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HELICOIDAL SURFACES IN THE 3-DIMENSIONAL LORENTZ-MINKOWSKI SPACE $\mathbf{E}_1^3$ SATISFYING $\Delta^{III} r = Ar$

By

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Abstract. In this paper the helicoidal surfaces in the 3-dimensional Lorentz-Minkowski space are classified under the condition $\Delta^{III} r = Ar$, where A is a real 3×3 matrix and Δ^{III} is the Laplace operator with respect to the third fundamental form.

Introduction

Let $\mathbf{E}_1^3$ be a three-dimensional Lorentz-Minkowski space with the scalar product of index 1 given by

$$g_L = ds^2 = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) are the canonical coordinates in $\mathbf{R}^3$.

Let $r = r(u, v)$ be a regular parametric representation of a surface M in the 3-dimensional Lorentz-Minkowski space $\mathbf{E}_1^3$ which does not contain parabolic points.

The notion of finite type submanifolds in Euclidean space or pseudo-Euclidean space was introduced by B.-Y. Chen [5]. A surface M is said to be of finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian Δ. B.-Y. Chen posed the problem of classifying the finite type surfaces in the 3-dimensional Euclidean space $\mathbf{E}^3$. Further, the notion of finite type can be extended to any smooth functions on a submanifold of a Euclidean space or a pseudo-Euclidean space.

If H is the mean curvature vector of the immersion r, we know that:

$$\Delta r = -2H.$$

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In [12] M. Choi, Y. H. Kim and G. C. Park classified helicoidal surfaces with pointwise 1-type Gauss maps and harmonic Gauss maps. In [8] G. Kaimakamis and B. J. Papantoniou classified the first three types of surfaces of revolution without parabolic points in the 3-dimensional Lorentz-Minkowski space, which satisfy the condition

$$\Delta^{II} r = Ar, \quad A \in Mat(3, \mathbf{R}),$$

where $Mat(3, \mathbf{R})$ is the set of 3×3 real matrices. They proved that such surfaces are either minimal or Lorentz hyperbolic cylinders or pseudospheres of real or imaginary radius.

In [1] Ch. Baba-Hamed and M. Bekkar studied the helicoidal surfaces without parabolic points in $\mathbf{E}_1^3$, which satisfy the condition

$$\Delta^{II} r_i = \lambda_i r_i, \quad 1 \leq i \leq 3.$$

In [3] Chr. Beneki, G. Kaimakamis and B. J. Papantoniou obtained a classification of surfaces of revolution with constant Gauss curvature in $\mathbf{E}_1^3$ and in [4] defined four kinds of helicoidal surfaces in $\mathbf{E}_1^3$. C. W. Lee, Y. H. Kim and D. W. Yoon [13] studied the ruled surfaces in $\mathbf{E}_1^3$ which satisfy the condition

$$\Delta^{III} r = Ar, \tag{1}$$

where $A \in Mat(3, \mathbf{R})$.

S. Stamatakis and H. Al-Zoubi in [11] classified the surfaces of revolution with non zero Gaussian curvature in $\mathbf{E}^3$ under the condition (1).

In [9] G. Kaimakamis, B. J. Papantoniou and K. Petoumenos classified and proved that such surfaces of revolution in the 3-dimensional Lorentz-Minkowski space $\mathbf{E}_1^3$ satisfying (1) are either minimal or Lorentz hyperbolic cylinders or pseudospheres of real or imaginary radius.

Recently, the authors [2] studied the translation surfaces in $\mathbf{E}_1^3$ satisfying (1).

In this work we classify the helicoidal surfaces with non-degenerate third fundamental form in the 3-dimensional Lorentz-Minkowski space under the condition (1).

1. Preliminaries

A vector X of $\mathbf{E}_1^3$ is said to be timelike if $g_L(X, X) < 0$, spacelike if $g_L(X, X) > 0$ or $X = 0$ and lightlike or null if $g_L(X, X) = 0$ and $X \neq 0$. A timelike or light-like vector in $\mathbf{E}_1^3$ is said to be causal.

