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PERIODICITY AND EIGENVALUES OF MATRICES OVER QUASI-MAX-PLUS ALGEBRAS

By

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Abstract. We extend the notion of a max-plus algebra and study periodicity and eigenvalues of matrices over this new structure thereby generalizing some well-known results on matrices over a max-plus algebra.

1. Introduction

The concept of the standard max-plus algebra over the real numbers has turned out to be a useful tool in applications (discrete event systems, optimal control, game theory) and several other fields of mathematics (matrix theory, combinatorics, asymptotic analysis, geometry). The reader is referred to [4, 9, 12] for details. M. Gavalec [10] introduced the notion of a max-plus algebra in a broader framework (see Section 2 for details) and proved among other things that every irreducible matrix over a max-plus algebra in this new setting is almost linear periodic. We establish a generalization of this result (see Theorem 3.5 below).

Our main concern is an extension of the notion 'max-plus algebra' by introducing a quasi-max-plus algebra (see the Definition 2.3). This concept is inspired by a certain dioid over the integers which was introduced in [1] for the description of primitive matrices over polynomial rings. We study periodicity properties and eigenvalues of matrices over a quasi-max-plus algebra (see Theorems 3.7, 3.8 and 3.10). In an appendix we mainly collect some results which are well-known under stronger prerequisites, but which are needed here in a more general setting.

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2. Definition and Examples of Quasi-Max-Plus Algebras

In this paper we always let $(\mathscr{D}, \oplus, \odot)$ be a commutative dioid with neutral elements ε and e, respectively, i.e., $\mathscr{D}$ is a commutative unital semiring with idempotent addition (see [2, Definition 4.1]).[1] As customary we often omit the multiplication sign $\odot$ if there is no fear of confusion. In particular, we write

$$a^n = \underbrace{a \odot \cdots \odot a}_{n \text{ factors}} \quad (a \in \mathscr{D},\, n \in \mathbf{N}_{>0})$$

and $a^0 = e$. If not stated otherwise we always use the natural (partial) order on $\mathscr{D}$, i.e., $a \leq b$ if and only if $a \oplus b = b$ (see [2, Section 4.3.2]). In particular, the natural order is compatible with addition and ε is the minimal element of $\mathscr{D}$ (see [2, Theorem 4.28]).

For the sake of completeness we recall several definitions. The dioid $\mathscr{D}$ is called

- entire if for all $\alpha, \beta \in \mathscr{D}$ the equation $\alpha\beta = \varepsilon$ implies $\alpha = \varepsilon$ or $\beta = \varepsilon$ (see [2, Definition 4.11]),
- cancellative if for $\alpha, \beta, \gamma \in \mathscr{D}$ with $\alpha\beta = \alpha\gamma$ and $\alpha \neq \varepsilon$ we have $\beta = \gamma$,
- archimedean if for all $\alpha, \beta \in \mathscr{D}$ there is some $\gamma \in \mathscr{D}$ such that $\gamma\beta \geq \alpha$ provided $\beta \neq \varepsilon$ (see [2, Definition 4.33]),
- algebraically closed[2] if for every $\alpha \in \mathscr{D}$ and $n \in \mathbf{N}_{>0}$ the equation $x^n = \alpha$ admits a solution in $\mathscr{D}$.

Finally we say that $\mathscr{D}$ satisfies the weak stabilization condition (see [8, Definition 1.1.5]) if for all $\alpha, \beta, \lambda, \mu \in \mathscr{D}$ there exist $\gamma, \nu \in \mathscr{D}$ and $N \in \mathbf{N}$ such that for all $n \geq N$ we have

$$\alpha\lambda^n \oplus \beta\mu^n = \gamma\nu^n.$$

A particular class of dioids was introduced under the name 'extremal algebra' by J. Nedoma [14] in 1974 and under the name 'max-plus algebra' by M. Gavalec [10] in 2000. Here we prefer the more suggestive latter notion. Let $(G, +, \leq)$ be an abelian linearly ordered divisible group with neutral element 0 and $\varepsilon \notin G$ a new element. We call the dioid $(G \cup \{\varepsilon\}, \oplus, \odot)$ the max-plus algebra generated by G where the operations are given by $\oplus = \max$ and $\odot = +$ and ε enjoys the

[1] By abuse of notation we use the same symbols ε and e for the neutral elements of all dioids which occur in the subsequent text.

[2] In [4] this property is called radicable. In the translation [7] this property is inadvertently named 'algebraic completeness'.

properties

$$\varepsilon < g \quad (g \in G),$$

and

$$\varepsilon + g = g + \varepsilon = \varepsilon \quad (g \in G \cup \{\varepsilon\}).$$

Correspondingly, the dioid $\mathscr{D}$ is called a max-plus algebra if there exists a commutative linearly ordered divisible group G such that $\mathscr{D}$ is the max-plus algebra generated by G.

Example 2.1. Let G be an additive divisible subgroup of the real numbers $\mathbf{R}$. Then $(G \cup \{-\infty\}, \max, +)$ is a max-plus algebra.[3] In particular, the standard max-plus algebra (or simply max-algebra) fits into these settings. For details and applications we refer the reader to [4].

The following elementary properties of max-plus algebras can be verified in a straightforward manner.

Proposition 2.2. *Let $\mathscr{D}$ be a max-plus algebra.*

(i) *$\mathscr{D}$ is a commutative linearly ordered idempotent semifield[4] of characteristic zero and with neutral elements ε and e, respectively. Moreover, ε is the minimal element of $\mathscr{D}$.*

(ii) *$\mathscr{D}$ is cancellative.*

(iii) *For every $\alpha \in \mathscr{D}$ and $n \in \mathbf{N}_{>0}$ the equation $x^n = \alpha$ has a unique solution in $\mathscr{D}$, namely*

$$\alpha/n := \frac{\alpha}{n}.$$

In particular, $\mathscr{D}$ is algebraically closed, and we have $e/n = e$.

(iv) *Let $\alpha \in \mathscr{D}$ and $n, m \in \mathbf{N}_{>0}$. Then*

$$\frac{\alpha^m}{nm} = \frac{\alpha}{n}.$$

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[3] In connection with the set $\mathbf{R} \cup \{-\infty, \infty\}$ we always use the conventions $(-\infty) \pm (-\infty) = -\infty$ and $-\infty < x$, $|-\infty| > x$ for all $x \in \mathbf{R}$. Furthermore, every positive real divides $\pm\infty$.

[4] See [2, Definition 3.1].

(v) *Let* $\alpha, \beta \in \mathscr{D}$ *and* $n, m \in \mathbf{N}_{>0}$. *Then*

$$\frac{\alpha}{n} < \frac{\alpha + \beta}{n + m}$$

implies $\alpha/n < \beta/m$.

It is well-known that new dioids can be formed on cartesian products of $\mathscr{D}$: Either we equip $\mathscr{D}^n$ with componentwise sum and product or we form the free symmetrized dioid on $\mathscr{D}$ (see [8, Sections 1.1 and 2.2]). Based on ideas of [1] we now introduce a different construction thereby extending the notion of a max-plus algebra.

