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WEIGHTED $L^p - L^q$ ESTIMATES OF THE STOKES SEMIGROUP IN SOME UNBOUNDED DOMAINS

By

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Abstract. We consider the Navier-Stokes equations in half-space and a perturbed half-space in L^p space with Muckenhoupt weight. As the first step, we shall describe the Helmholtz decompostion of L^p space with Muckenhoupt weight and the weighted resolvent estimates for the Stokes equations. Next we shall show the $L^p - L^q$ estimates of Stokes semigroup with $\langle x \rangle^s$ type weight. Finally as the application of the weighted $L^p - L^q$ estimates, we shall obtain the weighted asymptotic behavior of the solution to the Navier-Stokes equations.

1 Introduction

Let $n \geq 2$. Let $\Omega \subset \mathbf{R}^n$ be the half-space H or a perturbed half-space with smooth boundary $\partial\Omega$. To be precise, the half-space H is defined by $H = \{x = (x', x_n) \in \mathbf{R}^n \mid x_n > 0\}$ and the perturbed half-space is a unbounded domain which has a positive number R satisfying

$$\Omega \backslash B_R = H \backslash B_R, \tag{1.1}$$

where $B_R = \{x \in \mathbf{R}^n \mid |x| < R\}$.

In this paper, we consider the following Navier-Stokes equations in Ω:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla \pi = 0 & \text{in } (0, \infty) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0, \infty) \times \Omega, \\ \lim_{|x| \to \infty} u(t, x) = 0, \quad u(t, x) = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ u(0, x) = a(x) & \text{in } \Omega. \end{cases} \tag{NS}$$

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Here $u(t,x) = (u_1(t,x),\dots,u_n(t,x))$ and $\pi(t,x)$ denote unknown velocity field and scalar pressure and $a(x)$ is a given vector function.

The Navier-Stokes equations (NS) have been already studied by many authors in some bounded domains and unbounded domains. In particular, we have many results concerning to (NS) in L^p-framework.

The results of Farwig and Sohr [14] and Miyakawa [33] are the first step to discuss the nonstationary problem (NS) in the L^p-space. They showed the Helmholtz decomposition of the L^p-space of vector fields $L^p(\Omega) = L^p_\sigma(\Omega) \oplus G^p(\Omega)$ for $n \ge 2$ and $1 < p < \infty$, where $L^p_\sigma(\Omega)$ and $G^p(\Omega)$ denote as follows:

$$L^p_\sigma(\Omega) = \overline{\{u \in C_0^\infty(\Omega) \mid \nabla \cdot u = 0 \text{ in } \Omega\}}^{\|\cdot\|_{L^p(\Omega)}},$$

$$G^p(\Omega) = \{\nabla \pi \in L^p(\Omega) \mid \pi \in L^p_{\mathrm{loc}}(\overline{\Omega})\}.$$

Let P be a continuous projection from $L^p(\Omega)$ to $L^p_\sigma(\Omega)$ associated with the Helmholtz decomposition. The Stokes operator A is defined by $A = -P\Delta$ with some domain. It is proved by Farwig and Sohr [14] that $-A$ generates a bounded analytic semigroup e^{-tA} on $L^p_\sigma(\Omega)$.

When we prove the existence theorem of global solution to (NS) with small data, the following $L^p - L^q$ estimates of the Stokes semigroup play the important role:

$$\|e^{-tA}f\|_{L^q} \le Ct^{-n(1/p-1/q)/2}\|f\|_{L^p}, \tag{1.2}$$

$$\|\nabla e^{-tA}f\|_{L^q} \le Ct^{-n(1/p-1/q)/2-1/2}\|f\|_{L^p} \tag{1.3}$$

for $f \in L^p_\sigma(\Omega)$ and $t > 0$, where $1 \le p \le q \le \infty$, $(p \ne \infty, q \ne 1)$ for (1.2) and $1 \le p \le q < \infty$, $(q \ne 1)$ for (1.3). The $L^p - L^q$ estimates of the Stokes semigroup have been already studied by many authors in some domains. In fact, when Ω is the whole space, applying the Young inequality to the concrete solution formula, we have (1.2) and (1.3) for $1 \le p \le q \le \infty$ $(p \ne \infty, q \ne 1)$. When Ω is the half-space, it is proved by Ukai [35] and Borchers and Miyakawa [4] that (1.2) and (1.3) hold for $1 \le p \le q \le \infty$ $(p \ne \infty, q \ne 1)$ (cf. Desch, Hieber and Prüss [11]). When Ω is an infinite layer case, Abe and Shibata [1] proved that (1.2) and (1.3) hold for $1 < p \le q < \infty$. When Ω is a bounded domain, (1.2) and (1.3) for $1 < p \le q < \infty$ follow from the result of Giga [22] on a characterization of the domains of fractional powers of the Stokes operator. In an infinite layer case and a bounded domain case, an exponential decay property of the semigroup is available.

When Ω is an exterior domain, (1.2) holds for $1 \le p \le q \le \infty$ $(p \ne \infty, q \ne 1)$ but (1.3) holds only for $1 \le p \le q \le n$ $(q \ne 1)$. At first Iwashita [25] proved that (1.2) holds for $1 < p \le q < \infty$ and (1.3) for $1 < p \le q \le n$ when $n \ge 3$. The refinement of his result was done by the following authors: Chen [6] $(n = 3, q = \infty)$, Shibata [34] $(n = 3, q = \infty)$, Borchers and Varnhorn [5] ($n = 2$, (1.2) for $p = q$), Dan and Shibata [8], [9] $(n = 2)$, Dan, Kobayashi and Shibata [10] $(n = 2, 3)$, and Maremonti and Solonnikov [31] $(n \ge 2)$. Especially, it was shown by Maremonti and Solonnikov [31] that Iwashita's restriction $q \le n$ in (1.3) is unavoidable.

When Ω is a perturbed half-space, Kubo and Shibata [30] proved (1.2) for $1 \le p \le q \le \infty$ $(p \ne \infty, q \ne 1)$ and (1.3) for $1 \le p \le q < \infty$ $(q \ne 1)$ when $n \ge 2$. When Ω is an aperture domain, Abels [2], Hishida [24] and Kubo [29] proved (1.2) for $1 \le p \le q \le \infty$ $(p \ne \infty, q \ne 1)$ and (1.3) for $1 \le p \le q < \infty$ $(q \ne 1)$ when $n \ge 2$.

In usual L^p-framework, it is well-known that we can prove the global existence of the solution to the Navier-Stokes problem with small L^n data. In fact, the time-global existence was proved by many authors in the following domain cases: Giga and Miyakawa [23] for bounded domains, Kato [27] for the whole space, Ukai [35] and Kozono [28] for the half-space, Iwashita [25] for the exterior domain, Abe and Shibata [1] for the infinite layer, Kubo and Shibata [30] for the perturbed half-space and so on.

On the other hand, the results on the weighted L^p space case are not so much than one of the L^p space case. For the whole space and an exterior domain case, Farwig and Sohr [13] proved the Helmholtz decomposition of the L^p space with Muckenhoupt weight. Moreover they considered the resolvent Stokes equation in the weighted L^p space and showed the weighted resolvent estimate and that the Stokes operator generates an analytic semigroup in L^p space with Muckenhoupt weight. The result on the weighted $L^p - L^q$ estimate of Stokes semigroup was not obtained. For the half-space case, H. O. Bae [3] proved the Helmholtz decomposition of L^p space with some weights (for example, $w(x) = (1 + |x|)^s$ for $0 \le s < 1/p'$) and he obtained the certain $L^p - L^q$ estimate of Stokes semigroup with the certain weight. A. Frölich [16] proved the one of L^p space with the Muckenhoupt weight and the weighted resolvent estimate of the resolvent Stokes equation in half-spaces and aperture domains (see [16], [17] for detail). However, he did not obtain the results on the weighted $L^p - L^q$ estimate of Stokes semigroup.

This paper consists of six sections. In the next section, after notation is fixed we present the statement of our main results: Theorem 2.3 on the resolvent estimate with Muckenhoupt weight, Theorem 2.4 on the Helmholtz decomposition of the weighted L^p space, Theorem 2.5 on the generation of the Stokes semigroup as one of the corollary of Theorem 2.3, Theorem 2.7 on $L^p - L^q$ estimates of Stokes semigroup with $\langle x\rangle^s$ $(0 \le s < n-2+1/n)$ which plays important role when we prove the asymptotic behavior for the solution to (NS) and Theorem 2.8 on the asymptotic behavior for the solution to (NS) obtained by Kozono [28] or Kubo and Shibata [30].

In section 3, we introduce the known results concerning the weighted L^p space which we use through this paper. In secton 4, we shall show the Helmholtz decomposition of $L^p_w(\Omega)$ in perturbed half-space. Moreover we shall consider the resolvent Stokes equations corresponding to (NS) and shall show the resolvent estimate. Our proof is based on the method due to Farwig and Sohr [13]. Since the results on the bounded domains and half-space are proved by Fröhlich [18] and [20], by cut-off technique with their results, we can prove the resolvent estimate for large λ, which implies that the Stokes operator $-A$ generates analytic semigroup in $L^q_{w,\sigma}(\Omega)$.

In secton 5, we shall prove the weighted $L^p - L^q$ estimate of Stokes semigroup obtained in section 4. First we consider the whole space case and the half-space case. For the whole space case, we can easily prove by Young's inequality. For the half-space case, using Ukai's solution formula (see [35]), we can reduce to the whole space case. For a perturbed half-space case, we derive the weighted $L^p - L^q$ estimates from the results for the half-space case and the estimate for $\Omega \cap B_R$ which is proved by Kubo and Shibata [30]. Finally, we consider the application of the weighted $L^p - L^q$ estimates to the Navier-Stokes equations in section 6. As we mentioned, the Navier-Stokes equations in the half-space and a perturbed half-space admits a unique strong solution u when the initial data is sufficient small. As the application of the weighted $L^p - L^q$ estimates, we consider the case where the initial data belongs to $L^n_w(\Omega) \cap L^n(\Omega)$, where $w(x) = \langle x\rangle^{sn}$ for $0 \le s < n-2+1/n$.

2 Main Theorems and Notations

In this paper, we shall consider the Navier-Stokes equations in the half-space and a perturbed half-space. For this purpose, we first introduce the definition of their domains. Let H denote the half-space by $H = \{x = (x', x_n) \in \mathbf{R}^n \mid x_n > 0\}$. We call a domain Ω perturbed half-space if there exists a positive number R such that

$$\Omega \backslash B_R = H \backslash B_R, \tag{2.1}$$

where $B_R = \{x \in \mathbf{R}^n \mid |x| < R\}$. We next introduce the class of weight functions and weighted L^p spaces.

