).
- [15] T. Kuusalo and M. Näätänen, Geometric uniformization in genus 2., *Ann. Acad. Sci. Fenn., Ser. A I, Math.* **20** (1995), no. 2, 401–418 (English).
- [16] Raghavan Narasimhan, Compact Riemann surfaces., *Lectures in Mathematics, ETH Zürich*. Basel: Birkhäuser. 120 p., 1992 (English).
- [17] H. E. Rauch and J. Lewittes, The Riemann surface of Klein with 168 automorphisms., *Probl. Analysis, Sympos. in Honor of Salomon Bochner, Princeton Univ.* 1969, 297–308 (1970), 1970.
- [18] Rubí E. Rodríguez and Víctor González-Aguilera, Fermat’s quartic curve, Klein’s curve and the tetrahedron., Quine, J. R. (ed.) et al., *Extremal Riemann surfaces. From the proceedings of the AMS special session with related papers, January 4–5, 1995, San Francisco, CA, USA*. Providence, RI: American Mathematical Society. *Contemp. Math.* **201**, 43–62 (1997), 1997.
- [19] B. Schindler, Jacobische varietäten hyperelliptischer kurven und einiger spezieller kurven vom geschlecht 3, 1991.
- [20] Bernhard Schindler, Period matrices of hyperelliptic curves., *Manuscr. Math.* **78** (1993), no. 4, 369–380 (English).
- [21] M. Streit, Period matrices and representation theory., *Abh. Math. Semin. Univ. Hamb.* **71** (2001), 279–290 (English).
- [22] Yuuki Tadokoro, A nontrivial algebraic cycle in the Jacobian variety of the Klein quartic., *Math. Z.* **260** (2008), no. 2, 265–275 (English).
- [23] Yoshiaki Tashiro, Seishi Yamazaki, Minoru Ito, and Teiichi Higuchi, On Riemann’s period matrix of $y^2 = x^{2n+1} - 1$., *RIMS Kokyuroku* **963** (1996), 124–141 (English).

- [24] C. L. Tretkoff and M. D. Tretkoff, Combinatorial group theory, Riemann surfaces and differential equations., Contributions to group theory, Contemp. Math. 33, 467–519 (1984), 1984.
- [25] Jürgen Wolfart, Regular dessins, endomorphisms of Jacobians, and transcendence., Wüstholz, Gisbert (ed.), A panorama in number theory or The view from Baker’s garden. Based on a conference in honor of Alan Baker’s 60th birthday, Zürich, Switzerland, 1999. Cambridge: Cambridge University Press. 107–120 (2002), 2002.
- [26] Katsuaki Yoshida, Klein’s surface of genus three and associated theta constants., Tsukuba J. Math. **23** (1999), no. 2, 383–416 (English).

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