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NON-FIBER PRESERVING ACTIONS ON PRISM MANIFOLDS

By

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Abstract. In this paper we classify the finite groups of isometries which act on a prism manifolds $M(b,d)$ and do not preserve any fibering. We construct nine distinct finite groups of isometries which act on $M(1,2)$, and do not preserve any fibering. We then show that if a finite group of isometries G acts on $M(b,d)$ and does not preserve any fibering, then $M(b,d) = M(1,2)$ and G is conjugate to one of these nine groups which are: $\mathbf{Z}_3 \times T$, T, O, $S_3 \times O$, $\mathbf{Z}_3 \circ O$, $S_3 \times T$, $\mathbf{Z}_3 \times O$, $\mathbf{Z}_3 \times I$, and $S_3 \times I$, where T, O, I and S_3 are the tetrahedral, octahedral, icosahedral, and symmetric groups respectively.

0. Introduction

In [1], W. D. Dunbar investigated which finite subgroups of $SO(4)$ acting on the three sphere $\mathbf{S}^3$ preserve no fibration of $\mathbf{S}^3$ by circles. He identified 21 conjugacy classes of isometries acting on $\mathbf{S}^3$ which preserve no fibration of $\mathbf{S}^3$ by circles, and he computed the quotient type of each such action. Each quotient type is a spherical orbifold, whose underlying space is $\mathbf{S}^3$, and contains an embedded trivalent graph for its exceptional set. These spherical orbifolds obviously cannot be fibered. For a very nice discussion of Seifert fibered spaces and distinct fiberings of 3-manifolds, see the work of W. Jaco [2] and K. Morimoto [6] which is written in Japanese.

This paper investigates which prism manifolds admit finite groups of isometries which do not preserve any fibering, and classifies these groups up to conjugacy. A prism manifold admits only two distinct non-isotopic fiberings, the

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meridian fibering and the longitudinal fibering. (See [2] or [5].) Let $M(b,d)$ be a prism manifold and let G be a finite group of isometries acting on $M(b,d)$. We show that if $M(b,d) \neq M(1,2)$, then G preserves either the meridian or longitudinal fibering. If G is a finite group of isometries acting on $M(1,2)$ which does not preserve any of the two fiberings, we show that G is conjugate to one of the following groups: $\mathbf{Z}_3 \times T$, T, O, $S_3 \times O$, $\mathbf{Z}_3 \circ O$, $S_3 \times T$, $\mathbf{Z}_3 \times O$, $\mathbf{Z}_3 \times I$, and $S_3 \times I$, where T, O, I and S_3 are the tetrahedral, octahedral, icosahedral, and symmetric groups respectively. In addition, we give explicit descriptions of the generators of these groups, which are projections of isometries of $\mathbf{S}^3$ to $M(1,2)$.

We now define a prism manifold. Let $T = S^1 \times S^1$ be a torus where $S^1 = \{z \in \mathbf{C} : |z| = 1\}$ is viewed as the set of complex numbers of norm 1 and $I = [0,1]$. The twisted I-bundle over a Klein bottle is the quotient space $W = T \times I/(u,v,t) \simeq (-u,\bar{v},1-t)$. Let D^2 be a unit disk with $\partial D^2 = S^1$ and let $V = S^1 \times D^2$ be a solid torus. Then the boundary of both V and W is a torus $S^1 \times S^1$. For relatively prime integers b and d, there exist integers a and b such that $ad - bc = -1$. We consider the manifold obtained by glueing ∂V and ∂W by the homeomorphism $\psi : \partial V \to \partial W$ defined by $\psi(u,v) = (u^a v^b, u^c v^d)$ for $(u,v) \in \partial V = S^1 \times S^1$. Then, since $(1,v)$ represents the meridian loop of V and $\psi(1,v) = (v^b, v^d)$, the manifold $V \cup_\psi W$ is determined by the pair (b,d) and is called the *prism manifold* $M(b,d)$. Using Van Kampen's Theorem the fundamental group $\pi_1(M(b,d)) = \langle c_0, c_1 \mid c_1 c_0 c_1^{-1} = c_0^{-1}, c_1^{2b} c_0^d = 1 \rangle$.

An embedded Klein bottle K in $M(b,d)$ is called a *Heegaard Klein bottle* if for any regular neighborhood $N(K)$ of K, $N(K)$ is a twisted I-bundle over K and the closure of $M(b,d) - N(K)$ is a solid torus. Any G-action which leaves a Heegaard Klein bottle invariant is said to *split*. In [4] the authors classify, up to conjugacy, the finite group actions on a prism manifold which split. It follows from this classification that if an action splits, then it preserves both the longitudinal and meridian fiberings. Thus the non-fiber preserving actions described on $M(1,2)$ do not leave any Heegaard Klein bottle invariant. We note that $M(1,2)$ is the Seifert fibered space over the 2-sphere which has three exceptional fibers with the Seifert invariants $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ and obstruction class -1, and that $M(1,2)$ is also the Seifert space over the projective plane which has no exceptional fiber with obstruction class -2. Thus we have $M(1,2) \cong \mathbf{S}^2\left(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \cong \mathbf{P}^2(-2; —)$.

The standard elliptic structure on the 3-sphere $\mathbf{S}^3$ is associated with the orthogonal group $O(4)$ under its action on $\mathbf{S}^3$, and therefore giving $O(4)$ as the group of isometries of $\mathbf{S}^3$ and $SO(4)$ as the orientation preserving subgroup. A 3-orbifold (or 3-manifold) M has an elliptic structure if there exists a finite group of isometries $\Gamma \leq O(4)$ such that there is an orbifold (or 3-manifold) covering

$\mathbf{S}^3 \to \mathbf{S}^3/\Gamma = M$. An isometry of M is a homeomorphism of M which lifts to an isometry of $\mathbf{S}^3$.

1. Group Isomorphisms of Fundamental Groups of Prism Manifolds in $\mathbf{S}^3 \times \mathbf{S}^3$

In this section, we will view the fundamental group of a prism manifold $\pi(n,m) = \langle c_0, c_1 \mid c_1 c_0 c_1^{-1} = c_0^{-1}, c_1^{2n} c_0^m = 1 \rangle$ as a subgroup of $\mathbf{S}^3 \times \mathbf{S}^3$, where we view $\mathbf{S}^3 = \{u + vj \mid u, v \in \mathbf{C} \text{ and } |u|^2 + |v|^2 = 1\}$.

Let $D_{4m}^* = \langle x, y \mid x^2 = y^m = (xy)^2 \rangle$ and $\mathbf{Z}_n = \langle t \mid t^n = 1 \rangle$ be a subgroups of $\mathbf{S}^3$ where $x = j$, $y = e^{\pi i/m}$, and $t = e^{2\pi i/n}$.

Proposition 1. *If n is odd, then $D_{4m}^* \times \mathbf{Z}_n$ is isomorphic to $\pi(n,m)$.*

Proof. We first note that $(x,t)^4 = (1, t^4)$. Since n is odd, t^4 generates $\mathbf{Z}_n$. Furthermore $(x,t)^n = (x^n, 1)$, which equals either $(x,1)$ or $(x^{-1},1)$ since $x = j$. Therefore, (x,t) and $(y,1)$ generate $D_{4m}^* \times \mathbf{Z}_n$. Observe that $(x,t)(y,1)(x,t)^{-1} = (xyx^{-1},1) = (y^{-1},1) = (y,1)^{-1}$, and $(x,t)^{2n}(y,1)^m = ((-1)^n,1)(-1,1) = (1,1)$. Consequently, there is an isomorphism from $D_{4m}^* \times \mathbf{Z}_n$ to $\pi(n,m)$ by sending (x,t) to c_1 and $(y,1)$ to c_0. □

Define groups $B_{2^{k+3}a} = \langle x, y \mid xyx^{-1} = y^{-1}, x^{2^{k+3}} = 1, y^a = 1 \rangle$ and $\mathbf{Z}_{m''} = \langle t \mid t^{m''} = 1 \rangle$. It follows that the order of the group $|B_{2^{k+3}a}| = 2^{k+3}a$.