Definition 2.1 (Muckenhoupt class $\mathscr{A}_q(\mathbf{R}^n)$). Let $1 < q < \infty$. A weighted function $0 \le w \in L^1_{\text{loc}}(\mathbf{R}^n)$ belongs to Muckenhoupt class $\mathscr{A}_q$ if the function w satisfies

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(q-1)}\,dx\right)^{q-1} \le C < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbf{R}^n$ and $|Q|$ denotes the Lebesgue measure of Q.

For example, the weighted function $w(x) = (1+|x|)^\alpha$ or $w(x) = |x|^\alpha$ $(-n < \alpha < n(q-1))$ belong to Muckenhoupt class $\mathscr{A}_q(\mathbf{R}^n)$. For a perturbed half space Ω we introduce a restricted class of $\mathscr{A}_q$ on Ω.

Definition 2.2. Let Ω be a perturbed half space with C^2-boundary $\partial\Omega$ with B_R satisfies (2.1). Then for $1 \le q < \infty$, $\mathscr{A}_q = \mathscr{A}_q(\Omega)$ is the class of weighted function defined as follows: The each element w of $\mathscr{A}_q$ belongs to $\mathscr{A}_q(\mathbf{R}^n)$ and has the bounded domain $G = G(w) \subset \Omega \cap B_{R+1}$ such that $w \in C^0(\overline{G})$ and $w|_{\overline{G}} > 0$.

We define a weighted L^q space with Muckenhoupt weight $w \in \mathscr{A}_q$ as follows:

$$L^q_w(\Omega) = \left\{u \in L^1_{\text{loc}}(\overline{\Omega}) \mid \|u\|_{L^q_w(\Omega)} = \|uw^{1/q}\|_{L^q(\Omega)} = \left(\int_\Omega |u|^q w\,dx\right)^{1/q} < \infty\right\}$$

for $1 < q < \infty$. Similarly, we define the weighted spaces as follows:

$$W^{k,q}_w(\Omega) = \{u \in L^q_w(\Omega) \mid \nabla^\alpha u \in L^q_w(\Omega), |\alpha| \le k\},$$

$$\hat{W}^{k,q}_w(\Omega) = \{u \in L^1_{\text{loc}}(\overline{\Omega}) \mid \nabla^\alpha u \in L^q_w(\Omega), |\alpha| = k\}$$

and

$$W^{k,q}_{0,w}(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{W^{k,q}_w(\Omega)}}$$

for $1 < q < \infty$, $k \in \mathbf{N}$ and $w \in \mathscr{A}_q$. The space $W^{k,q}_w(\Omega)$ equipped with the norm

$$\|u\|_{W^{k,q}_w(\Omega)} := \left(\sum_{|\alpha| \le k} \|\nabla^\alpha u\|^q_{L^q_w(\Omega)}\right)^{1/q}$$

is a reflexive Banach space.

For $1 < q < \infty$, let q' denote the dual exponent: $1/q + 1/q' = 1$ and w' denote the dual weight: $w' = w^{-1/(q-1)}$. The dual space of $\hat{W}^{1,q'}_{w'}(\Omega)$ is denote by $\hat{W}^{-1,q}_{w}(\Omega) = \hat{W}^{1,q'}_{w'}(\Omega)^*$ and endowed with the norm

$$\|F\|_{\hat{W}^{-1,q}_{w}(\Omega)} = \sup_{0 \neq \varphi \in \hat{W}^{1,q'}_{w'}(\Omega)} \frac{|\langle F, \varphi \rangle|}{\|\nabla \varphi\|_{L^{q'}_{w'}(\Omega)}}$$

for $F \in \hat{W}^{-1,q}_{w}(\Omega)$.

For the half-space and the perturbed half-space, we shall investigate the Navier-Stokes equations (NS). As first step of analysis of Navier-Stokes equations (NS) in Kato's argument [27], we need the weighted $L^p - L^q$ estimates of Stokes semigroup. To this end, we consider the generalized resolvent Stokes equations corresponding to (NS):

$$(\lambda - \Delta)u + \nabla \pi = f, \quad \operatorname{div} u = g \text{ in } \Omega \quad u = 0 \text{ on } \partial\Omega, \tag{GS}$$

where $f \in L^q_w(\Omega)$, $g \in W^{1,q}_w(\Omega) \cap \hat{W}^{-1,q}_w(\Omega)$ and $\lambda \in \Sigma_\varepsilon = \{\lambda \in \mathbf{C}\backslash\{0\} \mid |\arg \lambda| < \pi - \varepsilon\}$ $(0 < \varepsilon < \pi/2)$. Then the following resolvent estimate holds.

THEOREM 2.3. *Let $\Omega \subset \mathbf{R}^n$ be a perturbed half-space with C^2-boundary and let $1 < q < \infty$, $w \in \mathscr{A}_q$, $0 < \varepsilon < \pi/2$ and $\delta > 0$. For every $f \in L^q_w(\Omega)$, $g \in W^{1,q}_w(\Omega) \cap \hat{W}^{-1,q}_w(\Omega)$ and $\lambda \in \Sigma_\varepsilon$, $|\lambda| \geq \delta$, the problem* (GS) *has a unique solution $(u, \pi) \in W^{2,q}_w(\Omega) \times \hat{W}^{1,q}_w(\Omega)$. Furthermore (u, π) satisfies the a priori estimate*

$$\|(|\lambda| u, \nabla^2 u, \nabla \pi)\|_{L^q_w(\Omega)} \leq C(\varepsilon, \delta)(\|(f, \nabla g)\|_{L^q_w(\Omega)} + \|\lambda g\|_{\hat{W}^{-1,q}_w(\Omega)}). \tag{2.2}$$

In order to define the Stokes operator, we need the Helmholtz decomposition of the weighted space $L^q_w(\Omega)$ for perturbed half-spaces.

THEOREM 2.4. *Let $\Omega \subset \mathbf{R}^n$ be a perturbed half-space with C^2-boundary and let $1 < q < \infty$ and $w \in \mathscr{A}_q$.*

(i) *$L^q_w(\Omega)$ has a unique algebraic and topological decomposition*

$$L^q_w(\Omega) = L^q_{w,\sigma}(\Omega) \oplus \nabla \hat{W}^{1,q}_w(\Omega),$$

where $L^q_{w,\sigma}(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega) \mid \nabla \cdot u = 0\}$ with respect to the norm $\|\cdot\|_{L^q_w(\Omega)}$. In particular there exists a unique bounded projection operator

$$P_{q,w} : L^q_w(\Omega) \to L^q_{w,\sigma}(\Omega)$$

with null space $\nabla \hat{W}^{1,q}_w(\Omega) = \{\nabla \pi \mid \pi \in \hat{W}^{1,q}_w(\Omega)\}$ and range $L^q_{w,\sigma}(\Omega)$.

(ii) $(P_{q,w})^* = P_{q',w'}$ *and* $(L^q_{w,\sigma}(\Omega))^* = L^{q'}_{w',\sigma'}$.
(iii) *If* $u \in L^{q_1}_{w_1}(\Omega) \cap L^{q_2}_{w_2}(\Omega)$ *for* $q_j \in (1,\infty)$ *and* $w_j \in \mathscr{A}_{q_j}$ $(j = 1,2)$, *then* $P_{q_1,w_1}u = P_{q_2,w_2}u$.

Given the Helmoltz projection $P_{q,w}$, the Stokes operator $A_{q,w}$ in $L^q_{w,\sigma}(\Omega)$ for a perturbed half-space is defined by $A_{q,w} = -P_{q,w}\Delta$ with domain

$$\mathscr{D}(A_{q,w}) = \{u \in W^{2,q}_w(\Omega) \cap L^q_{w,\sigma}(\Omega) \mid u = 0 \text{ on } \partial\Omega\}.$$

For the Stokes operator, we obtain the following results which say that the Stokes operator generates an analytic semigroup in $L^q_{w,\sigma}(\Omega)$:

Theorem 2.5. *Let* $\Omega \subset \mathbf{R}^n$ *be a perturbed half-space with* C^2*-boundary and let* $1 < q < \infty$, $w \in \mathscr{A}_q$ *and* $0 < \varepsilon < \pi/2$. *Then for every* $\lambda \in \Sigma_\varepsilon$ *and every* $f \in L^q_{w,\sigma}(\Omega)$ *the resolvent problem*

$$\lambda u + A_{q,w}u = f, \quad u \in \mathscr{D}(A_{q,w})$$

has a unique solution $u \in \mathscr{D}(A_{q,w})$.

(i) *For* $\lambda \in \Sigma_\varepsilon$ *and* $|\lambda| \geq \delta > 0$, *this solution satisfies the resolvent estimate*

$$\|(\lambda u, A_{q,w}u)\|_{L^q_w(\Omega)} \leq C(\varepsilon,\delta)\|f\|_{L^q_w(\Omega)}.$$

(ii) *The Stokes operator generates an analytic semigroup* $\{e^{-tA_{q,w}}\}_{t\geq 0}$.
(iii) *Moreover* $A_{q,w}$ *is a closed operator and* $(A_{q,w})^* = A_{q',w'}$.

In this paper, for simplicity, we use the abbreviations A for $A_{q,w}$ and P for $P_{q,w}$ if there is no confusion.

Remark 2.6. The same results as Theorem 2.3–2.5 for half-space and exterior domains have been proved by Fröhlich [18] and Farwig and Sohr [13].

We next consider the weighted $L^p - L^q$ estimates for the Stokes semigroup e^{-tA}. As well-known, The $L^p - L^q$ estimates play an important role when we prove the unique existence of a global solution to (NS). Here setting the weighted function w as $w(x) = \langle x\rangle^{sp} = (1 + |x|^2)^{sp/2}$ $(1 < p < \infty)$, we obtain the following weighted $L^p - L^q$ estimates for the Stokes semigroup in the half-space and a perturbed half-space:

Theorem 2.7 (Weighted $L^p - L^q$ estimates). *Let* $n \geq 2$ *and let* Ω *be the half-space or a perturbed half-space with* C^2*-boundary. Let* $1 < p \leq q < \infty$,

$0 \le s < (n-1)(1-1/p)$ *and* $w(x) = \langle x \rangle^{sp}$. *Then for* $a \in L^p_w(\Omega) \cap L^p(\Omega)$, *we have*

$$\|\langle x \rangle^s e^{-tA} Pa\|_{L^q} \le Ct^{-n(1/p-1/q)/2} \|\langle x \rangle^s a\|_{L^p} + Ct^{-n(1/p-1/q)/2+s/2} \|a\|_{L^p},$$

$$\|\langle x \rangle^s \nabla e^{-tA} Pa\|_{L^q} \le Ct^{-n(1/p-1/q)/2-1/2} \|\langle x \rangle^s a\|_{L^p} + Ct^{-n(1/p-1/q)/2+(s-1)/2} \|a\|_{L^p}$$

for $t > 0$.

