

INTERACTIVE INFINITE MARKOV PARTICLE SYSTEMS WITH JUMPS

By

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Abstract. In [2] we investigated independent infinite Markov particle systems as measure-valued Markov processes with jumps, and we gave sample path properties and martingale characterizations. In particular, we investigated the exponent of Hölder-right continuity in case that the motion process is *absorbing α -stable motion on $(0, \infty)$* with $0 < \alpha < 2$, that is, time-changed absorbing Brownian motions on $(0, \infty)$ by the increasing $\alpha/2$ -stable Lévy processes.

In the present paper we shall extend the results to the case of simple interactive infinite Markov particle systems. We also consider the absorbing stable motion on a half space $H = \mathbf{R}^{d-1} \times (0, \infty)$ as a motion process.

1. Settings and Previous Results

In this section we give the general settings and the known results which are given in [2] in order to describe the main results in §3 and §4.

Let S be a domain of \mathbf{R}^d . Let $(w(t), P_x)_{t \geq 0, x \in S}$ be a S -valued Markov process having life time $\zeta(w) \in (0, \infty]$ such that $w \in \mathbf{D}([0, \zeta(w)) \rightarrow S)$, i.e., $w : [0, \zeta(w)) \rightarrow S$ is right continuous and has left-hand limits. For convenience, we fix an extra point $\Delta \notin S$ and set $w(t) = \Delta$ if $t \geq \zeta(w)$. Moreover we shall extend functions f on S to on $\{\Delta\}$ by $f(\Delta) = 0$, if necessary.

We use the following notations: Let $S \subset \mathbf{R}^d$ be a domain.

- If $x = (x_1, \dots, x_d) \in \mathbf{R}^d$, then $\partial_{i_1 \dots i_k}^k = \partial^k / (\partial x_{i_1} \dots \partial x_{i_k})$, $\partial_i^k = \partial^k / (\partial x_i^k)$ and $\partial_i = \partial_i^1$ for each $k = 0, 1, \dots, i = 1, \dots, d$. Moreover $\partial_t = \partial / \partial t$ for time $t \geq 0$.

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- $f \in C_c \equiv C_c(S) \stackrel{\text{def}}{\iff} f$ is a continuous function on S with compact support in S , and $C_c^\infty \equiv C_c^\infty(S) := C_c(S) \cap C^\infty(S)$.
- For each integer $k \geq 0$, $C_b^k := C_b^k(\mathbf{R}^d)|_S$, that is, $f \in C_b^k \stackrel{\text{def}}{\iff} f$ is a restriction to S of k -times continuously differentiable function on \mathbf{R}^d with bounded derivatives of order between 0 and k . Moreover $f \in C_0 \stackrel{\text{def}}{\iff} f$ is continuous on S and $f(x) \rightarrow 0$ whenever $x \rightarrow \partial S$ or $|x| \rightarrow \infty$. Furthermore $C_b := C_b^0$, $C_b^\infty := \bigcap_k C_b^k$, $C_0^k := C_0 \cap C_b^k$ and $C_0^\infty := \bigcap_k C_0^k$.
- For a function space D on S , $f \in D^+ \stackrel{\text{def}}{\iff} f \in D$; $f \geq 0$.
- $\langle \mu, f \rangle := \int_S f(x) \mu(dx)$ for a function f on S and a measure μ on S .

The following two assumptions are the same as in [2].

ASSUMPTION 1. Let $(P_t)_{t \geq 0}$ be the transition semigroup of $(w(t), P_x)$, i.e., $P_t f(x) = E_x[f(w(t)) : t < \zeta]$.

- (i) (P_t) is a strongly continuous nonnegative contraction semigroup on $(C_0, \|\cdot\|_\infty)$ with generator $(A, \mathcal{D}(A))$, where $\|f\|_\infty = \sup_{x \in S} |f(x)|$.
- (ii) $C_c^\infty \subset \mathcal{D}(A)$ and there is a strictly positive function $g_0 \in C_0^\infty$ such that $g_0 \in \mathcal{D}(A)$ and that $g_0^{-1} A f \in C_b$ with $g_0^{-1} = 1/g_0$ for every $f \in C_c^\infty$ and $f = g_0$.
- (iii) $\sup_{t \leq T} \|g_0^{-1} P_t g_0\|_\infty < \infty$ for every $T > 0$.