For two vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ in $\mathbf{E}_1^3$ the Lorentz cross product of X and Y is defined by

$$X \wedge_L Y = (x_3 y_2 - x_2 y_3, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The pseudo-vector product operation $\wedge_L$ is related to the determinant function by

$$\det(X, Y, Z) = g_L(X \wedge_L Y, Z).$$

The matrices

$$\begin{pmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix}$$

are called the *Lorentzian rotation matrix* in $\mathbf{E}_1^3$, where $\theta \in \mathbf{R}$.

For an open interval $I \subset \mathbf{R}$, let $\gamma : I \to \Pi$ be a curve in a plane Π in $\mathbf{E}_1^3$ and let L be a straight line in Π which does not intersect the curve γ (axis). A helicoidal surface in Minkowski space $\mathbf{E}_1^3$ is a surface invariant by a uni-parametric group

$$G_{L,c} = \{g_v / g_v : \mathbf{E}_1^3 \to \mathbf{E}_1^3; v \in \mathbf{R}\}$$

of helicoidal motions. Each helicoidal surface is given by a group of helicoidal motions and a generating curve. A helicoidal surface parametrizes as

$$r(u, v) = g_v(\gamma(u)), \quad (u, v) \in I \times \mathbf{R}.$$

Each group of helicoidal motions is characterized by an axis L and a pitch $c \neq 0$. Depending on the axis L being spacelike, timelike or null, there are three types of motion.

If the axis L is spacelike (resp. timelike), then L is transformed to the y-axis or z-axis (resp. x-axis) by the Lorentz transformation. Therefore, we may consider z-axis (resp. x-axis) as the axis if L is spacelike (resp. timelike). If the axis L is lightlike, then we may suppose that the axis is the line spanned by the vector $(1, 1, 0)$. We distinguish helicoidal surfaces in $\mathbf{E}_1^3$ into the following types.

Case 1. The axis L is spacelike, i.e., $(L = \langle (0, 0, 1) \rangle)$.

Without loss of generality we may assume that the profile curve γ lies in the yz-plane or xz-plane. Hence, the curve γ can be represented by

$$\gamma(u) = (0, f(u), g(u)) \quad \text{or} \quad \gamma(u) = (f(u), 0, g(u)),$$

where f is a smooth positive function and g is a smooth function on I.

The helicoidal surfaces M in $\mathbf{E}_1^3$ given by [4] are defined by

$$r(u,v) = (f(u)\sinh v, f(u)\cosh v, cv + g(u)), \quad c \in \mathbf{R}^+ \tag{2}$$

or

$$r(u,v) = (f(u)\cosh v, f(u)\sinh v, cv + g(u)), \quad c \in \mathbf{R}^+. \tag{3}$$

We call (2) and (3) a helicoidal surface of type ***I*** and type ***II*** respectively.
Case 2. The axis L is time-like, i.e., $(L = \langle(1,0,0)\rangle)$.

In this case, we may assume that the profile curve γ lies in the xy-plane. So the curve γ is given by

$$\gamma(u) = (g(u), f(u), 0)$$

for a positive function $f = f(u)$ on I. Hence, the helicoidal surface M is given by [4]

$$r(u,v) = (g(u) + cv, f(u)\cos v, f(u)\sin v), \quad f(u) > 0,\ c \in \mathbf{R}^+. \tag{4}$$

We call (4) a helicoidal surface of type ***III***.
Case 3. The axis L is light-like, i.e., $(L = \langle(1,1,0)\rangle)$.

In this case, we may assume that the profile curve γ lies in the xy-plane. Then its parametrization is given by

$$\gamma(u) = (f(u), g(u), 0), \quad u \in I,$$

where f and g are functions on I, such that $f(u) \neq g(u), \forall u \in I$.

Therefore the helicoidal surface M may be parametrized as [4]

$$r(u,v) = \left(f(u) + \frac{v^2}{2}h(u) + cv, g(u) + \frac{v^2}{2}h(u) + cv, vh(u)\right), \quad c \in \mathbf{R}, \tag{5}$$

where $h(u) = f(u) - g(u)$. We call (5) a helicoidal surface of type ***IV***.