Finally we shall apply the weighted $L^p - L^q$ estimates to Navier-Stokes equations (NS). Following Kato's argument [27], we can prove the unique existence of global solution to (NS) with small initial data. By applying the Helmholtz projection P to (NS), we can rewrite (NS) as follows:

$$\partial_t u + Au + P[(u \cdot \nabla)u] = 0, \quad u(0) = a. \tag{P-NS}$$

By Duhamel's principle, we obtain the integral equation:

$$u(t) = e^{-tA} a - \int_0^t e^{-(t-\tau)A} P[(u \cdot \nabla)u](\tau)\, d\tau.$$

By the usual $L^p - L^q$ estimate and contraction mapping principle, we can prove that there exists the unique strong soluton u to (NS) with small initial data and the solution u satisfies the following asymptotic behavior as $t \to \infty$:

$$\|u(t)\|_{L^q(\Omega)} \le Ct^{-1/2+n/(2q)} \quad \text{for } n \le q \le \infty,$$

$$\|\nabla u(t)\|_{L^q(\Omega)} \le Ct^{-1+n/(2q)} \quad \text{for } n \le q < \infty$$

(see Kozono [28] and Kubo and Shibata [30]). Here for given $a \in L^n_\sigma(\Omega)$ and $0 < T \le \infty$ a measurable function u defined on $\Omega \times (0, T)$ is called a strong solution to (NS) on $(0, T)$ if u belongs to

$$u \in C([0, T); L^n_\sigma(\Omega)) \cap C((0, T); D(A)) \cap C^1((0, T); L^n_\sigma(\Omega))$$

together with $\lim_{t \to 0} \|u(t) - a\|_{L^n} = 0$ and satisfies (P-NS) for $0 < t < T$ in $L^n_\sigma(\Omega)$.

When the initial data belongs to $L^n_w(\Omega)$ $(w(x) = \langle x \rangle^{sn})$ additionally, we can show the following theorem on the weighted asymptotic behavior as $t \to \infty$ by the weighted $L^p - L^q$ estimates:

THEOREM 2.8. *Let* $n \ge 2$, Ω *be the half-space and a perturbed half-space. Let* $0 \le s < n - 2 + 1/n$ *and* $w(x) = \langle x \rangle^{sn}$. *If* $a \in L^n_w(\Omega) \cap L^n_\sigma(\Omega)$ *with small* $\|a\|_{L^n}$, *the solution* $u(t)$ *satisfies the following asymptotic behavior*:

$$\|\langle x\rangle^s u(t)\|_{L^q} \le Ct^{-1/2+n/(2q)+s/2} \quad \textit{for } n \le q < \infty, \tag{2.3}$$

$$\|\langle x\rangle^s \nabla u(t)\|_{L^q} \le Ct^{-1+n/(2q)+s/2} \quad \textit{for } n \le q < \infty \tag{2.4}$$

as $t \to \infty$.

3 Preliminaries

In this section, we shall introduce some facts and lemmas which we use in this paper. First we shall introduce the lemma concerning the Muckenhoupt weight function. The weights $w \in \mathscr{A}_q$ have the important property that regular singular integral operators are constinuous on $L^q_w(\mathbf{R}^n)$ into itself.

Lemma 3.1. *Let* $1 < q < \infty$, $w \in \mathscr{A}_q$ *and let* T *be a regular singular integral operator. Then* T *is bounded on* $L^q_w(\mathbf{R}^n)$. *More precisely, there is a positive constant* C *such that for all* $f \in L^q_w(\mathbf{R}^n)$, *we have*

$$\|Tf\|_{L^q_w} \le C\|f\|_{L^q_w}.$$

Proof. See [21, Chapter IV, Theorem 3.1]. □

By Lemma 3.1, the Riesz transforms $R_j f$ and the partial Riesz transform $S_j f$ define by

$$R_j f(x) := \mathscr{F}_\xi^{-1}\left[\frac{i\xi_j}{|\xi|}\mathscr{F}_x[f](\xi)\right] \quad j = 1, \dots, n, \tag{3.1}$$

$$S_j f(x) := \mathscr{F}_{\xi'}^{-1}\left[\frac{i\xi_j}{|\xi'|}\mathscr{F}_{x'}[f](\xi', x_n)\right] \quad j = 1, \dots, n-1 \tag{3.2}$$

are continuous on $L^q_w(\mathbf{R}^n)$ and $L^q_w(H)$ into itself respectively. Here $\mathscr{F}_x$ and $\mathscr{F}_{x'}$ denotes the Fourier transform with respect to x and the partial Fourier transform with respect to $x' = (x_1, \dots, x_{n-1})$ respectively. These Riesz transforms are used in Ukai's solution formula. Here the weight $w(x) = \langle x\rangle^s$ considerd for fixed x_n as weight in $\mathbf{R}^{n-1}$ is in the class $\mathscr{A}_q$ only $-(n-1)/q < s < (n-1)(1-1/q)$.

In this paper, we consider a perturbed half-space by using the cut-off technique. For this purpose, we introduce the cut-off function. We fix R_0 satisfying (2.1). Given $R \ge R_0$, let $\psi \in C^1(\mathbf{R})$ be nondecreasing with $\psi(\xi) = 1$ if $|\xi| \ge R$ and $\psi(\xi) = 0$ if $|\xi| \le R-1$ and set $\psi_R = \psi(|x|)$.

By this cut-off function, we can show the following lemma which means the interpolation between $W^{2,q}_w(\Omega)$ and $L^q_w(\Omega)$.

LEMMA 3.2. *Let Ω be a perturbed half-space with C^2-boundary and let $1 < q < \infty$ and $w \in \mathscr{A}_q$. Then there is a constant $c = c(q, w, \Omega) > 0$ such that for all $u \in W^{2,q}_{w,0}(\Omega)$ and all $\varepsilon \in (0,1)$,*

$$\|\nabla u\|_{L^q_w(\Omega)} \le c\left(\varepsilon\|\nabla^2 u\|_{L^q_w(\Omega)} + \frac{1}{\varepsilon}\|u\|_{L^q_w(\Omega)}\right). \tag{3.3}$$

PROOF. Since the half-space case is proved by Fröhlich [18, Corollary 4.5], it is sufficient to consider a perturbed half-space case. Let ψ_R be a cut-off function defined above. Recall that the following estimate holds in a bounded domain G with Lipschitz boundary:

$$\|\nabla u\|_{L^q_w(G)} \le C\left(\varepsilon\|\nabla^2 u\|_{L^q(G)} + \frac{1}{\varepsilon}\|u\|_{L^q(G)}\right) \tag{3.4}$$

for all $u \in W^{2,q}(G)$ and $0 < \varepsilon < 1$ (see Fröhlich [18]). Applying $u(1-\psi_R)$ to (3.4) and $u\psi_R$ to (3.3) for half-space H, we have

$$\begin{aligned}
\|\nabla u\|_{L^q_w(\Omega)} &\le \|\nabla(u(1-\psi_R))\|_{L^q_w(\Omega_R)} + \|\nabla(u\psi_R)\|_{L^q_w(\Omega)} \\
&\le C\left(\varepsilon\|\nabla^2(u(1-\psi_R))\|_{L^q(\Omega_R)} + \frac{1}{\varepsilon}\|u(1-\psi_R)\|_{L^q(\Omega_R)}\right) \\
&\quad + C\left(\varepsilon\|\nabla^2(u\psi_R)\|_{L^q_w(H)} + \frac{1}{\varepsilon}\|u\psi_R\|_{L^q_w(H)}\right) \\
&\le C\varepsilon\|\nabla^2 u\|_{L^q_w(\Omega)} + C\left(\varepsilon + \frac{1}{\varepsilon}\right)\|u\|_{L^q_w(\Omega)} + C\varepsilon\|\nabla u\|_{L^q(\Omega_R)}.
\end{aligned}$$

Applying the third term $\|\nabla u\|_{L^q_w(\Omega_R)}$ to (3.4), we obtain (3.3). □

The following four lemmas proved by Fröhlich [18], [16] and [20]. First lemma says that the weighted resolvent estimate holds in bounded domains.

LEMMA 3.3. *Let $G \subset \mathbf{R}^n$ be a bounded domain with boundary of class $C^{1,1}$ and let $1 < q < \infty$, $0 < \varepsilon < \pi/2$ and $w \in \mathscr{A}_q$. Then for every $f \in L^q_w(G)$, $g \in W^{1,q}_w(G) \cap \hat{W}^{-1,q}_w(G)$ and $\lambda \in \Sigma_\varepsilon \cup \{0\}$ the resolvent Stokes equation* (GS) *with boundary condition*: $u = 0$ *on* ∂G *has a unique solution* $(u,\pi) \in W^{2,q}_w(G) \times \hat{W}^{1,q}_w(G)$. *Further*

$$\|(\lambda u, \nabla^2 u, \nabla\pi)\|_{L^q_w(G)} \le C_\varepsilon(\|(f, \nabla g)\|_{L^q_w(G)} + \|\lambda g\|_{\hat{W}^{-1,q}_w(G)})$$

with a constant $C_\varepsilon > 0$ *independent of* f, g, λ *and* u, π. *Here* $\|\nabla^2 u\|_{L^q_w(G)}$ *may be replaced by* $\|u\|_{W^{2,q}_w(G)}$.

Second lemma means that the weighted resolvent estimate in half-space H.

Theorem 3.4. *Let* $n \geq 2$, $1 < q < \infty$, $w \in \mathscr{A}_q$ *and* $0 < \varepsilon < \pi/2$. *Then for every* $f \in L^q_w(H)$, $g \in W^{1,q}_w(H) \cap \hat{W}^{-1,q}_w(H)$ *and* $\lambda \in \Sigma_\varepsilon$, *there exists a unique solution* (u, π) *to the resolvent problem* (GS). *This solution satisfies the estimate*

$$\|(|\lambda| u, \nabla^2 u, \nabla\pi)\|_{L^q_w(H)} \leq C(\|(f, \nabla g)\|_{L^q_w(H)} + \|\lambda g\|_{\hat{W}^{-1,q}_w(H)}), \tag{3.5}$$

where $C > 0$ *dependents only on* n, q, ε. *Moreover if for some* $r \in (1, \infty)$ *and some* $v \in \mathscr{A}_r$ *additionally* $f \in L^r_v(H)$ *and* $g \in W^{1,r}_v(H) \cap \hat{W}^{-1,r}_v(H)$, *then* $(u, \pi) \in W^{2,r}_v(H) \times \hat{W}^{1,r}_v(H)$.

Next two lemmas are used when we consider the Helmholtz decomposition of the weighted L^p space.