Under this assumption we introduce a function space $D_{g_0} \subset \mathcal{D}(A)$ as follows:

$$f \in D_{g_0} \stackrel{\text{def}}{\iff} f \in \mathcal{D}(A) \quad \text{such that} \quad \|g_0^{-1} f\|_\infty < \infty \quad \text{and} \quad \|g_0^{-1} A f\|_\infty < \infty.$$

Clearly $g_0 \in D_{g_0}$ and $C_c^\infty \subset D_{g_0}$. Moreover since C_c^∞ is dense in C_0 and $P_t C_c^\infty \subset D_{g_0}$ for every $t \geq 0$, D_{g_0} is a core for A . However, D_{g_0} may be too large, so we further need the following assumption:

ASSUMPTION 2. There exist a bounded function $g_1 \in C^\infty$; $g_1 \geq g_0 (> 0)$ and a core $D \subset D_{g_0}$ (we denote $D = D_g$ with $g = (g_0, g_1)$) satisfying the following:

- (i) If $f \in D_g$, then $\lim_{t \downarrow 0} \frac{1}{t} (P_t f^2(x) - f(x)^2)$ exists for each $x \in S$ (we also denote the limit as $A f^2(x)$), $\partial_t P_t f^2(x) = A P_t f^2(x) = P_t A f^2(x)$ ($\rightarrow A f^2(x)$ as $t \downarrow 0$ for each $x \in S$), $A f^2 \in C_b$ and $\|g_1^{-1} A f^2\|_\infty < \infty$.
- (ii) For each $T > 0$, $\sup_{t \in [0, T]} \|g_1^{-1} P_t g_1\|_\infty < \infty$.
- (iii) For each $0 < s < T$, $\sup_{t \in [s, T]} \|g_0^{-1} P_t g_1\|_\infty < \infty$.
- (iv) There exist constants $0 \leq \gamma < 1$, $\delta > 0$ such that $\sup_{0 \leq t \leq \delta} t^\gamma \|g_0^{-1} P_t g_1\|_\infty < \infty$.
- (v) $g_0 \in D_g$.

All through the present paper we suppose that Assumption 1 and 2 are fulfilled. We shall sometime use the notation $\|\cdot\|_{g_0} = \|\cdot/g_0\|_\infty$.

Let $\mathcal{M}_{g_0} \equiv \mathcal{M}_{g_0}(S)$ be a space of counting measures on S defined as $\mu \in \mathcal{M}_{g_0} \stackrel{\text{def}}{\iff} \mu = \sum_n \delta_{x_n}$ such that $\langle \mu, g_0 \rangle < \infty$ and

$$\begin{aligned} \mu_n \rightarrow \mu \quad \text{in } \mathcal{M}_{g_0} &\stackrel{\text{def}}{\iff} \sup \langle \mu_n, g_0 \rangle < \infty, \\ \langle \mu_n, f \rangle &\rightarrow \langle \mu, f \rangle \quad \text{for all } f \in C_c \text{ and } f = g_0, \end{aligned}$$

where $C_c \equiv C_c(S)$ denotes the space of all continuous functions with compact supports on S . Then \mathcal{M}_{g_0} is metrizable and separable.

We mainly consider the case that the generator has the form

$$(1.1) \quad A = A^c + A^d,$$

with

$$\begin{aligned} A^c f(x) &= \frac{1}{2} \sum_{i,j=1}^d a^{ij}(x) \partial_{ij}^2 f(x) + \sum_{i=1}^d b^i(x) \partial_i f(x), \\ A^d f(x) &= \int_{S \setminus \{x\}} [f(y) - f(x) - \nabla f(x) \cdot (y-x) I(|y-x| < 1)] v(x, dy) \\ &\quad - k(x) f(x) + \sum_{i=1}^d c^i(x) \partial_i f(x) \end{aligned}$$

for $f \in D_g$, where $a^{ij}, b^i \in C_b(S)$, (a^{ij}) is positive definite, $k(x) \geq 0$ denotes the killing rate by jumps, $(c^i(x))$ depends on jumps, and $v(x, dy)$ is the Lévy kernel on $S \times (S \setminus \{x\})$ satisfying that

$$\sup_{x \in S} \int_{S \setminus \{x\}} (1 \wedge |y-x|^2) v(x, dy) < \infty.$$