Proposition 2. *If a and m'' are both odd, then $B_{2^{k+3}a} \times \mathbf{Z}_{m''}$ is isomorphic to $\pi(2^{k+1}m'', a)$.*

Proof. We will first show that $\langle c_0^2, c_1^{m''}, c_1^{2^{k+3}} \rangle = \pi(2^{k+3}m'', a)$. Since 2^{k+3} and m'' are relatively prime, there exist integers s and t such that $2^{k+3}s + m''t = 1$. Therefore $(c_1^{2^{k+3}})^s (c_1^{m''})^t = c_1$. Now $(c_1^{m''})^{-(2^{k+2})} = c_0^a = c_0^{2l+1}$ for some l since a is odd. Hence $(c_0^2)^{-l}(c_1^{m''})^{-(2^{k+2})} = c_0^{-2l} c_0^{2l+1} = c_0$. Note that $(c_1^{2^{k+3}})^{m''} = 1$, $(c_0^2)^a = 1$, and $c_1^{m''} c_0^2 c_1^{-m''} = c_0^{-2}$ since m'' is odd. Thus, we can define an isomorphism from $B_{2^{k+3}a} \times \mathbf{Z}_{m''}$ to $\pi(2^{k+1}m'', a)$ by sending $(x,1)$ to $c_1^{m''}$, $(y,1)$ to c_0^2, and $(1,t)$ to $c_1^{2^{k+3}}$. □

Let H be the subgroup of $\mathbf{S}^3 \times \mathbf{S}^3$ generated by $X = (j, e^{\pi i/2^{k+2}})$, $Y = (e^{2\pi i/a}, 1)$, and $T = (1, e^{2\pi i/m''})$.

Proposition 3. *The group H is isomorphic to $B_{2^{k+3}a} \times \mathbf{Z}_{m''}$.*

PROOF. Observe that X and Y both commute with T, $X^{2^{k+3}} = (j^{2^{k+3}}, 1) = (1,1)$, and $Y^a = (e^{2\pi i/a}, 1)^a = (1,1)$. Furthermore, $XYX^{-1} = (j, e^{\pi i/2^{k+2}})(e^{2\pi i/a}, 1) \cdot (j, e^{\pi i/2^{k+2}})^{-1} = (je^{2\pi i/a}j^{-1}, 1) = (e^{-2\pi i/a}, 1) = Y^{-1}$. Thus, there exists a surjection θ from $B_{2^{k+3}a} \times \mathbf{Z}_{m''}$ to G by sending x to X, y to Y, and t to T. Now, $\langle Y \rangle \simeq \mathbf{Z}_a$ is a normal subgroup of G, and $H/\langle Y \rangle \simeq \mathbf{Z}_{2^{k+3}} \times \mathbf{Z}_{m''}$. Hence $|H| = 2^{k+3}am'' = |B_{2^{k+3}a} \times \mathbf{Z}_{m''}|$, and therefore θ is an isomorphism. □

Let $\sigma : \mathbf{S}^3 \times \mathbf{S}^3 \to SO(4)$ be the homomorphism defined by $\sigma(q_1, q_2)(q) = q_1 q q_2^{-1}$. Now σ is onto with kernel $\mathbf{Z}_2 = \langle(-1,-1)\rangle$.

PROPOSITION 4. *Suppose a, m'', and n are odd. The element $(-1,-1)$ is not an element of H or $D^*_{4m} \times \mathbf{Z}_n$, and hence σ restricted to either of these groups is one-to-one.*

PROOF. Suppose $(-1,-1) \in H$. Then for some integers u, v, and w, we have $X^u Y^v T^w = (j, e^{\pi i/2^{k+2}})^u (e^{2\pi i/a}, 1)^v (1, e^{2\pi i/m''})^w = (-1,-1)$. We obtain the equations $j^u e^{2v\pi i/a} = -1$ and $e^{u\pi i/2^{k+2}} e^{2w\pi i/m''} = -1$. The first equation implies that $j^u = 1$ or -1. Suppose first that $j^u = 1$, and hence $e^{2v\pi i/a} = -1$. Becasuse a is odd, this is impossible since $e^{2\pi i/a}$ generates a cyclic group of odd order. Therefore, we must have $j^u = -1$ and $u = 2(2x+1)$ for some integer x. For the second equation, we have $e^{2(2x+1)\pi i/2^{k+2}} e^{2w\pi i/m''} = -1$, and simplifying we get $e^{(2x+1)\pi i/2^{k+1}} e^{2w\pi i/m''} = -1$. By raising both sides of this equation to the m''-th power and using the fact that m'' is odd, we obtain $e^{m''(2x+1)\pi i/2^{k+1}} = -1$. Since $2x+1$ and m'' are both odd, we again get a contradiction by raising both sides to the 2^{k+1} power.

Suppose that $(-1,-1) \in D^*_{4m} \times \mathbf{Z}_n$. This implies that there exist integers u, v and w, such that $(x^u y^v, t^w) = (-1,-1)$. Since $t = e^{2\pi i/n}$ and n is odd, $e^{2w\pi i/n} = -1$ is impossible, giving a contradiction. □

2. Fiber Preserving Actions on Prism Manifolds

In this section we indicate the elliptic structure and the two distinct fiberings on a prism manifold. We show that any finite group of isometries acting on a prism manifold $M(b,d)$ when $M(b,d) \neq M(1,2)$ is fiber preserving.

Let $\sigma : \mathbf{S}^3 \times \mathbf{S}^3 \to SO(4)$ be the homomorphism defined by $\sigma(q_1, q_2)(q) = q_1 q q_2^{-1}$. Now σ is onto with kernel $\mathbf{Z}_2 = \langle(-1,-1)\rangle$. For a more complete discussion, see [3] and [7]. Define a map $\kappa : \mathbf{S}^3 \to SO(3)$ by $\kappa(q)(p) = qpq^{-1}$ for any $p \in \mathbf{S}^2$ and $q \in \mathbf{S}^3$. By Dunbar [1], there exists a map $\rho : SO(4) \to SO(3) \times SO(3)$ such that $\rho \circ \sigma = \kappa \times \kappa$. Let $p_1 : SO(3) \times SO(3) \to SO(3)$ be the projection onto the first coordinate, and let $p_2 : SO(3) \times SO(3) \to SO(3)$ be the projection onto the second coordinate.

For the subgroup $S^1 = \langle e^{i\theta} \mid \theta \in \mathbf{R} \rangle$ of $\mathbf{S}^3$, let $\mathbf{F}_l = \langle pS^1 \rangle_{p \in \mathbf{S}^3}$ and $\mathbf{F}_r = \langle S^1 p \rangle_{p \in \mathbf{S}^3}$ be the left and right Hopf fibrations of $\mathbf{S}^3$ respectively. View $\mathbf{S}^2 = \mathbf{C} \cup \{\infty\}$ where $\mathbf{C}$ is the complex plane. Define $H_l : \mathbf{S}^3 \to \mathbf{S}^2$ and $H_r : \mathbf{S}^3 \to \mathbf{S}^2$ by $H_l(u + vj) = u/\bar{v}$ and $H_r(u + vj) = u/v$ respectively. See [5] for a good reference. Let $D^*_{4m} = \langle x, y \mid x^2 = y^m = (xy)^2 \rangle$ and $\mathbf{Z}_n = \langle t \mid t^n = 1 \rangle$ be subgroups of $\mathbf{S}^3$ where $x = j$, $y = e^{\pi i/m}$, and $t = e^{2\pi i/n}$. We will assume that n is odd and relatively prime to m. Since the group generated by $x^2 = -1$ is a normal subgroup of D^*_{4m}, let $D_{2m} = D^*_{4m}/\langle x^2 \rangle = \langle x, y \mid 1 = x^2 = y^m = (xy)^2 \rangle$. The subscripts indicate the order of these groups.