DEFINITION 2.3. The dioid $(\mathscr{D}, \oplus, \odot)$ is called a quasi-max-plus algebra if it satisfies the following properties:

(i) There exist a max-plus algebra F and a totally ordered set S with at most two elements such that $\mathscr{D} \subseteq F \times S$.

(ii) The projection on the first component $\pi : \mathscr{D} \to F$ is a dioid homomorphism which enjoys the following properties:
 (a) $\pi^{-1}(\{\varepsilon_F\}) = \varepsilon$ and $\pi^{-1}(\{e_F\}) = e$.
 (b) For all $\alpha, \beta \in \mathscr{D}$ the following implication holds.[5]

$$\pi(\alpha) = \pi(\beta) \Rightarrow (\alpha \oplus \beta)_2 = \max\{\alpha_2, \beta_2\} \quad (\alpha, \beta \in \mathscr{D}).$$

(iii) The second components of the neutral elements of $\mathscr{D}$ equal the minimal element of S, i.e., we have $\varepsilon_2 = e_2 = \min S$.

(iv) For all $\alpha \in \mathscr{D}$ we have $(\alpha_1, \min S) \in \mathscr{D}$.

(v) For all $\alpha, \beta \in \mathscr{D} \backslash \{\varepsilon\}$ we have

$$(\alpha\beta)_2 = \max\{\alpha_2, \beta_2\}.$$

(vi) $\mathscr{D}$ is algebraically closed.

EXAMPLE 2.4. Plainly, every max-plus algebra can be regarded as a quasi-max-plus algebra: Just take S to be a singleton.

[5] We write x_i for the i-th component of the element x of the cartesian product of a family of sets $(X_i)_{i \in I}$.

In addition to the natural order we impose the lexicographical order on a quasi-max-plus algebra by

$$\alpha <_{\mathrm{lex}} \beta \Leftrightarrow \alpha_1 < \beta_1 \quad \text{or} \quad (\alpha_1 = \beta_1 \text{ and } \alpha_2 < \beta_2).$$

The following basic properties of a quasi-max-plus algebra will be used in the sequel.

Proposition 2.5. *Let $\mathscr{D}$ the quasi-max-plus algebra contained in $F \times S$ where F is a max-plus algebra and S a totally ordered set with at most two elements.*

(i) *$\mathscr{D}$ is a commutative unital entire semiring of characteristic zero with idempotent addition, and its neutral elements are $(\varepsilon, \min S)$ and $(e, \min S)$, respectively.*

(ii) *$\mathscr{D}$ is totally ordered by the lexicographical order, and $(\varepsilon, \min S)$ is its minimal element.*

(iii) *Let $\alpha, \beta \in \mathscr{D}$ with $\alpha \le \beta$. Then we have $\alpha \le_{\mathrm{lex}} \beta$.*

(iv) *Let $\alpha, \beta \in \mathscr{D}$ with $\alpha <_{\mathrm{lex}} \beta$. If $\gamma \in \mathscr{D} \backslash \{\varepsilon\}$ then $\alpha\gamma <_{\mathrm{lex}} \beta\gamma$.*

(v) *For all $\alpha, \beta \in \mathscr{D}$ and $n \in \mathbf{N}_{>0}$ the equation $\alpha^n = \beta^n$ implies $\alpha = \beta$.*

(vi) *For every $\alpha \in \mathscr{D}$ and $n \in \mathbf{N}_{>0}$ the equation $x^n = \alpha$ admits a unique solution in $\mathscr{D}$, namely $(\alpha_1/n, \alpha_2)$, and we also write this solution as*

$$\alpha/n.$$

(vii) *$\pi(\mathscr{D})$ is an algebraically closed max-plus algebra.*

(viii) *Let $\mathscr{E}$ be a subdioid of a $\mathscr{D}$. Then $\mathscr{E}$ is quasi-max-plus algebra if and only if $\mathscr{E}$ is algebraically closed and $(\alpha_1, \min S) \in \mathscr{E}$ for all $\alpha \in \mathscr{E}$.*

Proof. (i), (vii) This can easily be checked.

(ii) Let $\alpha, \beta \in \mathscr{D}$. By Proposition 2.2 we may assume $\alpha_1 \le \beta_1$, hence $\alpha \le_{\mathrm{lex}} \beta$.

(iii) By assumption we have $\alpha \oplus \beta = \beta$, thus $\alpha_1 \oplus \beta_1 = \beta_1$, hence $\alpha_1 \le \beta_1$. Thus we are done provided $\alpha_1 < \beta_1$ or $S = \varnothing$. Therefore, let us assume $\alpha_1 = \beta_1$ and $S \ne \varnothing$. Then we have $\alpha_2 \le \beta_2$ by definition.

(iv) Assume $\alpha\gamma \ge_{\mathrm{lex}} \beta\gamma$. Then we have

$$\alpha_1\gamma_1 = (\alpha\gamma)_1 \ge (\beta\gamma)_1 = \beta_1\gamma_1$$

which implies $\alpha_1 \ge \beta_1$. But then we have $\alpha \ge_{\mathrm{lex}} \beta$: Contradiction.

(v) If $\alpha = \varepsilon$ then clearly $\beta = \varepsilon$. Therefore we let $\alpha, \beta \neq \varepsilon$ and assume $\alpha <_{\mathrm{lex}} \beta$. From (iv) we infer

$$\alpha^2 <_{\mathrm{lex}} \alpha\beta <_{\mathrm{lex}} \beta^2$$

and analogously $\alpha^n <_{\mathrm{lex}} \beta^n$: Contradiction. Similarly we show the impossibility of $\alpha >_{\mathrm{lex}} \beta$ and then deduce our assertion.

(vi) The existence of a solution is clear by divisibility. In case $\alpha = \varepsilon$ the only solution is ε. Otherwise, uniqueness follows from the fact that $(\mathscr{D} \backslash \{\varepsilon\})_1$ is contained in a group.

(vii), (viii) Clear by the above. □

We now construct our principal example, namely a quasi-max-plus algebra over the nonnegative real numbers. On the set

$$\mathscr{R} := ((\mathbf{R}_{\geq 0} \cup \{-\infty\}) \times \{0,1\}) \backslash \{(-\infty, 1), (0, 1)\}$$

endowed with the usual order relation we introduce two binary operations:

$$(r, a) \oplus (s, b) = (\max\{r, s\}, \delta_+((r, a), (s, b))), \quad \text{and}$$
$$(r, a) \odot (s, b) = (r + s, \delta_\times((r, a), (s, b))),$$

where the functions $\delta_+, \delta_\times : \mathscr{R} \times \mathscr{R} \to \{0, 1\}$ are defined as follows. First, $\delta_+((r, a), (s, b)) = 1$ if one of the following four conditions is satisfied:

(i) $\max\{|r - s|, a, b\} = 1$,
(ii) $r > s + 1$ and $a = 1$,
(iii) $s > r + 1$ and $b = 1$,
(iv) $0 < |r - s| < 1$,

otherwise $\delta_+((r, a), (s, b)) = 0$. Second, $\delta_\times((r, a), (s, b)) = \max\{a, b\}$ if $r, s \in \mathbf{R}$, and $\delta_\times((r, a), (s, b)) = 0$, otherwise. Obviously, these definitions extend the one given in [1, Section 6.3].