Lemma 3.5. *Let* $1 < q < \infty$ *and* $w \in \mathscr{A}_q$. *Then there is a constant* $c \in \mathbf{R}$ *such that*

$$\|\nabla\pi\|_{q,w} \leq C \sup_{0 \neq \phi \in \hat{W}^{1,q'}_{w'}(H)} \frac{|\langle \nabla\pi, \nabla\phi \rangle|}{\|\nabla\phi\|_{q',w'}}$$

for $\pi \in \hat{W}^{1,q}_w(H)$.

Lemma 3.6. *Let* $1 < q < \infty$ *and* $w \in \mathscr{A}_q$. *Then* $-\Delta_{q,w}$ *is an isomorphism, i.e. for any* $F \in \hat{W}^{-1,q}_w(H)$, *there exists* $\pi \in \hat{W}^{1,q}_w(H)$ *such that*

$$(\nabla\pi, \nabla\phi) = \langle F, \phi \rangle \quad \text{for } \phi \in \hat{W}^{1,q'}_{w'}(H)$$

and the weak solution π *satisfies*

$$\|\nabla\pi\|_{L^q_w(H)} \leq C\|F\|_{\hat{W}^{-1,q}_w(H)}.$$

4 Helmholtz Decomposition and Resolvent Estimate

The goal of this section is to prove Helmholtz decomposition of the weight L^p-space (Theorem 2.4) and the resolvent estimate (Theorem 2.3) in a perturbed half-space. Since their facts can be proved by the method due to Farwig and Sohr [13], we may omit their complete proof. Here we shall describe the outline of the proof.

We shall first show the Helmholtz decompositon (Theorem 2.4). In order to prove Theorem 2.4, we need the uniqness theorem of the corresponding weak Neumann problem which implies the Helmholtz decomposition (Theorem 2.4) (see [13] for detail).

THEOREM 4.1. *Let* Ω *be a perturbed half-space and let* $1 < q < \infty$ *and* $w \in \mathscr{A}_q$. *Then for every* $F \in \hat{W}_w^{-1,q}(\Omega)$ *the weak Neumann problem*

$$\int_\Omega \nabla\pi \cdot \nabla\psi \, dx = \langle F, \psi \rangle, \quad \psi \in \hat{W}_{w'}^{1,q'}(\Omega) \tag{4.1}$$

has a unique solution $\pi \in \hat{W}_w^{1,q}(\Omega)$. *Furthermore*

$$\|\nabla\pi\|_{L_w^q(\Omega)} \le C\|F\|_{\hat{W}_w^{-1,q}(\Omega)} \tag{4.2}$$

with a constant $C = C(\Omega, q, w) > 0$. *Moreover if* $F \in \hat{W}_{w_1}^{-1,q_1}(\Omega) \cap \hat{W}_{w_2}^{-1,q_2}(\Omega)$ *for weights* $w_j \in \mathscr{A}_{q_j}$, $q_j \in (1, \infty)$, $j = 1, 2$, *then the weak solution* u *of* (4.1) *satisfies* $\pi \in \hat{W}_{w_1}^{1,q_1}(\Omega) \cap \hat{W}_{w_2}^{1,q_2}(\Omega)$.

PROOF. This theorem can be proved by the method due to [13]. Here we shall remark the difference between the exterior domains case considered in [13] and the perturbed half-spaces. Compared with the exterior domains case, proof of the following preliminary estimate is different:

$$\|\nabla\pi\|_{L_w^q(\Omega)} \le C(\|F\|_{\hat{W}_w^{-1,q}(\Omega)} + \|\pi\|_{L_w^q(\Omega_R)}) \tag{4.3}$$

given $F \in \hat{W}_w^{-1,q}(\Omega)$ and $\pi \in \hat{W}_w^{1,q}(\Omega)$ satisfying (4.1). If we obtain (4.3), we can prove Theorem 4.1 in a same way as [13]. Therefore here we shall prove the (4.3). A well-known variational inequality on $W_w^{1,q}(\Omega_R)$ yields

$$\|\nabla(\pi(1 - \psi_R))\|_{L_w^q(\Omega_R)} \le C(\|F\|_{\hat{W}_w^{-1,q}(\Omega)} + \|\pi\|_{L_w^q(\Omega_R)}),$$

where ψ_R is the radially symmetric cut-off defined in section 3 (see Fröhlich [16] for example). Therefore it is sufficient to prove

$$\|\nabla(\pi\psi_R)\|_{L_w^q(H)} \le C(\|F\|_{\hat{W}_w^{-1,q}(\Omega)} + \|\pi\|_{L_w^q(\Omega_R)}). \tag{4.4}$$

To prove (4.4), we consider a test function $\phi \in C_0^\infty(\Omega)$ and define $\tilde{\phi} = \phi - |\Omega_R|^{-1} \int_{\Omega_R} \phi \, dx$. Then we see $\pi\psi_R \in \hat{W}_w^{1,q}(H)$ and

$$\begin{aligned}\int_H \nabla(\pi\psi_R) \cdot \nabla\phi \, dx &= \int_\Omega \nabla\pi \cdot \nabla(\psi_R\tilde{\phi}) \, dx - \int_{\Omega_R} \nabla\pi \cdot \tilde{\phi}\nabla\psi_R \, dx + \int_{\Omega_R} \pi\nabla\psi_R \cdot \nabla\phi \, dx \\ &= \int_\Omega \nabla\pi \cdot \nabla(\psi_R\tilde{\phi}) \, dx + \int_{\Omega_R} \pi \cdot \operatorname{div}(\tilde{\phi}\nabla\psi_R) \, dx + \int_{\Omega_R} \pi\nabla\psi_R \cdot \nabla\phi \, dx,\end{aligned}$$

where we used the fact: supp $\nabla\psi_R \subset \Omega_R$ and $e_n \cdot \nabla\psi_R = \partial_n\psi_R(x) = \psi'(|x|)x_n/|x|$ vanishes at $x_n = 0$. By using Lemma 3.6, we obtain (4.4). □

Theorem 2.4 can be proved by the method due to [13] with Theorem 4.1. The next lemma tells us a regularity property of the Helmholtz decomposition. By using their method, we can obtain the following lemma.

LEMMA 4.2. *Let $1 < q < \infty$, $w \in \mathscr{A}_q$ and $f \in L^q_w(\Omega)$ satisfying $\nabla(\nabla\cdot f) \in L^q_w(\Omega)$ and $N \cdot f = 0$ on $\partial\Omega$, where N denotes the outer normal vector on $\partial\Omega$. Further let $f = f_0 + \nabla\pi$ with $f_0 \in L^q_{w,\sigma}(\Omega)$, $\pi \in \hat{W}^{1,q}_w(\Omega)$ be the Helmholtz decomposition of f. Then $\nabla^2\pi \in L^q_w(\Omega)$ and $\nabla\cdot f \in L^q_w(\Omega)$.*

We next consider the weighted resolvent estimate (Theorem 2.3). For this purpose, we consider the generalized resolvent problem

$$(\lambda - \Delta)u + \nabla\pi = f, \quad \operatorname{div} u = g, \quad u = 0 \text{ on } \partial\Omega, \tag{GS}$$

where $f \in L^q_w(\Omega)$, $g \in W^{1,q}_w(\Omega) \cap \hat{W}^{-1,q}_w(\Omega)$ and $\lambda \in \Sigma_\varepsilon = \{\lambda \in \mathbf{C}\backslash\{0\} \mid |\arg\lambda| < \pi - \varepsilon\}$ $(0 < \varepsilon < \pi/2)$. Since we can prove Theorem 2.3 in the same way as [13], we shall show the outline of its proof. First step of its proof is to show the following lemma which tells us a priori estimates:

LEMMA 4.3. *For a given solution $(u,\pi) \in W^{2,q}_w(\Omega) \times \hat{W}^{1,q}_w(\Omega)$ to* (GS) *it holds the a priori estimates*

$$\begin{aligned}\|(\lambda u, \nabla^2 u, \nabla\pi)\|_{L^q_w(\Omega)} \le C(&\|(f,\nabla g)\|_{L^q_w(\Omega)} + \|\lambda g\|_{\hat{W}^{-1,q}_w(\Omega)} \\ &+ \|(u,\nabla u,\pi)\|_{L^q(\Omega_R)} + \|\lambda u\|_{W^{1,q'}(\Omega_R)^*})\end{aligned} \tag{4.5}$$

with a constant $C = C(\Omega, R, w, q, \varepsilon) > 0$ independent of $\lambda \in \Sigma_\varepsilon$. Here $W^{1,q'}(\Omega_R)^*$ is the dual space of $W^{1,q'}(\Omega_R)$.

Next step is to show that the operator $S_{q,w}(\lambda)$ defined as follows is injective: $S_{q,w}(\lambda)$ is the operator from $W^{2,q}_w(\Omega) \times \hat{W}^{1,q}_w(\Omega)$ to $L^q_w(\Omega) \times \hat{W}^{-1,q}_w(\Omega)$ by $S_{q,w}(\lambda)(u,\pi) = ((\lambda-\Delta)u + \nabla\pi, \nabla\cdot u)$ with domain $\mathscr{D}(S_{q,w}(\lambda)) = (W^{2,q}_w(\Omega) \cap W^{1,q}_{0,w}(\Omega)) \times \hat{W}^{1,q}_w(\Omega)$. The following lemma implies the uniquness of the solution to the resolvent problem.

LEMMA 4.4. *$S_{q,w}(\lambda)$ is injective and its range $\mathscr{R}(S_{q,w}(\lambda))$ is dense in $L^q_w(\Omega) \times \hat{W}^{-1,q}_w(\Omega)$ for all $\lambda \in \Sigma_\varepsilon$.*

Lemma 4.3 and Lemma 4.4 can be proved by the method due to Farwig and Sohr [13]. By two lemmas above and Lemma 3.2, we can prove Theorem 2.3 (see [13]).

5 Weighted $L^p - L^q$ Estimates of Stokes Semigroup

In this section, we shall prove weighted $L^p - L^q$ estimates of Stokes semigroup in half-sapces and perturbed half-spaces. To this end, we begin to prove the following lemma on weighted $L^p - L^q$ estimates in the whole space $\mathbf{R}^n$ and the half-space H.