Let $G(2^{k+1}a, m'')$ be the subgroup of $\mathbf{S}^3 \times \mathbf{S}^3$ generated by $X = (j, e^{\pi i/2^{k+2}})$, $Y = (e^{2\pi i/a}, 1)$, and $T = (1, e^{2\pi i/m''})$ where a and m'' are both odd and $k \geq 0$.

Proposition 5. *Let $m \geq 2$ and n be relatively prime positive integers with n odd. The group $\sigma(D^*_{4m} \times \mathbf{Z}_n)$ acts freely on $\mathbf{S}^3$ and preserves both the left and right Hopf fibrations. The manifold $\mathbf{S}^3/\sigma(D^*_{4m} \times \mathbf{Z}_n)$ is the prism manifold $M(n, m)$ with induced left and right Hopf fibrations. If $h_l : M(n, m) \to B_l$ and $h_r : M(n, m) \to B_r$ are the maps which identify fibers to points in the induced left and right fibrations respectively, then $B_l = \mathbf{S}^2(2, 2, m)$ and $B_r = \mathbf{P}^2(n)$.*

Proof. It is not hard to see that each of the actions $\sigma(j, 1)$, $\sigma(e^{\pi i/m}, 1)$, and $\sigma(1, e^{2\pi i/n})$ on $\mathbf{S}^3$ preserves both the left and right Hopf fibrations. We obtain the following commutative diagram where H is either H_l or H_r, h is either h_l or h_r respectively, and the vertical maps are covering maps where B is $\mathbf{S}^2$ modulo the induced action on $\mathbf{S}^2$.

$$\begin{array}{ccc} \mathbf{S}^3 & \xrightarrow{H} & \mathbf{S}^2 \\ \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle \bar{v}} \\ M(n,m) & \xrightarrow{h} & B \end{array}$$

Consider first the left Hopf fibration $\mathbf{F}_l = \langle pS^1 \rangle_{p \in \mathbf{S}^3}$ on $\mathbf{S}^3$. Let $p = u + vj \in \mathbf{S}^3$. Then $\sigma(j, 1)(pe^{i\theta}) = j(ue^{i\theta} + ve^{-i\theta}j) = \bar{u}e^{-i\theta}j + \bar{v}e^{i\theta}j^2 = -\bar{v}e^{i\theta} + \bar{u}e^{-i\theta}j$. Therefore $H_l\sigma(j, 1)(pe^{i\theta}) = H_l(-\bar{v}e^{i\theta} + \bar{u}e^{-i\theta}j) = -\bar{v}e^{i\theta}/\overline{\bar{u}e^{-i\theta}} = -\bar{v}/u$. For the induced action $\bar{\sigma}(j, 1)$ on $\mathbf{S}^2$, we have $\bar{\sigma}(j, 1)(u/\bar{v}) = -\bar{v}/u$. Therefore for any

$z \in \mathbf{S}^2$, it follows that $\bar{\sigma}(j,1)(z) = -1/z$ and the fixed points are i and $-i$. We also have $\sigma(e^{\pi i/m},1)(pe^{i\theta}) = e^{\pi i/m}(ue^{i\theta} + ve^{-i\theta}j) = ue^{i(\theta+\pi/m)} + ve^{i(-\theta+\pi/m)}j$. Thus $H_l\sigma(e^{\pi i/m},1)(pe^{i\theta}) = ue^{i(\theta+\pi/m)}/\overline{ve^{i(-\theta+\pi/m)}} = ue^{2\pi i/m}/\bar{v}$. For the induced action $\bar{\sigma}(e^{\pi i/m},1)$ on $\mathbf{S}^2$, it follows that $\bar{\sigma}(e^{\pi i/m},1)(z) = ze^{2\pi i/m}$ for any $z \in \mathbf{S}^2$. The fixed points are 0 and ∞. Now $\sigma(1,e^{2\pi i/n})(pe^{i\theta}) = (u+vj)e^{i\theta}e^{-2\pi i/n} = ue^{i(\theta-2\pi/n)} + ve^{i(-\theta+2\pi/n)}j$, and thus $H_l\sigma(1,e^{2\pi i/n})(pe^{i\theta}) = ue^{i(\theta-2\pi/n)}/\overline{ve^{i(-\theta+2\pi/n)}} = u/\bar{v}$. Therefore the induced map $\bar{\sigma}(1,e^{2\pi i/n})$ on $\mathbf{S}^2$ is the identity. It now follows that the orbifold $\mathbf{S}^2/\langle\bar{\sigma}(j,1),\bar{\sigma}(e^{\pi i/m},1)\rangle = \mathbf{S}^2(2,2,m)$.

Consider now the right Hopf fibering $\mathbf{F}_r = \langle S^1 p\rangle_{p\in\mathbf{S}^3}$ on $\mathbf{S}^3$. We see that $\sigma(j,1)(e^{i\theta}p) = j(e^{i\theta}u + e^{i\theta}vj) = -e^{-i\theta}\bar{v} + e^{-i\theta}\bar{u}j$, and thus $H_r\sigma(j,1)(e^{i\theta}p) = -e^{-i\theta}\bar{v}/e^{-i\theta}\bar{u} = -\bar{v}/\bar{u}$. For the induced action $\bar{\sigma}(j,1)$ on $\mathbf{S}^2$, we get $\bar{\sigma}(j,1)(z) = -1/\bar{z}$ for any $z \in \mathbf{S}^2$. Furthermore, $\bar{\sigma}(j,1)$ is fixed point free. We see that $\sigma(e^{\pi i/m},1)(e^{i\theta}p) = e^{\pi i/m}(e^{i\theta}u + e^{i\theta}vj) = e^{i(\theta+\pi/m)}u + e^{i(\theta+\pi/m)}vj$, and thus $H_r\sigma(e^{\pi i/m},1)(e^{i\theta}p) = u/v$. Therefore the induced map $\bar{\sigma}(e^{\pi i/m},1)$ on $\mathbf{S}^2$ is the identity. Now, $\sigma(1,e^{2\pi i/n})(e^{i\theta}p) = (e^{i\theta}u + e^{i\theta}vj)e^{-2\pi i/n} = e^{i(\theta-2\pi/n)}u + e^{i(\theta+2\pi/n)}vj$, and therefore $H_r\sigma(1,e^{2\pi i/n})(e^{i\theta}p) = e^{(i\theta-2\pi/n)}u/e^{i(\theta+2\pi/n)}v = e^{-4\pi i/n}u/v$. If $\bar{\sigma}(1,e^{2\pi i/n})$ is the induced action on $\mathbf{S}^2$, then $\bar{\sigma}(1,e^{2\pi i/n})(z) = e^{-4\pi i/n}z$ for any $z \in \mathbf{S}^2$ where 0 and ∞ are the fixed points. Since n is odd, this is a cyclic action of order n. Thus $\mathbf{S}^2/\langle\bar{\sigma}(j,1),\bar{\sigma}(1,e^{2\pi i/n})\rangle = \mathbf{P}^2(n)$. □

PROPOSITION 6. *Let m'' and a be relatively prime positive odd integers. The finite group $\sigma(G(2^{k+1}a,m''))$ acts freely on $\mathbf{S}^3$ and preserves both the left and right Hopf fibrations. The manifold $\mathbf{S}^3/\sigma(G(2^{k+1}a,m''))$ is the prism manifold $M(2^{k+1}m'',a)$ with induced left and right Hopf fibrations. If $h_l : M(2^{k+1}m'',a) \to B_l$ and $h_r : M(2^{k+1}m'',a) \to B_r$ are the maps which identify fibers to points in the induced left and right fibrations respectively, then $B_l = \mathbf{S}^2(2,2,a)$ and $B_r = \mathbf{P}^2(2^{k+1}m'')$.*