Now we collect some properties of subsets of $\mathscr{R}$.

THEOREM 2.6. *Let $T \neq \{0\}$ be an additively closed subset of $\mathbf{R}_{\geq 0}$ and set*

$$\mathscr{T} = ((T \times \{0,1\} \backslash \{(0,1)\}) \cup \{e, \varepsilon\}, \oplus, \odot)$$

with $\varepsilon = (-\infty, 0)$ and $e = (0, 0)$.

(i) *$\mathscr{T}$ is a commutative entire archimedian dioid with neutral elements ε and e, respectively.*

(ii) *For all* $\alpha, \beta \in \mathscr{T}$ *with* $\beta \neq \varepsilon, e$ *there exists* $N \in \mathbf{N}$ *such that for all* $n \geq N$ *we have*

$$\alpha \oplus \beta^n = \beta^n.$$

(iii) $\mathscr{T}$ *satisfies the weak stabilization condition.*

(iv) *If* $t/n \in T$ *for all* $t \in T$ *and* $n \in \mathbf{N}_{>0}$ *then* $\mathscr{T}$ *is an algebraically closed quasi-max-plus algebra.*

Proof. We leave the rather lengthy, but straightforward verification to the reader. □

Remark 2.7. In general, $\mathscr{T}$ cannot be embedded into the standard max-plus algebra. Otherwise, for $t \in T \cap (0,1]$ we would have

$$(t,1) = (t,0) \oplus (0,0) \in \{(t,0),(0,0)\}$$

which is impossible.

3. Matrices Over Quasi-Max-Plus Algebras

In this section we let $\mathscr{D} \subseteq F \times S$ be a quasi-max-plus algebra where F is a max-plus algebra and S a linearly ordered set with at most two elements. For $r \in \mathbf{N}_{>0}$ the set of matrices $\mathscr{D}^{r \times r}$ is a dioid and a $\mathscr{D}$-semimodule where the matrix operations are defined as usual.

Let $A \in \mathscr{D}^{r \times r}$. The digraph[6] $\mathscr{G}(A)$ is the weighted digraph $([r], E, w)$ with vertex set $[r]$, edge set

$$E = \{(i,j) \in [r]^2 : A_{ij} \neq \varepsilon\}$$

and weight function $w : E \to \mathscr{D} \backslash \{\varepsilon\}$ with

$$w(i,j) = A_{ij}$$

for all $(i,j) \in E$; here we use the abbreviation $[r] = \{1, \dots, r\}$. We write $q' \subseteq_c q$ if the path q is a cycle extension of the path q', and we denote by $|q|$ the length of q. We let $P_A^{(n)}(i,j)$ ($P_{A,\mathrm{el}}(i,j)$, respectively) be the set of all paths of length n (all elementary paths, respectively) from i to j.

[6] If not stated otherwise we use the terminology of [11].

Let $q = (i_0, i_1, \dots, i_n)$ be a path of positive length n in $\mathscr{G}(A)$. We call $\operatorname{supp}(q) = \{i_0, i_1, \dots, i_n\}$ the support,

$$w(q) = A_{i_0, i_1} \odot \cdots \odot A_{i_{n-1}, i_n}$$

the weight and

$$\overline{w}(q) = \frac{w(q)}{|q|}$$

the mean weight of q (see Proposition 2.5).

Let $\mathscr{K}$ be a subgraph of $\mathscr{G}(A)$. $\mathscr{K}$ is said to be nontrivial if it contains at least one cycle of positive length. The high period of $\mathscr{K}$ is defined by

$$\operatorname{hper}(\mathscr{K}) = \gcd\{|c| : c \text{ cycle of positive length in } \mathscr{K} \text{ and } \overline{w}(c) = \mathscr{G}(A)\},$$

if the set on the right hand side is nonvoid, and $\operatorname{hper}(\mathscr{K}) = 0$, otherwise; here[7]

$$\lambda(\mathscr{K}) = \max\{\overline{w}(c) : c \text{ cycle of positive length in } \mathscr{K}\}$$

is the maximal cycle mean weight of $\mathscr{K}$ (cf. [10, p. 169]). Every cycle c in $\mathscr{K}$ with $\overline{w}(c) = \lambda(\mathscr{K})$ is called a critical cycle in $\mathscr{K}$, and a cycle in $\mathscr{G}(A)$ is critical if it takes the maximal mean weight $\lambda(A) := \lambda(\mathscr{G}(A))$. Further, we use the abbreviation SCC $\mathscr{G}(A)$ (SCC* $\mathscr{G}(A)$, respectively) for the set of strongly connected components (nontrivial strongly connected components, respectively) of $\mathscr{G}(A)$.

Following [10, p. 169] we say that the vertices i and j are highly connected if i and j are contained in a critical cycle; in this case we write $i \equiv_h j$. The subgraphs of $\mathscr{G}(A)$ induced by the classes of $\equiv_h$ are called highly connected components of $\mathscr{G}(A)$. A highly connected component is called trivial if it does not possess a cycle of positive length with cycle mean weight equal to $\lambda(A)$. Analogously as above, we denote by HCC $\mathscr{G}(A)$ (HCC* $\mathscr{G}(A)$, respectively) the set of highly connected components (nontrivial highly connected components, respectively) of $\mathscr{G}(A)$.

LEMMA 3.1. *Assume $\lambda(\mathscr{K}) = \lambda$ for all $\mathscr{K} \in$ SCC* $\mathscr{G}(A)$.*

(i) *We have $\lambda(A) = \lambda$.*

(ii) *Let $\lambda = e$. Then we have $w(c)_1 \le e$ for all cycles c of $\mathscr{G}(A)$, and for all $\mathscr{K}' \in$ SCC* $\mathscr{G}(A)$ there exists a $\mathscr{K} \in$ HCC* $\mathscr{G}(A)$ with $\mathscr{K} \subseteq \mathscr{K}'$. Furthermore, if c is a cycle in $\mathscr{G}(A)$ which consists of vertices belonging to critical cycles then c is critical.*

[7] Throughout we use the convention $\max \varnothing = \min S$ if $(S, \le)$ is an ordered set with a minimal element.

Proof. (i) Let c be a critical cycle in $\mathscr{G}(A)$. It is easy to see that there is a $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}(A)$ with $\mathrm{supp}(c) \subseteq \mathscr{K}$. Thus c is a cycle of $\mathscr{K}$ and therefore $\lambda(\mathscr{K}) \geq_{\mathrm{lex}} \overline{w}(c)$. Now the proof can easily be completed.