Lemma 5.1. *Let $n \geq 2$, $1 < p, r \leq q < \infty$ and $w = \langle x \rangle^{sp}$. Let Ω be the whole space $\mathbf{R}^n$ or the half-space H. Let s be a positive number such that $0 \leq s < n(1-1/p)$ if $\Omega = \mathbf{R}^n$ or $0 \leq s < (n-1)(1-1/p)$ if $\Omega = H$. Then for $a \in L^p_w(\Omega) \cap L^r(\Omega)$, we have*

$$\|\langle x \rangle^s e^{-tA} Pa\|_{L^q(\Omega)} \leq Ct^{-n(1/p-1/q)/2} \|\langle x \rangle^s a\|_{L^p(\Omega)} + Ct^{-n(1/r-1/q)/2+s/2} \|a\|_{L^r(\Omega)},$$

$$\begin{aligned}\|\langle x \rangle^s \nabla e^{-tA} Pa\|_{L^q(\Omega)} &\leq Ct^{-n(1/p-1/q)/2-1/2} \|\langle x \rangle^s a\|_{L^p(\Omega)} \\ &\quad + Ct^{-n(1/r-1/q)/2+(s-1)/2} \|a\|_{L^r(\Omega)}\end{aligned}$$

for $t > 0$.

Proof. We shall first consider the whole space $\mathbf{R}^n$ case. In this case, it is well-known that the Stokes semigroup e^{-tA} is represented by

$$e^{-tA} f = (E_0 * f)(x) := \int_{\mathbf{R}^n} E_0(t, x-y) f(y)\, dy$$

for $f \in L^p_{w,\sigma}(\mathbf{R}^n)$, where $E_0(t)$ is heat kernel: $E_0(t) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$. Since $\langle x \rangle^s \leq \langle x-y \rangle^s + \langle y \rangle^s$, we have

$$\begin{aligned}|\langle x \rangle^s \nabla^{|\alpha|} e^{-tA} Pa| &\leq \frac{C}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} \langle x-y \rangle^s \left(\frac{x-y}{2t}\right)^{\alpha} e^{-|x-y|^2/4t} Pa(y)\, dy \\ &\quad + \frac{C}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} \left(\frac{x-y}{2t}\right)^{\alpha} e^{-|x-y|^2/4t} \langle y \rangle^s Pa(y)\, dy \\ &=: G_1 * Pa + G_2 * (\langle x \rangle^s Pa).\end{aligned}$$

We first consider the first term G_1. Since G_1 is estimated by

$$|G_1(t,x)| \leq Ct^{-n/2+(s-|\alpha|)/2} e^{-|x|^2/(8t)},$$

we obtain

$$\|G_1\|_{L^{\tilde{r}}} \le Ct^{-n(1-1/\tilde{r})/2+(s-|\alpha|)/2}$$

for $1 < \tilde{r} < \infty$. Thus by Young's inequality with $1 + 1/q = 1/r + 1/\tilde{r}$, we have

$$\|G_1 * Pa\|_{L^q} \le \|G_1\|_{L^{\tilde{r}}} \|Pa\|_{L^r} \le Ct^{-n(1/r-1/q)/2+(s-|\alpha|)/2} \|Pa\|_{L^r}.$$

We next consider the second term G_2. By Young's inequality, we have

$$\|G_2 * (\langle x\rangle^s Pa)\|_{L^q} \le \|G_2\|_{L^r} \|\langle x\rangle^s Pa\|_{L^p} \le Ct^{-n(1/p-1/q)/2-|\alpha|/2} \|\langle x\rangle^s Pa\|_{L^p}.$$

Since the weight function $\langle x\rangle^{sp}$ and the Helmholtz projection P can be commutable when the exponent s satisfies $-n/p < s < n(1-1/p)$, we obtain the desired result for the whole space.

Next we consider the half-space case. In half-space, we have the solution formula obtained by Ukai [35]. Let R_j and S_j be the Riesz transform and the partial Riesz transform defined by (3.1) and (3.2). And let $\gamma f = f|_H$, $\gamma f = f|_H$ and e zero extension operator from H to $\mathbf{R}^n$. Finally, let $E(t)$ be the solution operator for the heat equation in H, which is derived from $E_0(t)$ by odd extension from H to $\mathbf{R}^n$. Then the solution $(u(t), \pi(t))$ of the non-stationary Stokes equations in H is

$$u(t) = WE(t)Va, \quad \pi(t) = -D\gamma\partial_n E(t)V_1 a,$$

where

$$W = \begin{pmatrix} I & -SU \\ 0 & U \end{pmatrix}, \quad V = \begin{pmatrix} V_2 \\ V_1 \end{pmatrix}$$

with

$$S = {}^t(S_1, \dots, S_{n-1}), \quad U = rR' \cdot S(R' \cdot S + R_n)e, \quad R' = {}^t(R_1, \dots, R_{n-1}),$$

$$V_1 a = -S \cdot a' + a^n, \quad V_2 a = a' + Sa^n$$

and D is the Poisson operator for the Dirichlet problem of the Laplace equation in H. Taking the fact that R_j and S_j is bounded operator on $L^q_w(\mathbf{R}^n)$ and $L^q_w(\mathbf{R}^{n-1})$ to themselves respectively into account, we can reduce to the whole space case, so that we obtain the desired result for the half-space. □

Next we shall prove the perturbed half-space case by using cut-off technique with Lemma 5.1. We first consider the $L^p - L^q$ estimates for $t > 2$.

LEMMA 5.2. *Let $n \ge 2$, $1 < p, r \le q < \infty$ and Ω be a perturbed half-space with C^2-boundary. Let s be a positive number such that $0 \le s < (n-1)(1-1/p)$.*

Then for $a \in L^p_w(\Omega) \cap L^r(\Omega)$ *and* $w = \langle x \rangle^{sp}$, *we have*

$$\begin{aligned}\|\langle x\rangle^s e^{-tA} Pa\|_{L^q(\Omega)} &\le Ct^{-n(1/p-1/q)/2}\|\langle x\rangle^s a\|_{L^p(\Omega)} \\ &\quad + Ct^{-n(1/r-1/q)/2+s/2}\|a\|_{L^r(\Omega)}, \end{aligned} \tag{5.1}$$

$$\begin{aligned}\|\langle x\rangle^s \nabla e^{-tA} Pa\|_{L^q(\Omega)} &\le Ct^{-n(1/p-1/q)/2-1/2}\|\langle x\rangle^s a\|_{L^p(\Omega)} \\ &\quad + Ct^{-n(1/r-1/q)/2+(s-1)/2}\|a\|_{L^r(\Omega)} \end{aligned} \tag{5.2}$$

for $t > 2$.

Proof. By using the cut-off technique, we divide Ω in $\Omega\backslash\Omega_R$ and Ω_R. By the result on Kubo and Shibata [30] and Sobolev's embedding theorem, we can obtain

$$\|\langle x\rangle^s e^{-tA} f\|_{L^q(\Omega_R)} \le C\|e^{-tA} f\|_{L^q(\Omega_R)} \le Ct^{-n/(2p)-1/2}\|f\|_{L^p} \le Ct^{-n/(2p)-1/2}\|\langle x\rangle^s f\|_{L^p}$$

for $f \in L^p_{w,\sigma}(\Omega)$. This implies (5.1) for Ω_R. Similarly we can prove the (5.2) for Ω_R.

We shall next consider the $L^p - L^q$ estimate for $\Omega\backslash\Omega_R$. For $R \ge R_0 + 2$, set $g = e^{-A} f \in \mathscr{D}(A^N)$, $u(t) = e^{-(t+1)A} f$. We set

$$Z(t) = \psi_R u(t) - \mathbf{B}[(\nabla\psi_R)\cdot u(t)], \quad \Phi(t) = \psi_R \pi(t),$$

where $u(t)$ and $\pi(t)$ are the solution to Stokes equations with

$$\int_{D_R} \pi(t)\, dx = 0, \tag{5.3}$$

where $D_R = \{x \in \Omega \mid R - 1 < |x| < R\}$ and $\mathbf{B}$ is the Bogovskiĭ operator. It is observed that $(Z(t), \Phi(t))$ satisfies the equations:

$$\partial_t Z(t) - \Delta Z(t) + \nabla\Phi(t) = L(t), \quad \nabla\cdot Z(t) = 0, \quad \text{in } H,$$

$$Z(0) = \psi_R a - \mathbf{B}[(\nabla\psi_R)a] =: z_0,$$

where

$$L(t) = -2\nabla\psi_R : \nabla u(t) - (\Delta\psi_R)u(t) + (\partial_t - \Delta)\mathbf{B}[(\nabla\psi_R)\cdot u] + (\nabla\psi_R)\pi(t).$$

Since $Z(t) \in C^1([0,\infty) : L^p_{w,\sigma}(H)) \cap C([0,\infty); \mathscr{D}(A_H))$, we can write $Z(t)$ as follows:

$$Z(t) = e^{-tA_H} z_0 - \int_0^t e^{-(t-\tau)A_H} PL(\tau)\, d\tau = z_1 + z_2 \tag{5.4}$$

where e^{-tA_H} is the semigroup in half-space obtained by Fröhlich [18].

Given $\phi \in C_0^\infty(H)$, we set

$$\Theta = (\nabla \psi_R)\phi - \frac{1}{D_R}\int_{D_R} (\nabla \psi_R)\phi \, dx$$

and then $\int_{D_R} \Theta \, dx = 0$. By a property of Bogovskiĭ operator, we can choose $\chi \in W^{1,p'}(D_R)$ such that $\nabla \cdot \chi = \Theta$, $\chi|_{\partial D_R} = 0$ and

$$\|\chi\|_{W^{1,p'}(D_R)} \le C\|\Theta\|_{L^{p'}(D_R)} \le C\|\phi\|_{L^{p'}(D_R)}$$

for $1/p + 1/p' = 1$. On the other hand, by (5.3), we have

$$((\nabla \psi_R)\pi(t), \phi) = \int_{D_R} \pi(t)\Theta \, dx = (\pi, \nabla \cdot \chi) = -(\nabla u, \nabla \chi) - (\partial_t u, \chi).$$

Here we recall the estimate for Ω_R obtained by Kubo and Shibata [30]:

$$\|\nabla e^{-tA} Pf\|_{L^r(\Omega_R)} + \|\partial_t e^{-tA} Pf\|_{L^r(\Omega_R)} \le Ct^{-n/(2r)-1/2}\|f\|_{L^r(\Omega)}$$

for $t > 2$, $1 < r < \infty$ and $f \in L^r_\sigma(\Omega)$. This estimate implies that

$$\begin{aligned} |((\nabla \psi_R)\pi(t), \phi)| &\le \|\nabla u\|_{L^p(\Omega_R)}\|\nabla \chi\|_{L^{p'}(\Omega_R)} + \|\partial_t u\|_{L^p(\Omega_R)}\|\chi\|_{L^{p'}(\Omega_R)} \\ &\le C(1+t)^{-n/(2p)-1/2}\|f\|_{L^p}\|\chi\|_{W^{1,p'}(\Omega_R)}. \end{aligned}$$