If we take $c = 0$, then we obtain a rotations group related to axis L. The helicoidal surface is a generalization of rotation surface.

The immersion (M, r) is said to be of finite Chen-type if the position vector r admits the following spectral decomposition

$$r = r_0 + \sum_{i=1}^{k} r_i,$$

where r_i are $\mathbf{E}_1^3$-valued eigenfunctions of the Laplacian of (M, r): $\Delta r_i = \lambda_i r_i$, $\lambda_i \in \mathbf{R}$, $i = 1, 2, \ldots, k$ [5]. If λ_i are different, then M is said to be of k-type.

Let $\{x^i, x^j\}$ be a local coordinate system of M. For the components e_{ij} $(i, j = 1, 2)$ of the third fundamental form III on M we denote by (e^{ij}) the inverse matrix of the matrix (e_{ij}).

The Laplace operator Δ^{III} of the third fundamental form III on M is formally defined by

$$\Delta^{III} = \frac{-1}{\sqrt{|e|}}\left(\frac{\partial}{\partial x^i}\left(\sqrt{|e|}e^{ij}\frac{\partial}{\partial x^j}\right)\right), \tag{6}$$

where $e = \det(e_{ij})$.

The coefficients of the first fundamental form and the second fundamental form are

$$E = g_{11} = \langle r_u, r_u\rangle, \quad F = g_{12} = \langle r_u, r_v\rangle, \quad G = g_{22} = \langle r_v, r_v\rangle,$$

$$L = h_{11} = \langle r_{uv}, \mathbf{N}\rangle, \quad M = h_{12} = \langle r_{uv}, \mathbf{N}\rangle, \quad N = h_{22} = \langle r_{vv}, \mathbf{N}\rangle.$$

If $\varphi : M \to \mathbf{R}$, $(u, v) \to \varphi(u, v)$ is a smooth function and Δ^{III} the Laplace operator with respect the third fundamental form, then it holds [10]:

$$\Delta^{III}\varphi = \frac{-1}{\sqrt{|e|}}\left(\frac{\partial}{\partial u}\left(\frac{e_{22}\varphi_u - e_{12}\varphi_v}{\sqrt{|e|}}\right) - \frac{\partial}{\partial v}\left(\frac{e_{12}\varphi_u - e_{11}\varphi_v}{\sqrt{|e|}}\right)\right). \tag{7}$$

The Gaussian curvature K_G and the mean curvature H of M are given by

$$K_G = g_L(\mathbf{N}, \mathbf{N})\frac{(LN - M^2)}{EG - F^2}$$

$$H = \frac{(EN + GL - 2FM)}{2|EG - F^2|},$$

where $\mathbf{N}$ is the unit normal vector to M.

2. Helicoidal Surfaces of Type *I*, *II*

In this section we are concerned with non-degenerate helicoidal surfaces M without parabolic points satisfying the condition (1).

Suppose that M is given by (2), or equivalently by

$$r(u, v) = (u \sinh v, u \cosh v, cv + g(u)), \quad c \in \mathbf{R}^+. \tag{8}$$

We define smooth function W as:

$$W = \sqrt{\varepsilon g_L(r_u \wedge_L r_v, r_u \wedge_L r_v)} = \sqrt{\varepsilon(u^2(1 + g'^2) - c^2)}.$$

The coefficients of the first and the second fundamental form are:

$$E = 1 + g'^2, \quad F = cg', \quad G = c^2 - u^2$$

$$L = \frac{-ug''}{W}, \quad M = \frac{c}{W}, \quad N = \frac{u^2 g'}{W},$$

where $g' = \frac{dg}{du}$, $g'' = \frac{d^2 g}{du^2}$.