(ii) Clearly, we have

$$\frac{w(c)_1}{|c|} = \overline{w}(c)_1 \leq e,$$

thus $w(c)_1 \leq e^n = e$. Let $\mathscr{K}' \in \mathrm{SCC}^* \mathscr{G}(A)$ and choose a cycle c in $\mathscr{K}'$ with $\overline{w}(c) = e$. Then all vertices of c are highly connected, thus there is $\mathscr{K} \in \mathrm{HCC}^* G(A)$ with $\mathrm{supp}(c) \subseteq \mathscr{K}$. Let $k \in \mathscr{K}$. Then k is highly connected to a vertex of c, thus $k \in \mathscr{K}'$, and this shows $\mathscr{K} \subseteq \mathscr{K}'$. The last assertion is clear by [2, Theorem 3.96]. □

The following two technical lemmas provide an essential step in the proof of Theorem 3.5. For convenience we introduce

$$p_A = \mathrm{lcm}\{\mathrm{hper}(\mathscr{K}) : \mathscr{K} \in \mathrm{HCC}^* \mathscr{G}(A)\}.$$

Lemma 3.2. *There exists some $M \in \mathbf{N}$ such that for all $n \geq M$ and every $\mathscr{K} \in \mathrm{HCC}^* \mathscr{G}(A)$ there is a critical cycle c in $\mathscr{K}$ with the following properties.*

(i) $|c| = np_A$

(ii) *Every elementary critical cycle in $\mathscr{K}$ occurs in c at least once.*

Proof. This is a straightforward application of a well-known result of elementary number theory (e.g., see [15, Lemma A.3]). □

Lemma 3.3. *Let $\lambda(\mathscr{K}) = e$ for all $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}(A)$. There exists $N \in \mathbf{N}$ such that for all $n \geq N$, $i, j \in [r]$ and $q_0 \in P_{A,\mathrm{el}}(i, j)$ the following statements hold:*

(i) *Either there does not exist a cycle extension $q \in P_A(i, j)$ of q_0 with $|q| = n$ or every cycle extension $q \in P_A(i, j)$ of q_0 with $|q| \equiv n \pmod{p_A}$ and*

$$w(q) = \max\{w(q'') : q'' \in P_A^{(|q|)}(i, j), q_0 \subseteq_c q''\}$$

is of the form

$$q = q_0 c_1 \cdots c_t,$$

where $c_1, \ldots, c_t$ are critical cycles of $\mathscr{G}(A)$, and we have $w(q) = w(q_0)$ and $|q| \equiv |q_0| \pmod{p_A}$.

(ii) *If $q \in P_{A,*}^{(n)}(i, j)$ is a cycle extension of q_0, $m \in \mathbf{N}_{>0}$ and $\hat{q} := q_0 c_1 \cdots c_t \in P_A^{(n+mp_A)}(i, j)$ with critical cycles $c_1, \ldots, c_t$ then $\hat{q} \in P_{A,*}^{(n+mp_A)}(i, j)$. Here we* *set*

$$P^{(t)}_{A,*}(i,j) = \{q \in P^{(t)}(i,j) : w(q) \geq w(q') \ \text{for all} \ q' \in P^{(t)}_A(i,j)\}$$

for $t \in \mathbf{N}_{>0}$.

Proof. This can be proved analogously to [13, Lemma 3.2] using Lemma 3.2. $\square$

In order to mitigate the condition $\lambda(\mathscr{K}) = e$ for all $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}(A)$ we need some preparation. The reader is referred to the appendix for the periodicity notions we are using in the sequel.

Lemma 3.4. *Let $\mathscr{D}$ be a quasi-max-plus algebra, $A \in \mathscr{D}^{r\times r}$ and $B := \pi(A)$.*

(i) *$\mathscr{G}(A) = \mathscr{G}(\pi(A))$, in particular the cycles of $\mathscr{G}(A)$ and $\mathscr{G}(\pi(A))$ coincide.*

(ii) *$\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}(A)$ if and only if $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}(\pi(A))$.*

(iii) *Every critical cycle of $\mathscr{G}(A)$ is a critical cycle of $\mathscr{G}(\pi(A))$, and we have $\lambda(A)_1 = \lambda(\pi(A))$.*

(iv) *For each $\mathscr{K} \in \mathrm{HCC}^* \mathscr{G}(A)$ there is a $\mathscr{K}' \in \mathrm{HCC}^* \mathscr{G}(\pi(A))$ with $\mathscr{K} \subseteq \mathscr{K}'$.*

(v) *A is irreducible if and only if $\pi(A)$ is irreducible.*

(vi) *Assume that A has at least one cycle and $\lambda(A)_2 = \min S$. Then the critical cycles in $\mathscr{G}(A)$ and $\mathscr{G}(\pi(A))$ coincide.*

(vii) *If A is almost linear periodic then $\pi(A)$ is almost linear periodic, $\mathrm{ldef}(\pi(A)) \leq \mathrm{ldef}(A)$ and $\mathrm{lper}(\pi(A))$ divides $\mathrm{lper}(A)$, and $\pi(Q_{ij})$ are the entries of a linear factor matrix of $\pi(A)$ where Q is a linear factor matrix of A.*

(viii) *If $\lambda(\mathscr{K}) = \lambda(\mathscr{K}')$ for all $\mathscr{K}, \mathscr{K}' \in \mathrm{SCC}^* \mathscr{G}(A)$ then $\lambda(\mathscr{K}) = \lambda(\mathscr{K}')$ for all $\mathscr{K}, \mathscr{K}' \in \mathrm{SCC}^* \mathscr{G}(B)$.*

Proof. (i), (ii) This can easily be checked.

(iii) We only show $\lambda(A)_1 = \lambda(\pi(A))$ using (i). Assume $\lambda(A)_1 > \lambda(\pi(A)) =: \lambda$. Then there exists a cycle c in $\mathscr{G}(A)$ with $\overline{w}(c)_1 > \lambda$, but c is a cycle in $\mathscr{G}(\pi(A))$: Contradiction. Thus we have $\lambda(A)_1 \leq \lambda$. The assumption of strict inequality leads to a cycle c' in $\mathscr{G}(\pi(A))$ with $\overline{w}(c')_1 > \lambda(A)_1$, hence $\overline{w}(c) > \lambda(A)$: Contradiction.

(iv) Let $i, j \in \mathscr{K}$ and c a critical cycle with vertices i, j. Then $\overline{w}(c)_1 = \lambda(\pi(A))$ by (iii), thus there is a $\mathscr{K}' \in \mathrm{HCC}^* \mathscr{G}(\pi(A))$ which contains every vertex of c. We easily check $\mathscr{K} \subseteq \mathscr{K}'$.

(v) Clear by (i).