By duality argument, we see

$$\|(\nabla \psi_R)\pi(t)\|_{L^p} \le C(1+t)^{-n/(2p)-1/2}\|f\|_{L^p}.$$

Since we have supp $L(t) \subset \Omega_R$, we obtain

$$\|PL(t)\|_{L^r} \le C\|L(t)\|_{L^r} \le C\|L(t)\|_{L^p} \le C(1+t)^{-n/(2p)-1/2}\|f\|_{L^p} \quad t \ge 1$$

for $1 < r < p < \infty$. Therefore we see

$$\|\langle x\rangle^s PL(t)\|_{L^r} \le C\|\langle x\rangle^s L(t)\|_{L^r} \le C(1+t)^{-n/(2p)-1/2}\|\langle x\rangle^s f\|_{L^p}. \tag{5.5}$$

Now we consider the estimate $Z(t)$ by using (5.5). We can show the estimate of z_1 in (5.4) by using the weighted $L^p - L^q$ estimates in the half-space as follows:

$$\|\langle x\rangle^s e^{-tA} Pz_0\|_{L^q(H)} \le Ct^{-n(1/p-1/q)/2}\|\langle x\rangle^s z_0\|_{L^p(H)} + Ct^{-n(1/r-1/q)/2+s/2}\|z_0\|_{L^r(H)},$$

$$\begin{aligned} \|\langle x\rangle^s \nabla e^{-tA} Pz_0\|_{L^q(H)} &\le Ct^{-n(1/p-1/q)/2-1/2}\|\langle x\rangle^s z_0\|_{L^p(H)} \\ &\quad + Ct^{-n(1/r-1/q)/2+(s-1)/2}\|z_0\|_{L^r(H)}. \end{aligned}$$

We shall estimate z_2 in (5.4). To this end, we assume that $1 < p_1, r_1 < p, r < \infty$ with $p_1, r_1 < \min(p, n)$, $n(1/p - 1/q) < 2$ and $n(1/r - 1/q) < 2 + s$. By Lemma 5.1 we have

$$\begin{aligned}
\|\langle x\rangle^s z_2\|_{L^q(H)} &\le C\int_{t-1}^{t} (t-\tau)^{-n(1/p-1/q)/2} \|\langle x\rangle^s PL(\tau)\|_{L^p}\, d\tau \\
&\quad + C\int_{t-1}^{t} (t-\tau)^{-n(1/r-1/q)/2+s/2} \|PL(\tau)\|_{L^r}\, d\tau \\
&\quad + C\int_{0}^{t-1} (t-\tau)^{-n(1/p_1-1/q)/2} \|\langle x\rangle^s PL(\tau)\|_{L^{p_1}}\, d\tau \\
&\quad + C\int_{0}^{t-1} (t-\tau)^{-n(1/r_1-1/q)/2+s/2} \|PL(\tau)\|_{L^{r_1}}\, d\tau \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

By using (5.5), we can estimate I_1 and I_2 as follows:

$$\begin{aligned}
I_1 &\le C\int_{t-1}^{t} (t-\tau)^{-n(1/p-1/q)/2} (1+\tau)^{-n/(2p)-1/2}\, d\tau \|\langle x\rangle^s f\|_{L^p} \\
&\le Ct^{-n/(2p)-1/2} \|\langle x\rangle^s f\|_{L^p}
\end{aligned}$$

and

$$I_2 \le C\int_{t-1}^{t} (t-\tau)^{-n(1/r-1/q)/2+s/2} (1+\tau)^{-n/(2r)-1/2}\, d\tau \|f\|_{L^r} \le Ct^{-n/(2r)-1/2} \|f\|_{L^r}.$$

We next consider the estimate of I_3.

$$\begin{aligned}
I_3 &\le \int_{0}^{t/2} (1+t-\tau)^{-n(1/p_1-1/q)/2} (1+\tau)^{-n/(2p)-1/2}\, d\tau \|\langle x\rangle^s f\|_{L^p} \\
&\quad + \int_{0}^{t/2} (1+\tau)^{-n(1/p_1-1/q)/2} (1+t-\tau)^{-n/(2p)-1/2}\, d\tau \|\langle x\rangle^s f\|_{L^p}.
\end{aligned}$$

Taking $(1+\tau)^{-1} \ge (1+t-\tau)^{-1}$ for $0 \le \tau \le t/2$ into accont, we obtain

$$\begin{aligned}
I_3 &\le C(1+t)^{-n(1/p-1/q)/2} \int_{0}^{t/2} (1+\tau)^{-n/(2p_1)-1/2}\, d\tau \|\langle x\rangle^s f\|_{L^p} \\
&\le C(1+t)^{-n(1/q-1/p)/2} \|\langle x\rangle^s f\|_{L^p}.
\end{aligned}$$

In a similar way to estimate of I_3, we can get the estimate of I_4 as follows:

$$I_4 \le C(1+t)^{-n(1/r-1/q)/2+s/2}\|f\|_{L^r}$$

Summing up, we obtain (5.1) for $n(1/p-1/q)<2$ and $n(1/r-1/q)<2+s$.

Finally we remove the restriction $n(1/p-1/q)<2$ and $n(1/r-1/q)<2+s$ by using the property of semigroup. We choose $p_1,\dots,p_\ell$ in such a way that $p=p_1<p_2<\cdots<p_\ell=q$ and $n(1/p_{j-1}-1/p_j)<2$ for $j=2,\dots,\ell$ and $r_1,\dots,r_m$ in such a way that $r=r_1<r_2<\cdots<r_m=q$ and $n(1/r_{j-1}-1/r_j)<2+s$ for $j=2,\dots,m$. By (5.1), we have

$$\begin{aligned}\|\langle x\rangle^s e^{-tA}f\|_{L^q} &\le Ct^{-n(1/p_{\ell-1}-1/p_\ell)/2}\|\langle x\rangle^s e^{-(\ell-2)/(\ell-1)tA}f\|_{L^{p_{\ell-1}}} \\ &\quad + Ct^{-n(1/r_{\ell-1}-1/r_\ell)/2+s/2}\|e^{-(\ell-2)/(\ell-1)tA}f\|_{L^{r_{\ell-1}}} \\ &\le Ct^{-n(1/p-1/q)/2}\|\langle x\rangle^s f\|_{L^p} + Ct^{-n(1/r-1/q)/2+s/2}\|f\|_{L^r},\end{aligned}$$

which implies (5.1). Similarly, we can obtain (5.2). Therefore we obtain the weighted $L^p - L^q$ estimate of Stokes semigroup. □

Lemma 5.3. *Let $n\ge 2$, $1<p\le q<\infty$ and Ω be a perturbed half-space with C^2 boundary. Let s be a positive number such that $0\le s<(n-1)(1-1/p)$ and let $w=\langle x\rangle^{sp}$. Then for $a\in L^p_w(\Omega)$, we have*

$$\|\langle x\rangle^s e^{-tA}Pa\|_{L^q(\Omega)} \le Ct^{-n(1/p-1/q)/2}\|\langle x\rangle^s a\|_{L^p(\Omega)}, \tag{5.6}$$

$$\|\langle x\rangle^s \nabla e^{-tA}Pa\|_{L^q(\Omega)} \le Ct^{-n(1/p-1/q)/2-1/2}\|\langle x\rangle^s a\|_{L^p(\Omega)} \tag{5.7}$$

for $0<t\le 2$.

Proof. In view of the weighted resolvent estimate (2.2), we have

$$\|\nabla e^{-tA}f\|_{L^p_w(\Omega)} \le Ct^{-1/2}\|f\|_{L^p_w(\Omega)}$$

for $1<p<\infty$ and $0<t<2$. Therefore it is sufficient to prove (5.6). We set $\sigma=n(1/p-1/q)$. By Sobolev's embedding theorem, we have

$$\|\langle x\rangle^s e^{-tA}f\|_{L^q(\Omega)} \le C\|\langle x\rangle^s e^{-tA}f\|_{W^{\sigma,p}(\Omega)}$$

for $1<p\le q<\infty$ and $0<\sigma<2$. By real interpolation for $\sigma\in(0,2)$, we have

$$W^{\sigma,p}(\Omega) = (L^p(\Omega), W^{2,p}(\Omega))_{\sigma/2,p}.$$

Moreover by using the weighted resolvent estimate (2.3), we have

$$\|\nabla^k e^{-tA} f\|_{L^p_w(\Omega)} \le C t^{-k/2} \|f\|_{L^p_w(\Omega)} \tag{5.8}$$

for $k = 0, 1, 2$ and $0 < t < 2$. Therefore we see

$$\|\langle x\rangle^s e^{-tA} f\|_{W^{\sigma,p}(\Omega)} \le \|\langle x\rangle^s e^{-tA} f\|_{L^p}^{1-\sigma/2} \|\langle x\rangle^s e^{-tA} f\|_{W^{2,p}}^{\sigma/2}$$

for $1 < p < \infty$ and $0 < t < 2$. Here taking the fact

$$\|\nabla\langle x\rangle^s \nabla e^{-tA} f\|_{L^p} = \left\| \frac{sx}{(1+|x|^2)} \langle x\rangle^s \nabla e^{-tA} f \right\|_{L^p} \le C \|\langle x\rangle^s \nabla e^{-tA} f\|_{L^p}$$

into account, by using (5.8) we can estimate the second term $\|\langle x\rangle^s e^{-tA} f\|_{W^{2,p}}$ as follows:

$$\begin{aligned}
\|\langle x\rangle^s e^{-tA} f\|_{W^{2,p}} &\le C(\|\nabla^2\langle x\rangle^s e^{-tA} f\|_{L^p} + \|\nabla\langle x\rangle^s \nabla e^{-tA} f\|_{L^p} + \|\langle x\rangle^s \nabla^2 e^{-tA} f\|_{L^p}) \\
&\le C(\|\langle x\rangle^s e^{-tA} f\|_{L^p} + \|\langle x\rangle^s \nabla e^{-tA} f\|_{L^p} + \|\langle x\rangle^s \nabla^2 e^{-tA} f\|_{L^p}) \\
&\le C(\|f\|_{L^p_w(\Omega)} + t^{-1/2}\|f\|_{L^p_w(\Omega)} + t^{-1}\|f\|_{L^p_w(\Omega)}) \\
&\le C t^{-1} \|f\|_{L^p_w(\Omega)}
\end{aligned}$$

for $0 < t < 2$ and $1 < p < \infty$. Summing up, we obtain

$$\|\langle x\rangle^s e^{-tA} f\|_{L^q(\Omega)} \le C \|\langle x\rangle^s e^{-tA} f\|_{W^{\sigma,p}(\Omega)} \le C t^{-n(1/p-1/q)/2} \|\langle x\rangle^s f\|_{L^p}$$

for $1 < p \le q < \infty$ and $0 < n(1/p - 1/q) < 2$. We can remove the restriction $n(1/p - 1/q) < 2$ by using the property of semigroup. This completes the proof of the weighted $L^p - L^q$ estimate for $0 < t < 2$. □