PROOF. The proof is similar to that of Proposition 5. It is easy to verify that $\sigma(j,e^{\pi i/2^{k+2}})$, $\sigma(e^{2\pi i/a},1)$, and $\sigma(1,e^{2\pi i/m''})$ on $\mathbf{S}^3$ preserve both the left and right Hopf fibrations. For the first generator, we have $\sigma(j,e^{\pi i/2^{k+2}})((u+vj)e^{i\theta}) = (\bar{u}j + \bar{v}j^2)e^{i(\theta-\pi/2^{k+2})} = -\bar{v}e^{i(\theta-\pi/2^{k+2})} + \bar{u}e^{i(-\theta+\pi/2^{k+2})}$. Applying H_l, we see that $H_l\sigma(j,e^{\pi i/2^{k+2}})((u+vj)e^{i\theta}) = -\bar{v}e^{i(\theta-\pi/2^{k+2})}/\overline{\bar{u}e^{i(-\theta+\pi/2^{k+2})}} = -\bar{v}/u$. The induced map $\bar{\sigma}(j,e^{\pi i/2^{k+2}})$ on $\mathbf{S}^2$ sends z to $-1/z$. For $\sigma(e^{2\pi i/a},1)$, we have $\sigma(e^{2\pi i/a},1)\cdot((u+vj)e^{i\theta}) = ue^{i(\theta+2\pi/a)} + ve^{i(-\theta+2\pi/a)}j$. Applying H_l, we get $H_l\sigma(e^{2\pi i/a},1)\cdot((u+vj)e^{i\theta}) = ue^{i(\theta+2\pi/a)}/\overline{ve^{i(-\theta+2\pi/a)}} = ue^{4\pi i/a}/\bar{v}$. The induced action $\bar{\sigma}(e^{2\pi i/a},1)$ on $\mathbf{S}^2$ sends z to $ze^{4\pi i/a}$. It is not hard to check that $\sigma(1,e^{2\pi i/m''})$ induces the identity on $\mathbf{S}^2$, and hence $\mathbf{S}^2/\langle\bar{\sigma}(j,e^{\pi i/2^{k+2}}),\bar{\sigma}(e^{2\pi i/a},1)\rangle = \mathbf{S}^2(2,2,a)$.

Next, we compute

$$\sigma(j, e^{\pi i/2^{k+2}})(e^{i\theta}(u+vj)) = j(e^{i\theta}u + e^{i\theta}vj)e^{-\pi i/2^{k+2}} = (e^{-i\theta}\bar{v}j + e^{-i\theta}\bar{v}j^2)e^{-\pi i/2^{k+2}}$$
$$= -\bar{v}e^{i(-\theta-\pi/2^{k+2})} + \bar{u}e^{i(-\theta+\pi/2^{k+2})}j.$$

Therefore, $H_r\sigma(j, e^{\pi i/2^{k+2}})(e^{i\theta}(u+vj)) = -\bar{v}e^{i(-\theta-\pi/2^{k+2})}/\bar{u}e^{i(-\theta+\pi/2^{k+2})} = -\bar{v}e^{-\pi i/2^{k+1}}/\bar{u}$. The induced map $\bar{\sigma}(j, e^{\pi i/2^{k+2}})$ on $\mathbf{S}^2$ sends z to $(-1/\bar{z})e^{-i\pi/2^{k+1}}$. Although $\bar{\sigma}(j, e^{\pi i/2^{k+2}})$ is fixed point free, $\bar{\sigma}^2(j, e^{\pi i/2^{k+2}})$ sends z to $ze^{-2\pi i/2^{k+1}}$ fixing both 0 and ∞. For the map $\sigma(1, e^{2\pi i/m''})$, we see that $\sigma(1, e^{2\pi i/m''})(e^{i\theta}(u+vj)) = (e^{i\theta}u + e^{i\theta}vj)e^{-2\pi i/m''} = ue^{i(\theta-2\pi/m'')} + ve^{i(\theta+2\pi/m'')}j$. Then, $H_r\sigma(1, e^{2\pi i/m''})(e^{i\theta}(u+vj)) = ue^{-4\pi i/m''}/v$, and hence the induced map $\bar{\sigma}(1, e^{2\pi i/m''})$ on $\mathbf{S}^2$ sends z to $ze^{-4\pi i/m''}$ fixing both 0 and ∞. Similarly one can check that $\sigma(e^{2\pi i/a}, 1)$ induces the identity on $\mathbf{S}^2$. Therefore, $\mathbf{S}^2/\langle\bar{\sigma}(j, e^{\pi i/2^{k+2}}), \bar{\sigma}(1, e^{2\pi i/m''})\rangle = \mathbf{P}^2(2^{k+1}m'')$. □

Proposition 7. *$p_1 \circ (\kappa \times \kappa)(D^*_{4m} \times \mathbf{Z}_n) = D_{2m}$ and $p_2 \circ (\kappa \times \kappa)(D^*_{4m} \times \mathbf{Z}_n) = \mathbf{Z}_n$.*

Proof. Since the kernel of κ is $\langle -1 \rangle$ which is a subgroup D^*_{4m}, we see that $\kappa(D^*_{4m}) = D^*_{4m}/\langle -1 \rangle = D_{2m}$. Furthermore since n is odd, $\langle -1 \rangle \not\leq \mathbf{Z}_n$. Thus $\kappa(\mathbf{Z}_n) = \mathbf{Z}_n$. □

Proposition 8. *$p_1 \circ (\kappa \times \kappa)(G(2^{k+1}a, m'')) = D_{2a}$ and $p_2 \circ (\kappa \times \kappa) \cdot (G(2^{k+1}a, m'')) = \mathbf{Z}_{2^{k+2}m''}$.*

Proof. Since $\ker(\kappa) = \langle -1 \rangle$ and a is odd, we obtain $\kappa(\langle j, e^{2\pi i/a}\rangle) = D_{2a}$. Notice that as m'' is odd, $\kappa(e^{\pi i/2^{k+2}})$ and $\kappa(e^{2\pi i/m''})$ generate cyclic subgroups of order 2^{k+2} and m'' respectively. Therefore $\kappa(\langle e^{\pi i/2^{k+2}}, e^{2\pi i/m''}\rangle)$ generates the cyclic subgroup $\mathbf{Z}_{2^{k+2}m''}$. □

Theorem 9. *Let $M(n,m)$ be a prism manifold and let G be a finite group of isometries acting on $M(b,d)$. If $M(b,d) \neq M(1,2)$, then G preserves either the meridian or longitudinal fibering.*

Proof. Let G be a finite group action on $M(b,d)$, and suppose that G is not fiber preserving. Lift G to a finite group of isometries $\tilde{G}$ on $\mathbf{S}^3$ and note that $\tilde{G} \leq SO(4)$.

We suppose first that $\mathbf{S}^3/\sigma(D_{4m}^* \times \mathbf{Z}_n) = M(n,m)$, and therefore $\sigma(D_{4m}^* \times \mathbf{Z}_n)$ is a normal subgroup of $\tilde{G}$. By Dunbar [1], $(p_i \circ \rho)(\tilde{G})$ for $i = 1,2$ is neither cyclic or dihedral. Now the only subgroups of $SO(3)$ are cyclic, dihedral, the tetrahedral group T, the octahedral group O, or the icosadedral group J. The only nontrivial normal subgroup of T is D_4. The octahedral group O has two normal subgroups, D_4 and T. The icosadedral group J is a simple group. Now $p_1 \circ \rho \circ \sigma(D_{4m}^* \times \mathbf{Z}_n) = p_1 \circ (\kappa \times \kappa)(D_{4m}^* \times \mathbf{Z}_n) = D_{2m}$ is a normal subgroup of $(p_1 \circ \rho)(\tilde{G})$, and hence $D_{2m} = D_4$ and $m = 2$. We also have $p_2 \circ \rho \circ \sigma(D_{4m}^* \times \mathbf{Z}_n)$ $= p_2 \circ (\kappa \times \kappa)(D_{4m}^* \times \mathbf{Z}_n) = \mathbf{Z}_n$ being a normal subgroup of $(p_2 \circ \rho)(\tilde{G})$, which implies that $\mathbf{Z}_n$ is the trivial group. Thus $D_{4m}^* \times \mathbf{Z}_n = D_8^* \times \{1\}$, and $\mathbf{S}^3/\sigma(D_8^* \times \{1\}) = M(1,2)$.