(vi) In view of (iii) it suffices to show that every critical cycle c in $\mathscr{G}(\pi(A))$ is a critical cycle in $\mathscr{G}(A)$. Suppose to the contrary that there is a cycle c' in $\mathscr{G}(A)$ with $\overline{w}(c) < \overline{w}(c')$. The same relation then holds for the first components because

$$w(c)_2 \geq \lambda(A)_2 = w(c')_2,$$

and we deduce the contradiction

$$\overline{w}(\pi(c)) = \frac{w(\pi(c))}{|c|} = \frac{w(c)}{|c|} < \frac{w(c')_1}{|c'|} = \frac{w(\pi(c'))}{|c'|} = \overline{w}(\pi(c')).$$

(vii) By assumption the sequence $A^* := (A^n)_{n\in\mathbf{N}}$ is almost linear periodic. Let $i, j \in [r]$, $p_{ij} := \operatorname{lper}(A^*)_{ij} \in \mathbf{N}_{>0}$ and $\lambda_{ij} \in \mathscr{D}$ with

$$(A^{n+p_{ij}})_{ij} = (A^n)_{ij}\lambda_{ij}^{p_{ij}} \quad (n > \operatorname{ldef}(A^*)_{ij}).$$

In particular, this holds for the first components, hence $\pi(A)$ is almost linear periodic, and $\operatorname{ldef}(\pi(A)) \leq \operatorname{ldef}(A)$. Furthermore, using Proposition 2.5 we check that $\operatorname{lper}(\pi(A^*))_{ij}$ divides p_{ij}, hence

$$\operatorname{lper}(\pi(A)) = \operatorname{lcm}\{\operatorname{lper}(\pi(A^*))_{ij} : i, j \in [r]\} \mid \operatorname{lcm}\{p_{ij} : i, j \in [r]\} = \operatorname{lper}(A).$$

The proof can now easily be concluded.

(viii) Let $\mathscr{K}, \mathscr{K}' \in \operatorname{SCC}^* \mathscr{G}(B)$. Then $\mathscr{K}, \mathscr{K}' \in \operatorname{SCC}^* \mathscr{G}(A)$ by (ii) and $\lambda(\mathscr{K})_1 = \lambda(\mathscr{K}')_1$ by assumption. This shows that there cannot be a cycle c in $\mathscr{K}$ with $\overline{w}(\pi(c))_1 > \lambda(\mathscr{K}')_1$ which implies our assertion. □

Now we can state our first main result which slightly extends the structural part of [10, Theorem 3.1] and generalizes a classical theorem on matrices over the standard max-plus algebra [2, Section 3.7].

Theorem 3.5. *Let $\mathscr{D}$ be a max-plus algebra, $\lambda \in \mathscr{D}\backslash\{\varepsilon\}$ and $A \in \mathscr{D}^{r\times r}$ with $\lambda(\mathscr{K}) = \lambda$ for all $\mathscr{K} \in \operatorname{SCC}^* \mathscr{G}(A)$. Then A is almost linear periodic, $\operatorname{lper}(A) = p_A$ and $(\operatorname{lfac}(A))_{ij} = \lambda$ for all $i, j \in [r]$. More explicitly, we have*

$$(A^{n+p_A})_{ij} = \lambda^{p_A}(A^n)_{ij} \quad (i, j \in [r])$$

for all sufficiently large n.

Proof. By Lemma 3.1 we have $\lambda(A) = \lambda$, and for $B := (-\lambda)A$ we have $\lambda(B) = e$ by Lemma 4.7.

Let $p := p_A$ and N be a constant given by Lemma 3.3, $n \geq N$ and $i, j \in [r]$. Assume that there is some $q \in P_{A,\mathrm{el}}(i, j)$. By [10, Lemma 2.1] we have $(B^n)_{ij} = a_q^{(n)}$, and by Lemma 3.3 we have $(B^{n+p_B})_{ij} = a_q^{(n)}$, hence

$$(B^{n+p_B})_{ij} = (B^n)_{ij}. \tag{1}$$

On the other hand, if $P_{A,\mathrm{el}}(i, j) = \emptyset$ then both sides of (1) equal ε by Lemma 3.3.

Thus B is almost linear periodic with $\mathrm{lfac}(B) = e$ by Lemma 4.6 and $\mathrm{lper}(B)$ divides p_B by Proposition 2.2. Arguing as in the proof of [10, Lemma 3.3] we see that $\mathrm{lper}(B)$ cannot be smaller than p_B, thus $\mathrm{lper}(B) = p_B$.

An application of Lemma 4.7 concludes the proof. □

We can certainly recover the first part of [10, Theorem 3.1].

Corollary 3.6. *Let $\mathscr{D}$ be a max-plus algebra and $A \in \mathscr{D}^{r\times r}$ be irreducible. Then A is almost linear periodic, and we have $\mathrm{lper}(A) = p_A$ and $\mathrm{lfac}(A) = \lambda(A) \neq \varepsilon$.*

Proof. As $\mathrm{SCC}^* \mathscr{G}(A)$ is a singleton the assertion drops out of the Theorem. □

Now we establish the analog of [5, Theorem 2.4].

Theorem 3.7. *Let $\mathscr{D}$ be a quasi-max-plus algebra with weak stabilization condition. Let $\mathscr{E}$ be a subdioid of $\mathscr{D}$ and $A \in \mathscr{E}^{r\times r}$ be irreducible. Then A and $\pi(A)$ are almost linear periodic and $\lambda(A) \neq \varepsilon$. Furthermore, we have $\mathrm{ldef}(A) \geq \mathrm{ldef}(\pi(A))$, $p_{\pi(A)}$ divides $p := \mathrm{lper}(A)$, $(\lambda(A)_1, \min \mathscr{D}_2)^p \in \mathscr{E}$ and*

$$A^{n+p} = (\lambda(A)_1, \min \mathscr{D}_2)^p A^n \quad (n > \mathrm{ldef}(A)).$$

Proof. (i) Using Lemma 3.4 (v), (iii) and Corollary 3.6 we find that the matrix $B := \pi(A)$ is irreducible and almost linear periodic, and $\mathrm{lper}(B) = p_B$ and $\mathrm{lfac}(B) = \lambda_1$ with $\lambda := \lambda(A)$.

(ii) Let $i, j \in [r]$. By the above we have

$$(A^{n+tp_B})_{ij1} = (B^{n+tp_B})_{ij} = (B^n)_{ij}\lambda_1^{tp_B} = \lambda_1^{tp_B}(A^n)_{ij1} \quad (n > \mathrm{ldef}(B),\ t \in \mathbf{N}_{>0}). \tag{2}$$

Furthermore, from [8, Proposition 1.2.2] we infer the existence of $\lambda_{ij} \in \mathscr{D}$ and $N_{ij}, p_{ij} \in \mathbf{N}_{>0}$ such that

$$(A^{n+tp_{ij}})_{ij} = \lambda_{ij}^t (A^n)_{ij} \quad (n \geq N_{ij},\ t \in \mathbf{N}_{>0}). \tag{3}$$

(iii) Let $N = \max\{\mathrm{ldef}(B), N_{ij} : i, j \in [r]\}$ and $p = \mathrm{lcm}\{p_B, p_{ij} : i, j \in [r]\}$. By (2) and (3) we have for all $i, j \in [r]$

$$(A^{n+p})_{ij1} = \lambda_1^p (A^n)_{ij1} = \lambda_{ij1}^{p/p_{ij}} (A^n)_{ij1} \quad (n > N).$$

Now, Proposition 2.2 (ii) yields

$$\lambda_{ij1}^{p/p_{ij}} = \lambda_1^p,$$

hence by (3) and the properties of a quasi-max-plus algebra

$$(A^{n+p})_{ij} = (\lambda_1^p, \lambda_{ij2})(A^n)_{ij} = (\lambda_1, \lambda_{ij2})^p (A^n)_{ij} \quad (n > N).$$

Thus A is almost linear periodic, and from Lemma 3.4 (vii) we know that p_B divides $\mathrm{lper}(A)$ and $\mathrm{ldef}(A) \geq \mathrm{ldef}(B)$).