The components of the third fundamental form of the surface M is given, respectively, by

$$\begin{aligned} e_{11} &= \frac{\varepsilon}{W^4}(u^4 g''^2 - c^2(ug'' + g')^2 - c^2), \\ e_{12} &= \frac{-c}{W^2}(ug'' + g'), \quad e_{22} = \frac{1}{W^2}(c^2 - u^2 g'^2), \end{aligned} \tag{9}$$

hence

$$\sqrt{|e|} = \frac{\varepsilon_1 R}{W^3},$$

where $\varepsilon_1 = \pm 1$ and $R = u^3 g' g'' + c^2$.

From these we find that the curvature K_G and the mean curvature H of (8) are given by

$$K_G = \frac{u^3 g' g'' + c^2}{W^4}$$

and

$$H = -\frac{u^2 g'(1 + g'^2) - 2c^2 g' - ug''(c^2 - u^2)}{2W^3}. \tag{10}$$

We rewrite the above equation as [7]

$$H = \frac{1}{2u}\left(\frac{u^2 g'}{W}\right)'.$$

PROPOSITION 2.1. *If* $H = 0$, *then the function on the profile curve* $\gamma(u) = (0, u, g(u))$ *is as follows*

$$g(u) = \pm \int \sqrt{\frac{a^2(u^2 - c^2)}{\varepsilon u^4 - a^2 u^2}}\, du + b \tag{11}$$

in $\mathbf{E}_1^3$, *where* $a, b \in \mathbf{R}$.

Proof. If $H = 0$, then we obtain

$$u^2 g' = aW, \quad a \in \mathbf{R}.$$

Hence, if we solve

$$g'^2 = \frac{a^2(u^2 - c^2)}{\varepsilon u^4 - a^2 u^2},$$

then we have (11). □

If a surface M in $\mathbf{E}_1^3$ has no parabolic points, then we have

$$u^3 g' g'' + c^2 \neq 0, \quad \forall u \in I.$$

Suppose that $LN - M^2 > 0$ (we have the same result if $LN - M^2 < 0$).

By a straightforward computation, the Laplacian Δ^{III} of the third fundamental form III on M with the help of (9) and (7) turns out to be

$$\begin{aligned}\Delta^{III} = -\frac{\varepsilon W^3}{R}\Bigg(&\frac{\varepsilon\varepsilon_1}{WR^2}(-\varepsilon W^2 u^3 g' g'''(c^2 - u^2 g'^2) + c^4 u - 3c^2 u^3 g'^2\\ &+ 3c^4 g'^2 u - 3c^2 g'^4 u^3 + 6c^4 g' g'' u^2 - 4c^2 g' g'' u^4 + c^2 g'^2 g''^2 u^5\\ &- 2g'^4 g''^2 u^7 - g'^2 g''^2 u^7 - c^2 g''^2 u^5 + c^4 g''^2 u^3 - 6c^2 g'^3 g'' u^4)\frac{\partial}{\partial u}\\ &+ \frac{c\varepsilon\varepsilon_1}{WR^2}(\varepsilon W^2 u g'''(c^2 - g'^2 u^2) - g' g''^2 u^5 - 2g'' g'^2 u^4 - 2g'^4 g'' u^4\\ &+ 3c^2 g' g''^2 u^3 + 3c^2 g'' u^2 + c^2 g' u + 7c^2 g'' g'^2 u^2 + c^2 g'^3 u\\ &- 2c^4 g'' + c^2 g''^3 u^4 - g''^3 u^6)\frac{\partial}{\partial v}\\ &+ \frac{2\varepsilon_1 Wc(ug'' + g')}{R}\frac{\partial^2}{\partial u \partial v} + \frac{\varepsilon_1 W(c^2 - g'^2 u^2)}{R}\frac{\partial^2}{\partial u^2}\\ &+ \frac{\varepsilon\varepsilon_1(g''^2 u^4 - c^2(ug'' + g')^2 - c^2)}{WR}\frac{\partial^2}{\partial v^2}\Bigg)\Bigg). \end{aligned} \tag{12}$$