Next we suppose $\mathbf{S}^3/\sigma(G(2^{k+1}a, m'')) = M(2^{k+1}a, m'')$ where $G(2^{k+1}a, m'')$ is the subgroup of $\mathbf{S}^3 \times \mathbf{S}^3$ generated by $X = (j, e^{\pi i/2^{k+2}})$, $Y = (e^{2\pi i/a}, 1)$, and $T = (1, e^{2\pi i/m''})$ where a and m'' are both odd. As above, we have $p_1 \circ \rho \circ \sigma(G) = p_1 \circ (\kappa \times \kappa)(G) = D_{2a}$ being a normal subgroup of $(p_1 \circ \rho)(\tilde{H})$, and hence $D_{2a} = D_4$. This implies $a = 2$, which is impossible since a is odd. $\square$

3. Non-fiber Preserving Actions on the Prism Manifold $M(1,2)$

In this section we will construct nine non-fiber preserving groups of isometries acting on the prism manifold $M(1,2)$, and show that any finite group of isometries which does not preserve a fibering is conjugate to one of these. The fundamental group of $M(1,2)$ is $\pi(1,2) = \langle c_0, c_1 \mid c_1 c_0 c_1^{-1} = c_0^{-1}, c_1^2 c_0^2 = 1 \rangle$. These actions will originate in $\mathbf{S}^3 \times \mathbf{S}^3$. Form the semidirect product $\Gamma = (\mathbf{S}^3 \times \mathbf{S}^3) \circ \mathbf{Z}_4$ in which $\mathbf{Z}_4$ is generated by φ and $\varphi(q_1, q_2)\varphi^{-1} = (q_2, j q_1 j^{-1})$. Note that if $q = u + vj$, then $j(u + vj)j^{-1} = \bar{u} + \bar{v}j$. Define an epimorphism $\bar{\sigma} : \Gamma \to O(4)$ by $\bar{\sigma}(q_1, q_2)(q) = q_1 q q_2^{-1}$ for $q_1, q_2, q \in \mathbf{S}^3$, and $\bar{\sigma}(\varphi)(u + vj) = v + \bar{u}j$. The kernel of $\bar{\sigma}$ is an order four cyclic subgroup which is generated by $(j, j)\varphi^2$ and coincides with the center of Γ. Then $\sigma = \bar{\sigma}|_{\mathbf{S}^3 \times \mathbf{S}^3} : \mathbf{S}^3 \times \mathbf{S}^3 \to SO(4)$ is an epimorphism whose kernel is $\mathbf{Z}_2 = \langle (-1, -1) \rangle$.

The quaternion subgroup $\langle i, j \rangle$ of $\mathbf{S}^3$ is isomorphic to $\pi(1,2)$ by sending i to c_0 and j to c_1. Since the group generated by $\langle i, j \rangle \times \langle 1 \rangle$ and $(-1, -1)$ is $\langle i, j \rangle \times \langle -1 \rangle$, it follows that $\sigma(\langle i, j \rangle \times \langle -1 \rangle)$ is a free action on $\mathbf{S}^3$ with $\mathbf{S}^3/\sigma(\langle i, j \rangle \times \langle -1 \rangle) = M(1,2)$. We note that $\mathbf{S}^3/\sigma(\langle -1 \rangle \times \langle i, j \rangle)$ is also a prism manifold which is homeomorphic to $M(1,2)$. This follows by observing that $\varphi(\langle i, j \rangle \times \langle -1 \rangle)\varphi^{-1} = \langle -1 \rangle \times \langle i, j \rangle$, and therefore $\bar{\sigma}(\varphi)$ conjugates $\sigma(\langle i, j \rangle \times \langle -1 \rangle)$ to $\sigma(\langle -1 \rangle \times \langle i, j \rangle)$. Thus $\sigma(\langle i, j \rangle \times \langle -1 \rangle)$ and $\sigma(\langle -1 \rangle \times \langle i, j \rangle)$ are conjugate in $O(4)$ but not in $SO(4)$.

$\mathbf{Z}_3 \times T$*-action on* $M(1,2)$.

Let $T^* = \langle x, y \mid x^2 = y^3 = (xy)^3 \rangle$ be a subgroup of $\mathbf{S}^3$ where $x = j$ and $y = \frac{1}{\sqrt{2}}(e^{\pi i/4} + e^{\pi i/4} j)$. Note that $Q^* = \langle x, yxy^{-1} \rangle$ is a normal subgroup of T^*, and $T^*/Q^* \simeq \mathbf{Z}_3$. A computation shows that $yxy^{-1} = ij$, and therefore $Q^* = \langle i, j \rangle$ which is isomorphic to $\pi(1,2)$.

By Dunbar [1], $\sigma(T^* \times T^*)$ is a non-fiber preserving action on $\mathbf{S}^3$, and therefore the orbifold $\mathbf{S}^3/\sigma(T^* \times T^*)$ cannot be fibered. Note that $Q^* \times \langle 1 \rangle$ is a normal subgroup of $T^* \times T^*$ and $(-1,-1) \notin Q^* \times \langle 1 \rangle$. Thus $\sigma(Q^* \times \langle 1 \rangle)$ is a normal subgroup of $\sigma(T^* \times T^*)$ isomorphic to Q^*. Observe that $\langle Q^* \times \langle 1 \rangle, (-1,-1) \rangle = Q^* \times \langle -1 \rangle$, and $\sigma(Q^* \times \langle -1 \rangle)$ is a normal subgroup of $\sigma(T^* \times T^*)$ isomorphic to Q^*.

Note that $\langle (-1,-1) \rangle \trianglelefteq Q^* \times \langle -1 \rangle \trianglelefteq T^* \times T^*$; and it follows that

$$\begin{aligned} &\sigma(T^* \times T^*)/\sigma(Q^* \times \langle -1 \rangle) \\ &\quad \simeq [(T^* \times T^*)/\langle (-1,-1) \rangle]/[(Q^* \times \langle -1 \rangle)/\langle (-1,-1) \rangle]. \end{aligned}$$

By the Third Isomorphism Theorem $[(T^* \times T^*)/\langle (-1,-1) \rangle]/[(Q^* \times \langle -1 \rangle)/\langle (-1,-1) \rangle] \simeq (T^* \times T^*)/(Q^* \times \langle -1 \rangle)$. Now $(T^* \times T^*)/(Q^* \times \langle -1 \rangle) \simeq T^*/Q^* \times (T^*/\langle -1 \rangle) \simeq \mathbf{Z}_3 \times T$, where $T = \langle x, y \mid 1 = x^2 = y^3 = (xy)^3 \rangle$. Therefore $\sigma(T^* \times T^*)/\sigma(Q^* \times \langle -1 \rangle) \simeq \mathbf{Z}_3 \times T$.

Let $p : \mathbf{S}^3 \to \mathbf{S}^3/\sigma(Q^* \times \langle -1 \rangle) = M(1,2)$ be the universal covering of the prism manifold $M(1,2)$. Now $\sigma(T^* \times T^*)/\sigma(Q^* \times \langle -1 \rangle) \simeq \mathbf{Z}_3 \times T$ acts on $M(1,2)$, and the quotient orbifold is $M(1,2)/(\mathbf{Z}_3 \times T) \simeq \mathbf{S}^3/\sigma(T^* \times T^*)$ which is not fibered. Thus $\mathbf{Z}_3 \times T$ acts on $M(1,2)$ and does not preserve any fibering.

$S_3 \times O$*-action on* $M(1,2)$.

We now consider the binary octahedral group $O^* = \langle x, y \mid x^2 = y^3 = (xy)^4 \rangle$, which can be viewed as a subgroup of $\mathbf{S}^3$ by letting $x = \frac{1}{\sqrt{2}}(i + j)$ and $y = \frac{1}{\sqrt{2}}(e^{\pi i/4} + e^{\pi i/4} j)$. By Dunbar [1], $\sigma(O^* \times O^*)$ is a non-fiber preserving action on $\mathbf{S}^3$, and therefore the orbifold $\mathbf{S}^3/\sigma(O^* \times O^*)$ cannot be fibered.