(iv) In case $\mathrm{Card}\, \mathscr{D}_2 = 1$ we are done. Otherwise we write $\mathscr{D}_2 = \{0, 1\}$ and show that there is some $M \geq N$ such that

$$(A^{n+p})_{ij} = (\lambda_1, 0)^p (A^n)_{ij} \quad (i, j \in [r],\ n > M). \tag{4}$$

Note that by the definition of p we have $(\lambda_1, 0)^p \in \mathscr{E}$. We distinguish two cases.

Case 1. For all $i, j \in [r]$ and $n > N$ we have $(A^n)_{ij2} = 0$.

In this case we set $M = N$.

Case 2. There is some $i, j \in [r]$ and $n_{ij} > N$ such that $(A^n)_{ij2} = 1$.

For all i, j with this property we fix some n_{ij}. Now we choose M to be the maximal n_{ij} and check that (4) is satisfied. This completes the proof. □

Now we study eigenvalues of certain matrices over quasi-max-plus algebras.

Theorem 3.8. *Let $\mathscr{D}$ be a quasi-max-plus algebra and $A \in \mathscr{D}^{r \times r}$.*

(i) *If A is nilpotent then ε is the unique eigenvalue of A.*

(ii) *Let $\mathscr{D}$ satisfy the weak stabilization condition and A be irreducible. Then ε is not an eigenvalue of A, but $(\lambda(A)_1, \min \mathscr{D}_2)$ is an eigenvalue of A.*

(iii) *Suppose $\mathscr{D}_2 = \{0, 1\}$, $(\lambda(A)_1, 1) \in \mathscr{D}$ and $(v_{11}, 1), \ldots, (v_{r1}, 1) \in \mathscr{D}$ for some eigenvector $(v_1, \ldots, v_r) \in \mathscr{D}^r$ of A with eigenvalue $(\lambda(A)_1, 0)$. Then $(\lambda(A)_1, 1)$ is an eigenvalue of A with eigenvector $((v_{11}, 1), \ldots, (v_{r1}, 1))$.*

PROOF. (i) This is a well-known classical result.

(ii) The first part is clear by Lemma 4.3, and for the second part we closely follow [6, proof of Teorema 1]. By Theorem 3.7 there exist $n, m \in \mathbf{N}$ such that $m > n > 0$ and

$$A^m = \lambda^{m-n} A^n$$

where we set $\lambda := (\lambda(A)_1, \min \mathscr{D}_2) \in \mathscr{D} \backslash \{\varepsilon\}$. Then Theorem 4.5 yields our assertion.

(iii) For each $i \in [r]$ the equation

$$\bigoplus_{j \in [r]} A_{ij} v_j = (\lambda(A)_1, 0) v_i$$

implies

$$\bigoplus_{j \in [r]} A_{ij} (v_{j1}, 1) = (\lambda(A)_1, 1)(v_{i1}, 1). \qquad \square$$

We illustrate Theorem 3.8 by two easy examples.

EXAMPLE 3.9. (i) The only eigenvalue of the matrix $(0,0) \in \mathscr{R}$ is $(0,0)$, and every element in $\mathscr{R} \backslash \{(-\infty, 0)\}$ is an eigenvector.

(ii) The matrix $(1,0) \in \mathscr{R}$ has exactly two eigenvalues, namely $(1,0)$ and $(1,1)$. In both cases $(x, 1)$ is an eigenvector provided $x > 0$.

Finally, we extend Theorem 3.7 and [1, Proposition 6.19] for the particular dioid $\mathscr{R}$.

THEOREM 3.10. *Let $T \neq \{0\}$ be an additively closed subset of $\mathbf{R}_{\geq 0}$ and $\mathscr{T}$ be defined as in Theorem 2.6. Further, let $A \in \mathscr{T}^{r \times r}$ be a non-nilpotent matrix and assume $\lambda(\mathscr{K}) = \lambda(\mathscr{K}')$ for all $\mathscr{K}, \mathscr{K}' \in \mathrm{SCC}^* \mathscr{G}(A)$. Then A and $\pi(A)$ are almost linear periodic. Furthermore, we have $\mathrm{ldef}(A) \geq \mathrm{ldef}(\pi(A))$, $(\lambda(A)_1, 0)^p \in \mathscr{T}$ and $p := \mathrm{lper}(A)$ divides $p_{\pi(A)}$*

$$A^{n+p} = (\lambda(A)_1, 0)^p A^n \quad (n > \mathrm{ldef}(A)).$$

Thus

$$(\lambda(A)_1, 0)_{i,j \in [r]} \in \mathscr{R}^{r \times r}$$

defines a linear factor matrix of A.

PROOF. As A is non-nilpotent the digraph $\mathscr{G}(A)$ has at least one cycle of positive length. In view of Lemma 3.4 (viii) and Lemma 3.1 (i) we have $\lambda(\mathscr{A}) = \lambda(\mathscr{K}) = \lambda(\mathscr{K}')$ for all $\mathscr{K}, \mathscr{K}' \in \mathrm{SCC}^* \mathscr{G}(B)$ where we set $B := \pi(A)$. We infer from Theorem 3.5 that B is almost linear periodic, $\lambda := \lambda(\mathscr{A})_1$ defines the linear factor of B, and $\mathrm{lper}(B) = p_B$. Thus $(\lambda, 0)^{p_B} \in \mathscr{T}$ and there exists $N \in \mathbf{N}$ such that

$$((A^{n+p_B})_{ij})_1 = (B^{n+p_B})_{ij} = (B^n)_{ij}\lambda^{p_B} = ((A^n)_{ij})_1\lambda^{p_B} = ((A^n)_{ij_1}(\lambda, 0)_1)^{p_B}. \tag{5}$$

Now, Lemma 3.4 shows that (5) also holds for the second components provided n is large enough. This means that A is almost linear periodic, $(\lambda, 0)_{i,j \in [r]}$ defines a linear factor matrix of A and the relation between the linear defects of A and B is clear by Lemma 3.4. Finally, Lemma 4.1 yields that $\mathrm{lper}(A)$ divides p_B. □

A central result of [13, Theorem 3.1] can now easily be generalized.