By using (8) and (12) we get

$$\begin{cases} \Delta^{III}(u \sinh v) = P(u)\cosh v + Q(u)\sinh v \\ \Delta^{III}(u \cosh v) = Q(u)\cosh v + P(u)\sinh v \\ \Delta^{III}(cv + g(u)) = T(u) \end{cases} \tag{13}$$

where

$$P(u) = -\frac{\varepsilon W^2}{R^3}(\varepsilon c W^2 u^2 g'''(c^2 - g'^2 u^2) - cg''^3 u^7 + c(1 + 2g'^2)g' g''^2 u^6 + c^3 g''^3 u^5 + c^3 g' g''^2 u^4 + c^3(7g'^2 + 5)g'' u^3 + 3c^3(1 + g'^2)g' u^2 - 4c^5 g'' u - 2c^5 g'),$$

$$Q(u) = -\frac{\varepsilon W^2}{R^3}(\varepsilon W^2 u^3 g' g'''(g'^2 u^2 - c^2) + 2c^4 g'^2 u + 4c^4 g'' g' u^2 - 3c^2(g'^2 + g'^4)u^3 - c^2(7g'^3 g'' + 5g'' g')u^4 - c^2 g'^2 g''^2 u^5 - c^2 g''^3 g' u^6 - (2g'^4 g''^2 + g'^2 g''^2)u^7 + g''^3 g' u^8), \tag{14}$$

$$T(u) = -\frac{\varepsilon W^2}{R^3}(\varepsilon W^2 u g'''(c^2 - g'^2 u^2)^2 + (-3g'^2 - 2)g'^3 g''^2 u^7 - c^2 g''^3 u^6 + c^2(3g'^2 - 1)g' g''^2 u^5 + c^2(c^2 g''^2 - 7g'^2 - 9g'^4)g'' u^4 + 3c^2(c^2 g''^2 - g'^4 - g'^2)g' u^3 + c^4(15g'^2 + 4)g'' u^2 + 2c^4(2g'^2 + 1)g' u - 3c^6 g'').$$

Remark 2.2. We observe that

$$\begin{aligned} &ug' P(u) + cQ(u) = 0 \\ &\left(\frac{\varepsilon K_G}{2cW}\right)((c^2 - g'^2 u^2)P(u) - cuT(u)) = H. \end{aligned} \tag{15}$$

The equation (1) by means of (8) and (13) gives rise to the following system of ordinary differential equations

$$\begin{cases} (P(u) - a_{12}u)\cosh v + (Q(u) - a_{11}u)\sinh v - a_{13}(cv + g) = 0 \\ (Q(u) - a_{22}u)\cosh v + (P(u) - a_{21}u)\sinh v - a_{23}(cv + g) = 0 \\ a_{31}u \sinh v + a_{32}u \cosh v + a_{33}(cv + g) = T(u), \end{cases} \tag{16}$$

where a_{ij} $(i, j = 1, 2, 3)$ denote the components of the matrix A given by (1).

But $\sinh v$ and $\cosh v$ are linearly independent functions of v, so we finally obtain $a_{32} = a_{31} = a_{33} = a_{13} = a_{23} = 0$.

We put $a_{11} = a_{22} = \lambda$ and $a_{12} = a_{21} = \mu$, $\lambda, \mu \in \mathbf{R}$. Therefore, this system of equations is equivalently reduced to

$$\begin{cases} Q(u) = \lambda u \\ P(u) = \mu u \\ T(u) = 0. \end{cases} \tag{17}$$

Therefore, the problem of classifying the helicoidal surfaces M in $\mathbf{E}_1^3$ given by (8) and satisfying (1) is reduced to the integration of this system of ordinary differential equations.

Next we study this system according to the values of the constants λ, μ.

Case 1. Let $\lambda = 0$ and $\mu \neq 0$.

The system of equations (17) takes the form

$$\begin{cases} g' P(u) = 0 \\ P(u) = \mu u \\ T(u) = 0. \end{cases} \tag{18}$$

Then $g'(u) = 0$, which is a contradiction. Hence there are no helicoidal surfaces of $\mathbf{E}_1^3$ in this case which satisfy (1).

Case 2. Let $\lambda \neq 0$ and $\mu = 0$.