Consider the subgroup $H^* = \langle (xy)^2, x(xy)^2 x^{-1} \rangle$. A computation shows that $(xy)^2 = -i$ and $x(xy)^2 x^{-1} = -j$, and thus $H^* = \langle i, j \rangle$. It can be shown that H^* is a normal subgroup of O^* which is isomorphic to $\pi(1,2)$. Observe that

$$\begin{aligned} O^*/\langle (xy)^2, x(xy)^2 x^{-1} \rangle &= \langle x, y \mid x^2 = y^3 = 1, (xy)^2 = 1 \rangle \\ &= \langle x, y \mid x^2 = y^3 = 1, xy = y^2 x \rangle. \end{aligned}$$

This is the symmetric group on three letters which we denote by S_3.

Now $H^* \times \langle 1 \rangle$ is a normal subgroup of $O^* \times O^*$, and since $(-1,-1) \notin H^* \times \langle 1 \rangle$, $\sigma(H^* \times \langle 1 \rangle)$ is a normal subgroup of $\sigma(O^* \times O^*)$ isomorphic to H^*. Observe that $\langle H^* \times \langle 1 \rangle, (-1,-1) \rangle = H^* \times \langle -1 \rangle$, and $\sigma(H^* \times \langle -1 \rangle)$ is a normal subgroup of $\sigma(O^* \times O^*)$ isomorphic to H^*.

Note that $\langle (-1,-1) \rangle \trianglelefteq H^* \times \langle -1 \rangle \trianglelefteq O^* \times O^*$; and it follows that

$$\begin{aligned}&\sigma(O^* \times O^*)/\sigma(H^* \times \langle -1 \rangle)\\ &\qquad \simeq [(O^* \times O^*)/\langle (-1,-1) \rangle]/[(H^* \times \langle -1 \rangle)/\langle (-1,-1) \rangle].\end{aligned}$$

We apply the Third Isomorphism Theorem to obtain

$$\begin{aligned}&[(O^* \times O^*)/\langle (-1,-1) \rangle]/[(H^* \times \langle -1 \rangle)/\langle (-1,-1) \rangle]\\ &\qquad \simeq (O^* \times O^*)/(H^* \times \langle -1 \rangle) \simeq (O^* \times O^*)/(H^* \times \langle -1 \rangle)\\ &\qquad \simeq O^*/H^* \times (O^*/\langle -1 \rangle) \simeq S_3 \times O,\end{aligned}$$

where $O = \langle x, y \mid 1 = x^2 = y^3 = (xy)^4 \rangle$. Therefore $\sigma(O^* \times O^*)/\sigma(H^* \times \langle -1 \rangle) \simeq S_3 \times O$.

Let $p : \mathbf{S}^3 \to \mathbf{S}^3/\sigma(H^* \times \langle -1 \rangle) = M(1,2)$ be the universal covering of the prism manifold $M(1,2)$. Note that $\sigma(O^* \times O^*)/\sigma(H^* \times \langle -1 \rangle) \simeq S_3 \times O$ acts on $M(1,2)$, and the quotient orbifold is $M(1,2)/(S_3 \times O) \simeq \mathbf{S}^3/\sigma(O^* \times O^*)$ which is not fibered. Thus the $S_3 \times O$-action does not preserve any fibering on $M(1,2)$.

$S_3 \times T$ and $\mathbf{Z}_3 \times O$-actions on $M(1,2)$.

It follows by Dunbar [1] that the two non-equivalent (in $SO(4)$) group actions $\sigma(T^* \times O^*)$ and $\sigma(O^* \times T^*)$ on $\mathbf{S}^3$ do not preserve any fibering. We note that these actions are equivalent in $O(4)$. Recall that H^* and Q^* are normal subgroups of O^* and T^* respectively which are isomorphic to $\pi(1,2)$.

As above we have $\langle (-1,-1) \rangle \trianglelefteq H^* \times \langle -1 \rangle \trianglelefteq O^* \times T^*$, and $\langle (-1,-1) \rangle \trianglelefteq Q^* \times \langle -1 \rangle \trianglelefteq T^* \times O^*$. Applying the Third Isomorphism Theorem we have $[(O^* \times T^*)/\langle (-1,-1) \rangle]/[(H^* \times \langle -1 \rangle)/\langle (-1,-1) \rangle] \simeq (O^* \times T^*)/(H^* \times \langle -1 \rangle)$, which is isomorphic to $O^*/H^* \times T^*/\langle -1 \rangle$; similarly $[(T^* \times O^*)/\langle (-1,-1) \rangle]/[(Q^* \times \langle -1 \rangle)/\langle (-1,-1) \rangle] \simeq (T^* \times O^*)/(Q^* \times \langle -1 \rangle)$, which is isomorphic to $T^*/Q^* \times O^*/\langle -1 \rangle$. Since $O^*/H^* \times T^*/\langle -1 \rangle$ and $T^*/Q^* \times O^*/\langle -1 \rangle$ are isomorphic to $S_3 \times T$ and $\mathbf{Z}_3 \times O$ respectively, we obtain as above $S_3 \times T$ and $\mathbf{Z}_3 \times O$-actions on $M(1,2)$ which do not preserve any fibering.

$\mathbf{Z}_3 \times I$ -action on $M(1,2)$.

Let $I^* = \langle x, y \,|\, x^2 = y^3 = (xy)^5 \rangle$ be the binary icosahedral subgroup of $\mathbf{S}^3$ where $x = j$ and $y = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 2\cos\left(\frac{2\pi}{5}\right)}\,i + \cos\left(\frac{\pi}{5}\right) j$. By Dunbar [1], the two non-equivalent (in $SO(4)$) group actions $\sigma(T^* \times I^*)$ and $\sigma(I^* \times T^*)$ do not preserve fiberings of $\mathbf{S}^3$, although the two actions are equivalent in $O(4)$. As above we have $\langle(-1,-1)\rangle \trianglelefteq Q^* \times \langle -1 \rangle \trianglelefteq T^* \times I^*$, and using the Third Isomorphism Theorem we obtain a non-fiber preserving $\mathbf{Z}_3 \times I$-action on $M(1,2)$ where I is the icosahedral group $\langle x, y \,|\, 1 = x^2 = y^3 = (xy)^5 \rangle$.

$S_3 \times I$-action on $M(1,2)$.

By Dunbar [1], the two non-equivalent (in $SO(4)$) group actions $\sigma(O^* \times I^*)$ and $\sigma(I^* \times O^*)$ do not preserve any fibering of $\mathbf{S}^3$. These actions are equivalent in $O(4)$. We have $\langle(-1,-1)\rangle \trianglelefteq H^* \times \langle -1 \rangle \trianglelefteq O^* \times I^*$, and as above we obtain a non-fiber preserving $S_3 \times I$-action on $M(1,2)$.

T-action on $M(1,2)$.

Using Dunbar's notation [1], let $T^* \times_{C_3} T^*$ be the subgroup of $T^* \times T^*$ generated by $(x,1)$, $(1,x)$, and (y,y). Note that $Q^* \times \langle -1 \rangle$ is a normal subgroup of $T^* \times_{C_3} T^*$. By Dunbar [1], $\sigma(T^* \times_{C_3} T^*)$ is a non-fiber preserving action on $\mathbf{S}^3$, and therefore the orbifold $\mathbf{S}^3/\sigma(T^* \times_{C_3} T^*)$ cannot be fibered.