COROLLARY 3.11. *Let $A \in \mathscr{R}^{r\times r}$ and assume that $\mathscr{G}(A)$ has at least one cycle of positive length. The matrix A is eventually periodic if and only if $\lambda(\mathscr{K}) = e$ for all $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}(A)$. In this case* per A *divides p_A.*

PROOF. We first observe that [13, Theorem 3.1] holds for any matrix over a max-plus algebra whose graph has at least one cycle of positive length.

Let A be eventually periodic. Then $B := \pi(A)$ is eventually periodic, hence for any $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}(\pi(A))$ we have $\lambda(\mathscr{K})_1 = e$ by [13, Theorem 3.1]. Then Lemma 3.4 (ii) yields $\lambda(\mathscr{K}) = e$ for all $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}(A)$.

Conversely, if $\lambda(\mathscr{K}) = e$ for all $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}(A)$ then Theorem 3.10 yields

$$(A^{n+p})_{ij} = (A^n)_{ij} \quad (n \geq N,\ i, j \in [r])$$

with some $N \in \mathbf{N}$, and an application of Lemma 4.1 completes the proof. □

4. Appendix

Let $(S, \cdot)$ be an abelian semigroup and $a^* = (a_n)_{n\in\mathbf{N}}$ be a sequence of elements of S. Following [10, Definition 2.3, 2.4] we say that a^* is almost linear periodic[8] if there are $N \in \mathbf{N}$, $p \in \mathbf{N}_{>0}$ and $b \in S$ such that

$$a_{n+p} = b^p \cdot a_n \quad (n > N). \tag{6}$$

In this case the smallest $p \in \mathbf{N}_{>0}$ such that there are $N \in \mathbf{N}$ and $b \in S$ with (6) is called the linear period of a^*, and we write $p = \operatorname{lper} a^*$. The minimal $N \in \mathbf{N}$ such that there is some $b \in S$ which satisfies (6) for $p = \operatorname{lper} a^*$ is called the linear defect of a^*, and we write $N = \operatorname{ldef} a^*$. Finally, an element b with (6) for $p = \operatorname{lper} a^*$ and $N = \operatorname{ldef} a^*$ is called a linear factor of a^*. In case b is unique we write $b = \operatorname{lfac} a^*$.

LEMMA 4.1. *Let $(S, \cdot)$ be an abelian cancellative semigroup and assume that for all $x, y \in S$ and $n \in \mathbf{N}_{>0}$ the equation $x^n = y^n$ implies $x = y$. Further, let $a^* = (a_k)_{k \in \mathbf{N}}$ be an almost linear periodic sequence in S, i.e., there exist $b \in S$, $N \in \mathbf{N}$ and $m \in \mathbf{N}_{>0}$ with*

$$a_{n+m} = a_n b^m$$

for every $n > N$.

(i) *We have*

$$a_{n+tm} = a_n b^{tm}$$

for every $t \in \mathbf{N}_{>0}$ and $n > N$.

(ii) $\operatorname{lfac}(a^*) = b$.

(iii) $\operatorname{lper}(a^*)$ *divides m.*

PROOF. (i) Clear by induction.

(ii) Assume that there are $c \in S$, $M \in \mathbf{N}$ and $k \in \mathbf{N}_{>0}$ with

$$a^{n+k} = a^n c^k$$

for all $n > M$. For large enough n we then have by (i)

$$a_n c^{km} = a_{n+km} = a_n b^{km},$$

which yields $c^{km} = b^{km}$ and then $c = b$.

(iii) Clearly, we have $p := \operatorname{lper}(a^*) \le m$. Write $m = qp + r$ with $q, r \in \mathbf{N}$, $q > 0$ and $r < p$. If $r = 0$ we are done. Otherwise, applying (i) again we find for large enough n

$$a_n b^{qp+r} = a_n b^m = a_{n+m} = a_{(n+r)+qp} = a_{n+r} b^{qp},$$

[8] Our definition slightly differs from the one given in [1, Definition 6.1].

hence

$$a_n b^r = a_{n+r},$$

contradicting the definition of p. $\square$

We collect some well-known results for commutative unital semirings. The first one is stated in [3, Theorem 3.24] for matrices over the complex numbers. However, it is easy to see that it remains true in a more general setting (cf. [5, Theorem 2.5], [7, Assertion 7.2]).

Theorem 4.2 [3, Theorem 3.24]. *Let S be a commutative unital semiring and $M \in S^{r\times r}$. Then there exists a permutation matrix $P \in S^{n\times n}$ such that*

$$PMP^T = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ 0 & M_{22} & \cdots & M_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & M_{nn} \end{pmatrix}. \tag{7}$$

Here $n \geq 1$ and the blocks $M_{11}, \dots, M_{nn}$ that occur in the diagonal in (7) *are square matrices which are either* 0 *or irreducible.*[9] *The blocks $M_{11}, \dots, M_{nn}$ are uniquely determined to within simultaneous permutation of their rows and columns, but their ordering in* (7) *is not necessarily unique. The form on the right hand side of* (7) *is called the Frobenius normal form of M.*

For the study of the ultimate behavior of the sequence of powers of a matrix over S we recall the following definitions. The matrix $A \in S^{r\times r}$ is called

- eventually periodic if the sequence $A^* := (A^n)_{n\in\mathbf{N}}$ is eventually periodic (see for instance [13, Definition 2.4] where the notion 'almost periodic' was coined for this property); in this case we write

$$\operatorname{per}(A) = \operatorname{per}(A^*).$$

- almost linear periodic if for all $i, j \in [r]$ the sequence

$$(A^*)_{ij} = ((A_n)_{ij})_{n\in\mathbf{N}} \in S^{\mathbf{N}}$$

[9] Observe the different notion of irreducibility in [3].

is almost linear periodic. In this case we write

$$\mathrm{ldef}(A) = \mathrm{ldef}(A^*), \quad \mathrm{lper}(A) = \mathrm{lper}(A^*).$$

If there is a unique linear factor we write $\mathrm{lfac}(A) = \mathrm{lfac}(A^*)$.

From now on we let $\mathscr{D}$ be a commutative entire dioid. The following result is well-known for the standard max-plus algebra (e.g., see [4]).

Lemma 4.3. *Let $A \in \mathscr{D}^{r\times r}$ be irreducible and assume $A \neq (\varepsilon)$. Further, let $\lambda \in \mathscr{D}$ be an eigenvalue of A with eigenvector $v \in \mathscr{D}^r$. Then $\lambda \neq \varepsilon$ and $v_i \neq \varepsilon$ for all $i \in [r]$.*

Proof. By definition we have $v_j \neq \varepsilon$ for some $j \in [r]$. For all $i,k \in [r]$ we have

$$A_{ki}v_i \leq \lambda v_k. \tag{8}$$

Let us assume $\lambda = \varepsilon$. Then by (8) we find

$$A_{ki}v_i = \varepsilon$$

for all $i,k \in [r]$, thus in particular $A_{kj}v_j = \varepsilon$ and then $A_{kj} = \varepsilon$ for all $k \in [r]$ which is impossible.