In this case the system (17) is reduced equivalently to

$$\begin{cases} g' P(u) = -\lambda c \\ P(u) = 0 \\ T(u) = 0. \end{cases}$$

But this is not possible. So, in this case there are no helicoidal surfaces of $\mathbf{E}_1^3$.

Case 3. Let $\lambda = \mu = 0$ then $A = diag(0, 0, 0)$.

In this case the system (17) is reduced equivalently to

$$\begin{cases} P(u) = 0 \\ Q(u) = 0 \\ T(u) = 0. \end{cases}$$

From (15) we have $H = 0$. If we substitute (11) in (14) we get $Q(u) = 0$. By using (15) we get $P(u) = 0$ and $T(u) = 0$. Consequently M is a minimal surface.

Case 4. Let $\lambda \neq 0$ and $\mu \neq 0$.

In this case the system (17) is reduced equivalently to

$$g(u) = -\frac{\lambda c}{\mu} \ln(u) + k, \quad k \in \mathbf{R}. \tag{19}$$

If we substitute (19) in (14) we get $Q(u) = 0$. So we have a contradiction and therefore, in this case there are no helicoidal surfaces of $\mathbf{E}_1^3$.

THEOREM 2.3. *Let* $r : M \to \mathbf{E}_1^3$ *be an isometric immersion given by* (8). *Then* $\Delta^{III} r = Ar$ *if and only if* M *has zero mean curvature.*

3. Helicoidal Surfaces of Type *III*

In this section, we study the case of helicoidal surfaces M in $\mathbf{E}_1^3$ of type ***III***. Suppose that M is given by (4), or equivalently by

$$r(u, v) = (cv + g(u), u \cos v, u \sin v). \tag{20}$$

The coefficients of the first and the second fundamental form are:

$$E = 1 - g'^2, \quad F = -cg', \quad G = u^2 - c^2,$$

$$L = \frac{ug''}{W}, \quad M = -\frac{c}{W}, \quad N = \frac{u^2 g'}{W}.$$

The unit normal vector field $\mathbf{N}$ on M is given by

$$\mathbf{N} = \frac{-1}{W}(u, -c \sin v + g'u \cos v, c \cos v + g'u \sin v),$$

where $W = \sqrt{\varepsilon g_L(r_u \wedge_L r_v, r_u \wedge_L r_v)} = \sqrt{\varepsilon(u^2(1 - g'^2) - c^2)}$.

The components of the third fundamental form of the surface M is given, respectively, by

$$\begin{aligned} e_{11} &= \frac{\varepsilon}{W^4}(u^4 g''^2 - c^2(ug'' + g')^2 + c^2), \\ e_{12} &= \frac{-c}{W^2}(ug'' + g'), \quad e_{22} = \frac{1}{W^2}(u^2 g'^2 + c^2), \end{aligned} \tag{21}$$

hence

$$\sqrt{|e|} = \frac{\varepsilon_1 R}{W^3},$$

where $\varepsilon_1 = \pm 1$ and $R = u^3 g' g'' - c^2$.

By a direct computation, we can see that the Gauss curvature K_G and the mean curvature H of M are given by

$$K_G = \frac{u^3 g' g'' - c^2}{W^4}$$

and

$$H = \frac{u^2 g'(1 - g'^2) - 2c^2 g' - ug''(c^2 - u^2)}{2W^3}. \tag{22}$$

We rewrite the above equation as [7]

$$H = \frac{1}{2u}\left(\frac{u^2 g'}{W}\right)'.$$

Proposition 3.1. *If* $H = 0$, *then the function on the profile curve* $\gamma(u) = (g(u), u, 0)$ *is as follows*

$$g(u) = \pm \int \sqrt{\frac{a^2(u^2 - c^2)}{\varepsilon u^4 + a^2 u^2}}\, du + b \tag{23}$$

in $\mathbf{E}_1^3$, *where* $a, b \in \mathbf{R}$.