Now $\langle(-1,-1)\rangle \trianglelefteq Q^* \times \langle -1 \rangle \trianglelefteq T^* \times_{C_3} T^*$; and it follows that

$$\begin{aligned}
&\sigma(T^* \times_{C_3} T^*)/\sigma(Q^* \times \langle -1 \rangle) \\
&\qquad \simeq [(T^* \times_{C_3} T^*)/\langle(-1,-1)\rangle]/[(Q^* \times \langle -1 \rangle)/\langle(-1,-1)\rangle].
\end{aligned}$$

By the Third Isomorphism Theorem $[(T^* \times_{C_3} T^*)/\langle(-1,-1)\rangle]/[(Q^* \times \langle -1 \rangle)/\langle(-1,-1)\rangle] \simeq (T^* \times_{C_3} T^*)/(Q^* \times \langle -1 \rangle)$. The group $(T^* \times_{C_3} T^*)/(Q^* \times \langle -1 \rangle) = \langle (x,1)(Q^* \times \langle -1 \rangle), (1,x)(Q^* \times \langle -1 \rangle), (y,y)(Q^* \times \langle -1 \rangle) \rangle = \langle (1,x)(Q^* \times \langle -1 \rangle), (y,y)(Q^* \times \langle -1 \rangle) \rangle$. It is convenient at this point to use different letters for a tetrahedral group $T = \langle a, b \,|\, 1 = a^2 = b^3 = (ab)^3 \rangle$. Define a function $\theta : T \to (T^* \times_{C_3} T^*)/(Q^* \times \langle -1 \rangle)$ by sending a to $(1,x)(Q^* \times \langle -1 \rangle)$ and b to $(y,y)(Q^* \times \langle -1 \rangle)$. One can check that θ is an isomorphism.

Note that $\sigma(T^* \times_{C_3} T^*)/\sigma(Q^* \times \langle -1 \rangle) \simeq T$ acts on the prism manifold $M(1,2) = S^3/\sigma(Q^* \times \langle -1 \rangle)$, and the quotient orbifold $M(1,2)/T \simeq \mathbf{S}^3/\sigma(T^* \times_{C_3} T^*)$ is not fibered. No fibering is preserved when the tetrahedral group T acts on $M(1,2)$.

$\mathbf{Z}_3 \circ O$-action on $M(1,2)$.

The binary tetrahedral group T^* can be viewed as a normal subgroup of O^* where $T^* = \langle (xy)^2, y \rangle$, and recall $H^* = \langle (xy)^2, x(xy)^2 x^{-1} \rangle$. Let $O^* \times_{C_2} O^*$ be the subgroup of $O^* \times O^*$ generated by (x,x), $(1,y)$, $(y,1)$, $((xy)^2,1)$, and $(1,(xy)^2)$. By Dunbar [1], $\sigma(O^* \times_{C_2} O^*)$ is a non-fiber preserving action on $\mathbf{S}^3$. Now $J = H^* \times \langle -1 \rangle$ is a normal subgroup of $O^* \times_{C_2} O^*$ with $\sigma(J)$ isomorphic to $\pi(1,2)$. The quotient group $O^* \times_{C_2} O^*/J = \langle (x,x)J, (1,y)J, (y,1)J, (1,(xy)^2)J \rangle$. Since $-1 \in H^*$, we have $((x,x)J(1,y)J)^2 = (x^2,(xy)^2)J = (-1,(xy)^2)J = (1,(xy)^2)J$. Thus $O^* \times_{C_2} O^*/J = \langle (x,x)J, (1,y)J, (y,1)J \rangle$.

As above we have $\langle (-1,-1) \rangle \trianglelefteq H^* \times \langle -1 \rangle \trianglelefteq O^* \times_{C_2} O^*$, and

$$\begin{aligned} &\sigma(O^* \times_{C_2} O^*)/\sigma(H^* \times \langle -1 \rangle) \\ &\qquad \simeq [(O^* \times_{C_2} O^*)/\langle (-1,-1) \rangle]/[(H^* \times \langle -1 \rangle)/\langle (-1,-1) \rangle], \end{aligned}$$

which is isomorphic to $(O^* \times_{C_2} O^*)/(H^* \times \langle -1 \rangle)$.

Let $\mathbf{Z}_3 = \langle t \mid t^3 = 1 \rangle$, and using different letters for the octahedral group let $O = \langle a,b \mid 1 = a^2 = b^3 = (ab)^4 \rangle$. Form the semi-direct product $\mathbf{Z}_3 \circ O$ by letting $ata^{-1} = t^{-1}$, and $btb^{-1} = t$. Define a function $\theta : \mathbf{Z}_3 \circ O \to O^* \times_{C_2} O^*/J$ as follows: $\theta(a) = (x,x)J$, $\theta(b) = (1,y)J$, and $\theta(t) = (y,1)J$. Now $((x,x)J)^2 = (x^2,x^2)J = (-1,y^3)J = (1,y^3)J = ((1,y)J)^3$, and $((x,x)J(1,y)J)^4 = ((x,xy)J)^4 = (x^4,(xy)^4)J = (1,(xy)^4)J = (1,y^3)J$. A computation shows $x(xy)^2 x^{-1} = (yx)^2$; and using $x^2 = (xy)^4$ it can be verified that $xyx^{-1} = y^{-1}(yx)^{-2}$. We therefore have $(x,x)J(y,1)J((x,x)J)^{-1} = (xyx^{-1},1)J = (y^{-1}(yx)^{-2},1)J = (y^{-1},1)J((yx)^{-2},1)J = (y^{-1},1)J$. This proves that θ is an isomorphism.

Therefore $\sigma(O^* \times_{C_2} O^*)/\sigma(H^* \times \langle -1 \rangle) \simeq \mathbf{Z}_3 \circ O$ acts on the prism manifold $M(1,2) = \mathbf{S}^3/\sigma(H^* \times \langle -1 \rangle)$, and the quotient orbifold $M(1,2)/\mathbf{Z}_3 \circ O \simeq \mathbf{S}^3/\sigma(O^* \times_{C_2} O^*)$ is not fibered. Thus $\mathbf{Z}_3 \circ O$ acts on $M(1,2)$ and does not preserve any fibering.

O-action on $M(1,2)$.

The binary dihedral group $D_2^* = \langle (xy)^2, x(xy)^2 x^{-1} \rangle$ is a normal subgroup of O^* and O^*/D_2^* is the dihedral group D_3. Let $O^* \times_{D_3} O^*$ be the subgroup of $O^* \times O^*$ generated by (x,x), (y,y), $(1,(xy)^2)$, and $((xy)^2,1)$. By Dunbar [1], $\sigma(O^* \times_{D_3} O^*)$ is a non-fiber preserving action on $\mathbf{S}^3$. Now $J = H^* \times \langle -1 \rangle$ is a normal subgroup of $O^* \times_{D_3} O^*$ and $\sigma(J)$ is isomorphic to $\pi(1,2)$. It follows that $O^* \times_{D_3} O^*/J = \langle (x,x)J, (y,y)J, (1,(xy)^2)J \rangle$. Since $((x,x)J(y,y)J)^2 = ((xy)^2,(xy)^2)J = ((xy)^2,1)J(1,(xy)^2)J = (1,(xy)^2)J$, it follows that $O^* \times_{D_3} O^*/J = \langle (x,x)J, (y,y)J \rangle$, which is isomorphic to O.

As above we obtain $\sigma(O^* \times_{D_3} O^*)/\sigma(H^* \times \langle -1 \rangle) \simeq O$ acting on the prism manifold $M(1,2) = \mathbf{S}^3/\sigma(H^* \times \langle -1 \rangle)$, and the quotient orbifold $M(1,2)/O \simeq \mathbf{S}^3/\sigma(O^* \times_{D_3} O^*)$ is not fibered.

The following proposition will be useful in classifying the finite group actions on $M(1,2)$ which do not preserve a fibering.

Proposition 10. *The quaternion group $\langle i,j \rangle$ contained in T^* and O^* is unique.*

Proof. We give a brief outline of the proof. Now $\mathbf{Z}_2 = \langle -1 \rangle$ is a normal subgroup of $\langle i,j \rangle$, T^* and O^*, and $\langle i,j \rangle/\langle -1 \rangle$ is the Klein four-group $\mathbf{Z}_2 \times \mathbf{Z}_2$. Using the 4-th Isomorphism Theorem giving the lattice correspondence, and the fact that the Klein four-group is unique in $T = T^*/\langle -1 \rangle$ and $O = O^*/\langle -1 \rangle$, proves the result. $\square$

We now have the following theorem where T, O, I, and S_3 are the tetrahedral, octahedral, icosahedral, and symmetric groups respectively.