Let us now assume $v_k = \varepsilon$ for some $k \in [r]$ and pick $n \in \mathbf{N}_{>0}$ such that $(A^n)_{kj} \neq \varepsilon$. Then we are lead to the contradiction

$$\varepsilon < (A^n)_{kj}v_j \leq \lambda^n v_k = \varepsilon. \qquad \square$$

Lemma 4.4. *Let $M \in \mathscr{D}^{r\times r}$, $\lambda \in \mathscr{D}\backslash\{\varepsilon\}$ and $n,m \in \mathbf{N}$ such that $m > n > 0$, $M^n \neq \varepsilon$ and*

$$M^m = \lambda^{m-n}M^n.$$

Then λ is an eigenvalue of M.

Proof. Our proof is taken from [6, proof of Teorema 1]. For the convenience of the reader we give the details here. Let $z \in \mathscr{D}^r$ such that $y := M^n z \neq \varepsilon$. Then we have

$$M^{m-n}y = M^m z = \lambda^{m-n}M^n z = \lambda^{m-n}y. \tag{9}$$

Therefore, the vector

$$x := \bigoplus_{i=0}^{m-n-1} \lambda^i M^{m-n-1-i}y$$

is not the zero vector because otherwise we have $\lambda^i M^{m-n-1-i} y = \varepsilon$ for all $i = 0, \dots, m-n-1$ which implies

$$\lambda^i M^{m-1-i} z = \varepsilon \quad (i = 0, \dots, m-n-1)$$

yielding the contradiction

$$\lambda^{m-n-1} y = \lambda^{m-n-1} M^{m-1-(m-n-1)} z = \varepsilon.$$

Thus x is an eigenvector of M with eigenvalue λ because by (9) we have

$$Mx = \bigoplus_{i=0}^{m-n-1} \lambda^i M^{m-n-i} y = M^{m-n} y \oplus \bigoplus_{i=1}^{m-n-1} \lambda^i M^{m-n-i} y$$

$$= \lambda^{m-n} y \oplus \bigoplus_{i=1}^{m-n-1} \lambda^i M^{m-n-i} y = \bigoplus_{i=1}^{m-n} \lambda^i M^{m-n-i} y = \lambda x. \qquad \square$$

Theorem 4.5. *Let $\mathscr{D}$ be a commutative entire dioid and $M \in \mathscr{D}^{r \times r}$ be not nilpotent. Further, let $\lambda \in \mathscr{D} \backslash \{\varepsilon\}$ and $n, m \in \mathbf{N}$ such that $m > n > 0$ and*

$$M^m = \lambda^{m-n} M^n.$$

Then λ is an eigenvalue of M.

Proof. We use induction on r and closely follow the proof of [6, proof of Teorema 1]. If M is irreducible then $M^n \neq \varepsilon$ and we are done by Lemma 4.4. Now, let M be reducible, hence $r > 1$ and by Theorem 4.2 we find a permutation matrix $P \in \mathscr{D}^{r \times r}$ such that we can write

$$PMP^T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

with square matrices A, B of smaller size than M. At least one of these matrices is not nilpotent because otherwise M is nilpotent. W.l.o.g. we assume that $A \in \mathscr{D}^{k \times k}$ is not nilpotent. Then we find

$$A^m = \lambda^{m-n} A^n,$$

hence by induction hypothesis there is an eigenvector $a \in \mathscr{D}^k$ of A with eigenvalue λ. Now, the vector $P^T v^T$ with $v := (a, \varepsilon, \dots, \varepsilon) \in \mathscr{D}^r$ is an eigenvector of M with eigenvalue λ. $\square$

We formally state some results which were implicitly used in [10].

LEMMA 4.6. *Let $\mathscr{D}$ be a max-plus algebra and $A \in \mathscr{D}^{r \times r}$ be almost linear periodic. Then A has a unique linear factor.*

PROOF. Clear by Proposition 2.2 and Lemma 4.1. $\square$

LEMMA 4.7. *Let $\mathscr{D}$ be a max-plus algebra, $\alpha \in \mathscr{D} \backslash \{\varepsilon\}$, $A \in \mathscr{D}^{r \times r}$ and $B = \alpha A$.*
(i) *We have $\mathscr{G}(A) = \mathscr{G}(B) =: \mathscr{G}$ and $p_A = p_B$.*
(ii) *For every path q of positive length in $\mathscr{G}$ we have $\overline{w}_B(q) = \alpha \overline{w}_A(q)$.*
(iii) *The set of critical cycles w.r.t. $\overline{w}_A$ coincides with the set of critical cycles w.r.t. $\overline{w}_B$.*
(iv) *If $\mathscr{K} \in \mathrm{SCC}^* \mathscr{G}$ and $\alpha = -\lambda_A(\mathscr{K})$ then $\lambda_B(\mathscr{K}) = e$.*
(v) *A is almost linear periodic if and only if B is almost linear periodic. In this case we have $\mathrm{lper}(A) = \mathrm{lper}(B)$, $\mathrm{ldef}(A) = \mathrm{ldef}(B)$ and $\mathrm{lfac}(B) = \alpha\, \mathrm{lfac}(A)$.*

PROOF. (i) Obvious.
(ii) We have

$$\overline{w}_B(q) = \frac{w_B(q)}{|q|} = \frac{w_A(q)\alpha^{|q|}}{|q|} = \overline{w}_A(q)\alpha.$$

(iii) Clear by (i) and (ii).
(iv) This can easily be checked.
(v) Let A be almost linear periodic with $p = \mathrm{lper}(A)$ and $\lambda = \mathrm{lfac}(A)$, hence

$$B^{n+p} = \alpha^{n+p} A^{n+p} = \alpha^n A^n \lambda^p \alpha^p = B^n (\lambda\alpha)^p.$$

for $n > \mathrm{ldef}(A)$. Thus B is almost linear periodic with $\mathrm{lper}(B) \leq p$. However, strict inequality is impossible, hence $\mathrm{lper}(B) = p$. Similarly, we find $\mathrm{ldef}(B) = \mathrm{ldef}(A)$, and finally $\mathrm{lfac}(B) = \lambda\alpha$ in view of Lemma 4.6.

Conversely, let B be almost linear periodic with $q = \mathrm{lper}(B)$ and $\mu = \mathrm{lfac}(B)$, hence

$$\alpha^{n+q} A^{n+q} = B^{n+q} = B^n \mu^q = \alpha^n A^n \mu^q$$

for $n > \mathrm{ldef}(B)$. As $\mathscr{D}$ is cancellative we have

$$A^{n+q} = (-\alpha)^q \alpha^q A^{n+q} = A^n (-\alpha)^q \mu^q = A^n (-\alpha\mu)^q,$$

and the proof can be completed analogously as above. $\square$

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