Proof. If $H = 0$, then we obtain

$$u^2 g' = aW, \quad a \in \mathbf{R}.$$

Hence, if we solve

$$g'^2 = \frac{a^2(u^2 - c^2)}{\varepsilon u^4 + a^2 u^2},$$

then we have (23). □

If a surface M in $\mathbf{E}_1^3$ has no parabolic points, then we have

$$u^3 g' g'' - c^2 \neq 0.$$

Suppose that $LN - M^2 > 0$ (we have the same result if $LN - M^2 < 0$).

By a straightforward computation, the Laplacian Δ^{III} of the third fundamental form III on M with the help of (7) and (21) turns out to be

$$\begin{aligned}
\Delta^{III} = \frac{\varepsilon W^3}{R}\Bigg(&\frac{\varepsilon\varepsilon_1}{WR^2}(\varepsilon W^2 u^3 g' g'''(c^2 + g'^2 u^2) + (2g'^2 - 1)g'^2 g''^2 u^7 \\
&+ c^2(g'^2 + 1)g''^2 u^5 + c^2(4 - 6g'^2)g' g'' u^4 \\
&+ c^2(3g'^2 - 3g'^4 - c^2 g''^2)u^3 - 6c^4 g' g'' u^2 + c^4(1 - 3g'^2)u)\frac{\partial}{\partial u} \\
&+ \frac{\varepsilon\varepsilon_1 c}{WR^2}(\varepsilon W^2 u g'''(c^2 + g'^2 u^2) + g''^3 u^6 + g' g''^2 u^5 \\
&+ (2g'^2 - 2g'^4 - c^2 g''^2)g'' u^4 - 3c^2 g' g''^2 u^3 + c^2(3 - 7g'^2)g'' u^2 \\
&+ c^2(1 - g'^2)g' u - 2c^4 g'')\frac{\partial}{\partial v}
\end{aligned}$$

$$- \left(\frac{2\varepsilon_1 Wc(ug'' + g')}{R} \right) \frac{\partial^2}{\partial u \partial v} - \left(\frac{\varepsilon_1 W(c^2 + g'^2 u^2)}{R} \right) \frac{\partial^2}{\partial u^2}$$

$$- \left(\frac{\varepsilon\varepsilon_1(-g''^2 u^4 + c^2(ug'' + g')^2 - c^2)}{WR} \right) \frac{\partial^2}{\partial v^2} \Bigg) \Bigg) \tag{24}$$

By using (24) and (20) we get

$$\begin{cases} \Delta^{III}(cv + g(u)) = T(u) \\ \Delta^{III}(u \cos v) = P(u) \cos v + Q(u) \sin v \\ \Delta^{III}(u \sin v) = -Q(u) \cos v + P(u) \sin v, \end{cases} \tag{25}$$

where

$$P(u) = \frac{\varepsilon W^2}{R^3} (\varepsilon W^2 u^3 g' g'''(c^2 + g'^2 u^2) + g' g''^3 u^8 + (2g'^2 - 1) g'^2 g''^2 u^7$$
$$- c^2 g' g''^3 u^6 - c^2 g'^2 g''^2 u^5 + c^2(5 - 7g'^2) g' g'' u^4 + 3c^2(1 - g'^2) g'^2 u^3$$
$$- 4c^4 g' g'' u^2 - 2c^4 g'^2 u), \tag{26}$$

$$Q(u) = \frac{-\varepsilon W^2}{R^3} (\varepsilon c W^2 u^2 g'''(c^2 + g'^2 u^2) + c g''^3 u^7 + c(-1 + 2g'^2) g' g''^2 u^6$$
$$- c^3 g''^3 u^5 - c^3 g' g''^2 u^4 + (-7g'^2 + 5) c^3 g'' u^3 + 3c^3 g'(1 - g'^2) u^2$$
$$- 4c^5 g'' u - 2c^5 g'), \tag{27}$$

$$T(u) = \frac{\varepsilon W^2}{R^3} (\varepsilon W^2 u g'''(c^2 + g'^2 u^2)^2 + (3g'^5 g''^2 - 2g'^3 g''^2) u^7 + c^2 g''^3 u^6$$
$$+ (3c^2 g'^3 g''^2 + c^2 g' g''^2) u^5 + (-c^4 g''^3 + 7c^2 g'^2 g'' - 9c^2 g'^4 g'') u^4$$
$$+ (-3c^4 g' g''^2 - 3c^2 g'^5 + 3c^2 g'^3) u^3 + (-15c^4 g'^2 g'' + 4c^4 g'') u^2$$
$$+ (-4c^4 g'^3 + 2c^4 g') u - 3c^6 g'').$$