Theorem 11. *The following groups act on $M(1,2)$ and do not preserve any fibering: $\mathbf{Z}_3 \times T$, T, O, $S_3 \times O$, $\mathbf{Z}_3 \circ O$, $S_3 \times T$, $\mathbf{Z}_3 \times O$, $\mathbf{Z}_3 \times I$, and $S_3 \times I$. In addition, if G is any finite group acting on $M(1,2)$ which does not preserve any fibering, then G is conjugate to one of the groups listed above.*

Proof. Let G be a finite group action on $M(1,2) = \mathbf{S}^3/\sigma(\langle i,j \rangle \times \langle 1 \rangle) = \mathbf{S}^3/\sigma(\langle i,j \rangle \times \langle -1 \rangle)$ which does not preserve any fibering. Lift G to a finite group $\tilde{G}$ acting on $\mathbf{S}^3$, and observe that $\tilde{G}$ does not preserve any fibering of $\mathbf{S}^3$. By Dunbar [1] $\tilde{G}$ is conjugate in $SO(4)$ to exactly one of 21 groups in $SO(4)$. There is an epimorphism $\rho : SO(4) \to SO(3) \times SO(3)$, and these groups are either contained in or equal to, the pre-image under ρ of certain subgroups of $SO(3) \times SO(3)$. Furthermore there is an epimorphism $\kappa : \mathbf{S}^3 \to SO(3) = \mathbf{S}^3/\langle -1 \rangle$ defined by $\kappa(p)(v) = pvp^{-1}$, such that $\rho \circ \sigma = \kappa \times \kappa$.

Suppose $(q_1,q_2) \in \mathbf{S}^3 \times \mathbf{S}^3$ so that $\sigma(q_1,q_2)\tilde{G}\sigma(q_1,q_2)^{-1}$ yields one of these 21 groups. Observe that $\sigma(\langle i,j \rangle \times \langle -1 \rangle)$ is a normal subgroup of $\tilde{G}$ isomorphic to the quaternion group $\langle i,j \rangle$, and $(q_1,q_2)(\langle i,j \rangle \times \langle -1 \rangle)(q_1,q_2)^{-1} = q_1\langle i,j \rangle q_1^{-1} \times \langle -1 \rangle$. We consider each of these 21 groups separately.

Suppose that $\tilde{G}$ is conjugate to one of the lifts in Dunbar [1] $(\mathbf{T} \times_{\mathbf{T}} \mathbf{T})^1 \leq \rho^{-1}(T \times_T T)$, $(\mathbf{O} \times_{\mathbf{O}} \mathbf{O})^1 \leq \rho^{-1}(O \times_O O)$, or $(\mathbf{O} \times_{\mathbf{O}} \mathbf{O})^2 \leq \rho^{-1}(O \times_O O)$. Since these groups are isomorphic to T, O and O respectively, and the quaternion group is not a subgroup of T or O, these groups are excluded.

Suppose $\tilde{G}$ is conjugate to $\mathbf{T} \times_{\mathbf{T}} \mathbf{T} = \rho^{-1}(T \times_T T)$ or $\mathbf{O} \times_{\mathbf{O}} \mathbf{O} = \rho^{-1}(O \times_O O)$, which are equal to $\sigma(\langle T^* \times_{T^*} T^*, (-1,1)\rangle)$ and $\sigma(\langle O^* \times_{O^*} O^*, (-1,1)\rangle)$ respectively. Since $q_1\langle i,j\rangle q_1^{-1} \times \langle -1\rangle$ is not contained in the groups $\langle T^* \times_{T^*} T^*, (-1,1)\rangle$ or $\langle O^* \times_{O^*} O^*, (-1,1)\rangle$, these cases are also excluded.

We now suppose that $\tilde{G}$ is conjugate to $\mathbf{T} \times \mathbf{T} = \rho^{-1}(T \times T)$, which equals $\sigma(T^* \times T^*)$. Thus $(q_1\langle i,j\rangle q_1^{-1}) \times \langle -1\rangle \trianglelefteq T^* \times T^*$ and $q_1\langle i,j\rangle q_1^{-1}$ is a normal subgroup of T^* isomorphic to the quaternion group. Since by Proposition 10 the quaternion group is unique in T^*, we must have $q_1\langle i,j\rangle q_1^{-1} = \langle i,j\rangle = Q^*$. Now $\sigma(T^* \times T^*) = \sigma(q_1,q_2)\tilde{G}\sigma(q_1,q_2)^{-1}$ and $\sigma(q_1,q_2)\sigma(\langle i,j\rangle \times \langle -1\rangle)\sigma(q_1,q_2)^{-1} = \sigma(Q^* \times \langle -1\rangle) \trianglelefteq \sigma(T^* \times T^*)$. As indicated in the above cases, we obtain a $\mathbf{Z}_3 \times T$-action on $M(1,2)$. Furthermore $\sigma(q_1,q_2)$ induces a homeomorphism of $M(1,2)$ which conjugates G to $\mathbf{Z}_3 \times T$. If $\tilde{G}$ is conjugate to either $\mathbf{O} \times \mathbf{O}$, $\mathbf{O} \times \mathbf{T}$, $\mathbf{T} \times \mathbf{O}$, $\mathbf{T} \times \mathbf{I}$, or $\mathbf{O} \times \mathbf{I}$, which equals $\sigma(O^* \times O^*)$, $\sigma(O^* \times T^*)$, $\sigma(T^* \times O^*)$, $\sigma(T^* \times I^*)$ or $\sigma(O^* \times I^*)$ respectively, then a similar proof can be used to show G is conjugate to either $S_3 \times O$, $S_3 \times T$, $\mathbf{Z}_3 \times O$, $\mathbf{Z}_3 \times I$, or $S_3 \times I$ respectively. Note that if $\tilde{G}$ is conjugate to $\mathbf{I} \times \mathbf{T} = \sigma(I^* \times T^*)$, then $q_1\langle i,j\rangle q_1^{-1}$ would be a normal subgroup of I^* isomorphic to the quaternion group, but this is impossible. Similarly we may exclude the groups $\mathbf{I} \times \mathbf{O}$, $\mathbf{I} \times \mathbf{I}$, $\mathbf{I} \times_{\mathbf{I}} \mathbf{I}$, $(\mathbf{I} \times_{\mathbf{I}} \mathbf{I})^1$, $(\mathbf{I} \times_{\mathbf{I}}^* \mathbf{I})^1$, and $\mathbf{I} \times_{\mathbf{I}}^* \mathbf{I}$.

Assume that $\tilde{G}$ is conjugate to $\mathbf{T} \times_{\mathbf{C}_3} \mathbf{T} = \rho^{-1}(T \times_{C_3} T) = \sigma(T^* \times_{C_3} T^*)$. As above we have $q_1\langle i,j\rangle q_1^{-1} \times \langle -1\rangle \leq T^* \times_{C_3} T^*$ and $q_1\langle i,j\rangle q_1^{-1} = \langle i,j\rangle$ in T^*. In this case we obtain a T-action on $M(1,2)$, and G is conjugate to this T-action. The cases $\mathbf{O} \times_{\mathbf{C}_2} \mathbf{O}$ and $\mathbf{O} \times_{\mathbf{D}_3} \mathbf{O}$ are similar, and we obtain $\mathbf{Z}_3 \circ O$ and O-actions on $M(1,2)$ respectively. □

By combining theorems 9 and 11, together with theorems 10 and 11 in [4], we obtain the following theorem.

Theorem 12. *Let $M(b,d)$ be a prism manifold and let G be a finite group of isometries acting on $M(b,d)$ which does not preserve any fibering. Then $M(b,d) = M(1,2)$ and G is conjugate to one of the following group of isometries*: $\mathbf{Z}_3 \times T$, T, O, $S_3 \times O$, $\mathbf{Z}_3 \circ O$, $S_3 \times T$, $\mathbf{Z}_3 \times O$, $\mathbf{Z}_3 \times I$, *and* $S_3 \times I$. *Furthermore these actions do not leave any Heegaard Klein bottle invariant.*

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