6 Navier-Stokes Equations

In this section, we shall consider the application of the weighted $L^p - L^q$ estimates to Navier-Stokes equations. As we mentioned in Introduction (section 1), we know the unique existence results for Navier-Stokes equations in the half-space and a perturbed half-space (see Kozono [28] for half-space case and Kubo and Shibata [30] for a perturbed half-space case for detail). We consider the case where the initial data a belongs to $L^n(\Omega) \cap L^n_w(\Omega)$. Since $a \in L^n(\Omega)$, we know that there exists the unique strong solution u to (NS) and the solution u satisfies the following assymptotic behavior:

$$\|u(t)\|_{L^r} \le C_1 t^{-1/2+n/(2r)} \|a\|_{L^n} \quad \text{for } n \le r \le \infty, \tag{6.1}$$

$$\|\nabla u(t)\|_{L^r} \le C_2 t^{-1+n/(2r)} \|a\|_{L^n} \quad \text{for } n \le r < \infty. \tag{6.2}$$

In order to prove Theorem 2.8, we begin to prove that there is a constant M independent of $T \ge 4$ such that

$$\sup_{0 \le t \le 2} (t^{1/2-n/(2q)} \|\langle x\rangle^s u(t)\|_{L^q}) + \sup_{2 \le t \le T} (t^{1/2-n/(2q)-s/2} \|\langle x\rangle^s u(t)\|_{L^q}) \le M \tag{6.3}$$

for $q \ge n$. To this end, we set

$$\tilde{m} = \sup_{0 \le t \le 2} (t^{1/2-n/(2q)} \|\langle x\rangle^s u(t)\|_{L^q}), \quad \tilde{M} = \sup_{2 \le t \le T} (t^{1/2-n/(2q)-s/2} \|\langle x\rangle^s u(t)\|_{L^q}).$$

We first consider the case for $0 \le t \le T \le 2$. By using the weighted $L^p - L^q$ estimate (5.6) and the relation (6.2), we have

$$\begin{aligned}
&\|\langle x\rangle^s u(t)\|_{L^q} \\
&\quad \le \|\langle x\rangle^s e^{-tA} a\|_{L^q} + \int_0^t \|\langle x\rangle^s e^{-(t-s)A} P(u\cdot\nabla)u(\tau)\|_{L^q}\, d\tau \\
&\quad \le C t^{-1/2+n/(2q)} \|\langle x\rangle^s a\|_{L^n} + \int_0^t (t-\tau)^{-n(1/q+1/n-1/q)/2} \|\langle x\rangle^s u(\tau)\|_{L^q} \|\nabla u(\tau)\|_{L^n}\, d\tau \\
&\quad \le C\left(\|\langle x\rangle^s a\|_{L^n} + C_2 \|a\|_{L^n} \tilde{m} B\left(\frac{1}{2}, \frac{n}{2q}\right)\right) t^{-1/2+n/(2q)},
\end{aligned}$$

where $B(\cdot,\cdot)$ denotes the beta function. Choosing $\|a\|_{L^n}$ smaller if necessary, we obtain $\tilde{m} \le C\|\langle x\rangle^s a\|_{L^n}$.

Next we consider the case for $2 \le t \le T \le 4$. To this end, we set I_j $(j = 1, \dots, 4)$ as follows:

$$I_1 = \|\langle x\rangle^s e^{-tA} a\|_{L^q}, \quad I_2 = \int_0^{t-2} \|\langle x\rangle^s e^{-(t-\tau)A} P(u\cdot\nabla)u(\tau)\|_{L^q}\, d\tau,$$

$$I_3 = \int_{t-2}^2 \|\langle x\rangle^s e^{-(t-\tau)A} P(u\cdot\nabla)u(\tau)\|_{L^q}\, d\tau, \quad I_4 = \int_2^t \|\langle x\rangle^s e^{-(t-\tau)A} P(u\cdot\nabla)u(\tau)\|_{L^q}\, d\tau.$$

By the argument for $t \le 2$, we see that there exists the positive constant C such that

$$\sup_{0<t<2} (t^{1/2-n/(2q)} \|\langle x\rangle^s u(t)\|_{L^q}) \le C\|\langle x\rangle^s a\|_{L^n} \tag{6.4}$$

with the constant C indenpendent of t. I_1 is easily estimated as follows:

$$I_1 \le Ct^{-1/2+n/(2q)}\|\langle x\rangle^s a\|_{L^n} + Ct^{-1/2+n/(2q)+s/2}\|a\|_{L^n} \tag{6.5}$$

Since $\tau < 2$ and $t - \tau \ge 2$, I_2 is estimated by the weighted $L^p - L^q$ estimates (5.1) as follows:

$$\begin{aligned}
I_2 \le{} & \int_0^{t-2} C(t-\tau)^{-1/2}\|\langle x\rangle^s u(\tau)\|_{L^q}\|\nabla u(\tau)\|_{L^n}\,d\tau \\
& + \int_0^{t-2} C(t-\tau)^{-1+n/(2q)+s/2}\|u(\tau)\|_{L^n}\|\nabla u(\tau)\|_{L^n}\,d\tau \\
\le{} & CC_2\|a\|_{L^n}\int_0^{t-2}(t-\tau)^{-1/2}\tau^{-1/2-1/2+n/(2q)}\,d\tau\left(\sup_{0\le\tau\le2}\tau^{1/2-n/(2q)}\|\langle x\rangle^s u(\tau)\|_{L^q}\right) \\
& + CC_1C_2\|a\|_{L^n}^2\int_0^{t-2}(t-\tau)^{-1+n/(2q)+s/2}\tau^{-1/2}\,d\tau \\
\le{} & CC_2\|a\|_{L^n}\|\langle x\rangle^s a\|_{L^n}t^{-1/2+n/(2q)} + CC_1C_2\|a\|_{L^n}^2 t^{-1/2+n/(2q)+s/2}.
\end{aligned}$$

Similarly, we can estimate I_3 and I_4 as follows:

$$\begin{aligned}
I_3 \le{} & \int_{t-2}^{2} C(t-\tau)^{-1/2}\|\langle x\rangle^s u(\tau)\|_{L^q}\|\nabla u(\tau)\|_{L^n}\,d\tau \\
\le{} & CC_2\|a\|_{L^n}\int_0^{t-2}(t-\tau)^{-1/2}\tau^{-1+n/(2q)}\,d\tau\left(\sup_{0\le\tau\le2}\tau^{1/2-n/(2q)}\|\langle x\rangle^s u(\tau)\|_{L^q}\right) \\
\le{} & CC_2\|a\|_{L^n}\|\langle x\rangle^s a\|_{L^n}t^{-1/2+n/(2q)}
\end{aligned}$$

and

$$\begin{aligned}
I_4 \le{} & \int_2^t C(t-\tau)^{-1/2}\|\langle x\rangle^s u(\tau)\|_{L^q}\|\nabla u(\tau)\|_{L^n}\,d\tau \\
\le{} & CC_2\|a\|_{L^n}\int_2^t(t-\tau)^{-1/2}\tau^{-1+n/(2q)+s/2}\,d\tau\left(\sup_{2\le\tau\le T}\tau^{1/2-n/(2q)-s/2}\|\langle x\rangle^s u(\tau)\|_{L^q}\right) \\
\le{} & CC_2\|a\|_{L^n}\tilde{M}t^{-1/2+n/(2q)+s/2}
\end{aligned}$$

Summing up, we obtain

$$\begin{aligned}
\tilde{M} ={} & \sup_{2\le t\le T}(t^{1/2-n/(2q)-s/2}\|\langle x\rangle^s u(\tau)\|_{L^q}) \\
\le{} & C\|\langle x\rangle^s a\|_{L^n} + C\|a\|_{L^n} + C\|\langle x\rangle^s a\|_{L^n}\|a\|_{L^n} + C\tilde{M}\|a\|_{L^n}.
\end{aligned}$$

If necessary, choose $\|a\|_{L^n}$ small, we obtain

$$\tilde{M} = \sup_{2 \le t \le 4} (t^{1/2-n/(2q)-s/2} \|\langle x \rangle^s u(\tau)\|_{L^q})$$
$$\le \frac{C(\|\langle x \rangle^s a\|_{L^n} + \|a\|_{L^n} + \|\langle x \rangle^s a\|_{L^n} \|a\|_{L^n})}{1 - C\|a\|_{L^n}}. \tag{6.6}$$

By (6.4) and (6.6), we conclude that there exists a positive constant M independent of T satisfied (6.3).

In order to consider the case for $4 \le t \le T$, set I_j $(j = 1, \dots, 4)$ as follows:

$$I_1 = \|\langle x \rangle^s e^{-tA} a\|_{L^q}, \quad I_2 = \int_0^2 \|\langle x \rangle^s e^{-(t-\tau)A} P(u \cdot \nabla) u(\tau)\|_{L^q} \, d\tau,$$

$$I_3 = \int_2^{t-2} \|\langle x \rangle^s e^{-(t-\tau)A} P(u \cdot \nabla) u(\tau)\|_{L^q} \, d\tau,$$

$$I_4 = \int_{t-2}^{t} \|\langle x \rangle^s e^{-(t-\tau)A} P(u \cdot \nabla) u(\tau)\|_{L^q} \, d\tau.$$

We notice that from the argument above, (6.4) and (6.5) hold. We next estimate I_2 by using Theorem 2.7. Noticing that $\tau < 2$ and $t - \tau > 2$, we have

$$I_2 \le C \int_0^2 (t-\tau)^{-1/2} \|\langle x \rangle^s u(\tau)\|_{L^q} \|\nabla u(\tau)\|_{L^n} \, d\tau$$
$$+ C \int_0^2 (t-\tau)^{-1+n/(2q)+s/2} \|u(\tau)\|_{L^n} \|\nabla u(\tau)\|_{L^n} \, d\tau$$
$$\le CC_2 \|a\|_{L^n} \|\langle x \rangle^s a\|_{L^n} \int_0^2 (t-\tau)^{-1/2} \tau^{-1+n/(2q)} \, d\tau$$
$$+ CC_1C_2 \|a\|_{L^n}^2 \int_0^2 (t-\tau)^{-1+n/(2q)+s/2} \tau^{-1/2} \, d\tau$$
$$\le CC_2 \|a\|_{L^n} (\|\langle x \rangle^s a\|_{L^n} t^{-s/2} + C_1 \|a\|_{L^n}) t^{-1/2+n/(2q)+s/2}.$$