REMARK 3.2. We observe that

$$\begin{aligned} \left(\frac{\varepsilon K_G}{2cW} \right) (cuT(u) + (c^2 + g'^2 u^2) Q(u)) = -H \\ cP(u) + ug' Q(u) = 0. \end{aligned} \tag{28}$$

The equation (1) by means of (20) and (25) gives rise to the following system of ordinary differential equations

$$\begin{cases} a_{12}u \cos v + a_{13}u \sin v + a_{11}(cv + g) = T(u) \\ (P(u) - a_{22}u) \cos v + (Q(u) - a_{23}u) \sin v - a_{21}(cv + g) = 0 \\ (Q(u) + a_{32}u) \cos v - (P(u) - a_{33}u) \sin v + a_{31}(cv + g) = 0. \end{cases} \tag{29}$$

From (29) we easily deduce that $a_{11} = a_{12} = a_{13} = a_{21} = a_{31} = 0$, $a_{22} = a_{33}$ and $a_{32} = -a_{23}$. We put $a_{22} = a_{33} = \lambda$ and $-a_{32} = a_{23} = \mu$, $\lambda, \mu \in \mathbf{R}$. Therefore, this system of equations is equivalently reduced to

$$\begin{cases} P(u) = \lambda u \\ Q(u) = \mu u \\ T(u) = 0. \end{cases} \tag{30}$$

Therefore, the problem of classifying the helicoidal surfaces M in $\mathbf{E}_1^3$ given by (20) and satisfying (1) is reduced to the integration of this system of ordinary differential equations.

We discuss four cases according to the constants λ and μ.

Case 1. Let $\lambda = 0$ and $\mu \neq 0$.

$$\begin{cases} g'Q(u) = 0 \\ Q(u) = \mu u \\ cP(u) = 0. \end{cases}$$

From this system we get $g' = 0$, which is a contradiction. Hence there are no helicoidal surfaces of $\mathbf{E}_1^3$ in this case.

Case 2. Let $\lambda \neq 0$ and $\mu = 0$.

In this case the system (30) is reduced equivalently to

$$\begin{cases} g'Q(u) = -\lambda c \\ Q(u) = 0. \end{cases}$$

But this is not possible. So, in this case there are no helicoidal surfaces of $\mathbf{E}_1^3$.

Case 3. Let $\lambda = \mu = 0$ then $A = diag(0, 0, 0)$.

In this case the system (30) is reduced equivalently to

$$\begin{cases} g'Q(u) = 0 \\ Q(u) = 0 \\ T(u) = 0. \end{cases}$$

Then, the equation (28) gives rise to $H = 0$. If we substitute (23) in (26) we get $P(u) = 0$. By using (28) we get $Q(u) = 0$ and $T(u) = 0$. Consequently M is a minimal surface.

Case 4. Let $\lambda \neq 0$ and $\mu \neq 0$.

In this case the system (30) is reduced equivalently to

$$g(u) = -\frac{\lambda c}{\mu} \ln(u) + k, \quad k \in \mathbf{R}. \tag{31}$$

If we substitute (31) in (27) we get $Q(u) = 0$. So we have a contradiction and therefore, in this case there are no helicoidal surfaces of $\mathbf{E}_1^3$.

We are now ready to state the following theorem.

THEOREM 3.3. *Let $r : M \to \mathbf{E}_1^3$ be an isometric immersion given by* (20). *Then $\Delta^{III} r = Ar$ if and only if M has zero mean curvature.*

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