Similarly, we can estimate I_3 and I_4 as follows:

$$I_3 \le \int_2^{t-2} (t-\tau)^{-1/2} \|\langle x \rangle^s u(\tau)\|_{L^q} \|\nabla u(\tau)\|_{L^n} \, d\tau$$
$$+ \int_2^{t-2} (t-\tau)^{-1+n/(2q)+s/2} \|u(\tau)\|_{L^n} \|\nabla u(\tau)\|_{L^n} \, d\tau$$

$$
\begin{aligned}
&\le CC_2\|a\|_{L^n}\left[\tilde{M}\int_2^{t-2}(t-\tau)^{-1/2}\tau^{-1+n/(2q)+s/2}\,d\tau\right.\\
&\qquad\qquad\left.+C_1\|a\|_{L^n}\int_2^{t-2}(t-\tau)^{-1+n/(2q)+s/2}\tau^{-1/2}\,d\tau\right]\\
&\le CC_2\|a\|_{L^n}(\tilde{M}+C_1\|a\|_{L^n})t^{-1/2+n/(2q)+s/2}
\end{aligned}
$$

and

$$
I_4\le C\int_{t-2}^{t}(t-\tau)^{-1/2}\|\langle x\rangle^s u(\tau)\|_{L^q}\|\nabla u(\tau)\|_{L^n}\,d\tau\le CC_2\tilde{M}\|a\|_{L^n}t^{-1/2+n/(2q)+s/2}.
$$

Summing up, we obtain

$$
\tilde{M}\le C\|\langle x\rangle^s a\|_{L^n}+C\|a\|_{L^n}+CC_2\|a\|_{L^n}(\|\langle x\rangle^s a\|_{L^n}+C_1\|a\|_{L^n}+\tilde{M}).
$$

Choose $\|a\|_{L^n}$ small if necessaly, we have

$$
\begin{aligned}
&\sup_{2<t<T}(t^{1/2-n/(2q)-s/2}\|\langle x\rangle^s u(\tau)\|_{L^q})\\
&\qquad\le\frac{C(\|\langle x\rangle^s a\|_{L^n}+\|a\|_{L^n}+C_1C_2\|a\|_{L^n}^2+C_2\|a\|_{L^n}\|\langle x\rangle^s a\|_{L^n})}{1-CC_2\|a\|_{L^n}}.
\end{aligned}\tag{6.7}
$$

By (6.4) and (6.7), we conclude that there exists a positive constant M independent of T satisfied (6.3). Since we obatin a positive constant M independent of T, we can conclude

$$
\sup_{0<t<2}(t^{1/2-n/(2q)}\|\langle x\rangle^s u(t)\|_{L^q})+\sup_{2<t<\infty}(t^{1/2-n/(2q)-s/2}\|\langle x\rangle^s u(t)\|_{L^q})\le M,
$$

which implies the weighted asymptotic behavior (2.3).

Finally we shall prove the asymptotic behavior for $\|\langle x\rangle^s\nabla u(t)\|_{L^r}$ for $n\le r<\infty$ and $0\le s<(n-1)(1-1/n)$. To this end, fix s as the number satisfying $0\le s<(n-1)(1-1/n)$. Then we remark that there exists the positive number $\tilde{q}$ such that $s<(n-1)(1-1/n-1/\tilde{q})$ holds. We have

$$
\|\langle x\rangle^s\nabla u(t)\|_{L^r}\le\|\langle x\rangle^s\nabla e^{-tA}a\|_{L^r}+\int_0^t\|\langle x\rangle^s e^{-(t-\tau)A}P(u\cdot\nabla)u(\tau)\|_{L^r}\,d\tau=I+II.
$$

Since we can prove the asymptotic behavior for I easily by using Theorem 2.7, we shall estimate only the second term II. Since we see $s<(n-1)(1-(n+q)/nq)$ for $q>\max(\tilde{q},r)$, we obtain

$$
\begin{aligned}
II &\le C \int_0^t (t-\tau)^{-n(1/q+1/n-1/r)/2-1/2} \| \langle x \rangle^s P(u\cdot\nabla)u(\tau)\|_{L^{qn/(q+n)}}\, d\tau \\
&\le C \int_0^t (t-\tau)^{-1+n(1/r-1/q)/2} \| \langle x \rangle^s (u\cdot\nabla)u(\tau)\|_{L^{qn/(q+n)}}\, d\tau \\
&\le C \int_0^t (t-\tau)^{-1+n(1/r-1/q)/2} \| \langle x \rangle^s u(\tau)\|_{L^q} \|\nabla u(\tau)\|_{L^n}\, d\tau \\
&\le C \int_0^t (t-\tau)^{-1+n(1/r-1/q)/2} \tau^{-1+n/2q+s/2}\, d\tau \\
&\le C t^{-1+n/2r+s/2}.
\end{aligned}
$$

Therefore we obtain the asymptotic behavior (2.4).

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References

[1] T. Abe and Y. Shibata, On a resolvent estimate of the Stokes equation on an infinite layer, Part 2 $\lambda = 0$ case, Journal of Mathematical Fluid Mechanics, **5** (2003), 245–274.

[2] H. Abels, $L^q - L^r$ estimates for the non-stationary Stokes equations in an aperture domain, Z. Anal. Anwendungen **21** (2002), 159–178.

[3] H. O. Bae, Analyticity and asymptotics for the Stokes Solutions in weighted space, J. Math. Anal. Appl., **269** (2002), 149–171.

[4] W. Borchers and T. Miyakawa, L^2 decay for the Navier-Stokes flow in halfspaces. Math. Ann. **282** (1988), 139–155.

[5] W. Borchers and W. Varnhorn, On the boundedness of the Stokes semigroup in two dimensional exterior domains, Math. Z., **213** (1993), 275–299.

[6] Z. M. Chen, Solutions of the stationary and nonstationary Navier-Stokes equations in exterior domains, Pacific J. Math., **159** (1993), 227–240.

[7] S.-K. Chua, Extension theorems for weighted Sobolev spaces, Indiana Univ. Math. J. **41** (1992), 1027–1076.

[8] W. Dan and Y. Shibata, On the $L_q - L_r$ estimates of the Stokes semigroup in a two dimensional exterior domain, J. Math. Soc. Japan **51** (1999), 181–207.

[9] W. Dan and Y. Shibata, Remark on the $L_q - L_\infty$ estimate of the Stokes semigroup in a 2-dimensional exterior domain, Pacific J. Math. **189** (1999), 223–240.

[10] W. Dan, T. Kobayashi and Y. Shibata, On the local energy decay approach to some fluid flow in exterior domain, Recent Topics on Mathematical Theory of Viscous Incompressible Fluid, 1–51, Lecture Notes Numer. Appl. Math. **16**, Kinokuniya, Tokyo, 1998.

[11] W. Desch, M. Hieber and J. Prüss, L^p theory of the Stokes equation in a half space, J. Evol. Equations **1** (2001), 115–142.

[12] R. Farwig and H. Sohr, On the Stokes and Navier-Stokes System for Domains with Noncompact Boundary in L^q-spaces, Math. Nachr. **170** (1994), 53–77.

[13] R. Farwig and H. Sohr, Weighted L^q-theory for the Stokes resolvent in exterior domains, J. Math. Sco. Japan, **49** (1997), No. 2, 251–288.

[14] R. Farwig and H. Sohr, Generalized resolvent estimates for the Stokes system in bounded and unbounded domains , J. Math. Soc. Japan **46** No. 4 (1994), 607–643.

[15] R. Farwig and H. Sohr, Helmholtz decomposition and Stokes resolvent system for aperture domains in L^q-space, Analysis, **16** (1996), 1–26.

[16] A. Fröhlich, Helmholtz decomposition of Weighted L^q spaces for Muckenhoupt Weights, Ann. Univ. Ferrara-Sez. VII-Sc. Mat. Vol. XLVI 11–19 (2000).

[17] A. Fröhlich, Maximal regularity for the nonstationary Stokes system in an aperture domain, J. Evol. Equations **2** (2002), 471–493.

[18] A. Fröhlich, The Stokes operator in weighted L^q-spaces I: weighted estimates for the Stokes Resolvent Problem in a half-space, J. Math. Fluid Mech. **5** (2003), 166–199.

[19] A. Fröhlich, Solutions of the Navier-Stokes initial value problem in weighted L^q-spaces, Math. Nachr. **269–270** (2004), 150–166.

[20] A. Fröhlich, The Stokes operator in weighted L^q-spaces II: weighted resolvent estimates and maxmal L^p-regularity, Math. Ann. **339** (2007), 287–316.

[21] J. Garcia-Cuerva and J. L. Rubio de Francia, Weighted norm inequalities and related topics, North Holland, Amsterdam, 1985.

[22] Y. Giga, Domains of fractional powers of the Stokes operator in L^r spaces, Arch. Rational Mech. Anal. **89** (1985), 251–265.

[23] Y. Giga and T. Miyakawa, Solutions in L^r of the Navier-Stokes initial value problem, Arch. Rational Mech. Anal. **89** (1985), 267–281.

[24] T. Hishida, The nonstationary Stokes and Navier-Stokes flows through an aperture. Advances in Mathematical Fluid Mechanics (2004), 79–123.

[25] H. Iwashita, $L_q - L_r$ estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L^q spaces, Math. Ann. **285** (1989), 265–288.

[26] P. W. Johns, Quasiconformal mappings and extendability of functions in Sobolev spaces, Acta Math. **147** (1981), 71–88.

[27] T. Kato, Strong L^p-Solutions of the Navier-Stokes Equation in $\mathbf{R}^m$, with Applications to Weak Solutions. Math. Z., **187** (1984), 471–480.

[28] H. Kozono, Global L^n-solution and its decay property for the Navier-Stokes equations in half-space $\mathbf{R}^n_+$, J. Differential Equations, **79** (1989), 79–88.

[29] T. Kubo, The Stokes and Navier-Stokes equations in an aperture domain, J. Math. Soc. Japan, **59**, No. 3 (2007), 837–859.

[30] T. Kubo and Y. Shibata, On the Stokes and Navier-Stokes equations in a perturbed half-space, Advances in Differential Equations, Vol. 10, No. 6 (2005), 695–720.

[31] P. Maremonti and V. A. Solonnikov, On nonstationary Stokes problem in exterior domains, Ann. Sc. Norm. Super Pisa, **24** (1997), 395–449.

[32] M. McCracken, The resolvent problem for the Stokes equation on halfspaces in L^p, SIAM J. Math. Anal. **12** (1981), 201–228.

[33] T. Miyakawa, The Helmholtz decomposition of vecter fields in some unbounded domains, Math. J. Toyama Univ. **17** (1994), 115–149.

[34] Y. Shibata, On an exterior initial boundary value problem for Navier-Stokes equations, Quart. Appl. Math., **LVII** (1999), 117–155.

[35] S. Ukai, A solution formula for the Stokes equation in $\mathbf{R}^n_+$. Comm. Pure Appl. Math. **40** (1987), 611–621.

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