Independent IMPS; Results in [2]

Let (X_t, \mathbf{P}_μ) be an (indistinguishable) independent Markov particle system (IMP) associated with the motion process $(w(t), P_x)$, i.e., for many independent motions $(w_n(t), P_{x_n}) \stackrel{(d)}{=} (w(t), P_{x_n})$,

$$X_t = \sum_n \delta_{w_n(t)}|_S \quad \text{if } \mu = \sum_n \delta_{x_n} \text{ on } S, \quad \text{and } \mathbf{P}_\mu = \prod_n P_{x_n}.$$

The generator \mathcal{L}_0 of this particle system is given by the following: for $f \in C_c^\infty$,

$$\begin{aligned}\mathcal{L}_0 e^{-\langle \cdot, f \rangle}(\mu) &= -\langle \mu, e^f A(1 - e^{-f}) \rangle e^{-\langle \mu, f \rangle} \\ &= -\langle \mu, Af - \Gamma f \rangle e^{-\langle \mu, f \rangle},\end{aligned}$$

where $\Gamma f := Af - e^f A(1 - e^{-f})$. In fact, let $\{\mathcal{F}_t\}$ be the filtration generated by $\{X_t\}$ and set

$$V_t f(x) = -\log P_x[\exp -f(w(t))] = -\log\{1 - P_t(1 - e^{-f})(x)\}.$$

We have that if $0 \leq s < t$, then

$$\mathbf{E}_\mu[e^{-\langle X_t, f \rangle} | \mathcal{F}_s] = \exp[-\langle X_s, V_{t-s} f \rangle].$$

It is easy to see that $(V_t)_{t \geq 0}$ is a nonnegative contraction semigroup on C_0 and that by (ii) of Assumption 1 if $f \in C_c^\infty$, then $1 - e^{-f} \in C_c^\infty \subset D_g$, hence we have

$$\begin{aligned}\partial_t V_t f &= \frac{P_t A(1 - e^{-f})}{1 - P_t(1 - e^{-f})} = \frac{A P_t(1 - e^{-f})}{1 - P_t(1 - e^{-f})} = e^{V_t f} A(1 - e^{-V_t f}) \\ &\rightarrow e^f A(1 - e^{-f}) = Af - \Gamma f \quad (t \downarrow 0).\end{aligned}$$

Note that since $V_t f \leq P_t f$ (by Jensen's inequality), Γ is nonnegative;

$$\Gamma f = Af - \partial_t V_t f|_{t=0+} = \lim_{t \downarrow 0} \frac{1}{t} [(P_t f - f) - (V_t f - f)] \geq 0$$

and that for each $f \in C_c^\infty$, $v_t = V_t f$ is the unique solution to the following equation:

$$\partial_t v_t = e^{v_t} A(1 - e^{-v_t}), \quad v_0 = f$$

(because $u_t := 1 - e^{-v_t}$ satisfies $\partial_t u_t = A u_t$, $u_0 = 1 - e^{-f}$ and the unique solution is given as $u_t = P_t(1 - e^{-f})$). Moreover if $Av_t(x)$ is well-defined for $t > 0$, $x \in S$, then

$$\partial_t v_t = Av_t - \Gamma v_t, \quad v_0 = f.$$

By using the Markov property and by induction we have

PROPOSITION 1 (Prop. 1 in [1]). *For every $0 \leq t_1 \leq \dots \leq t_n$ and $f_i \in D_g^+$, $i = 1, 2, \dots, n$,*

$$\begin{aligned}
& \mathbf{E}_\mu[\langle X_{t_1}, f_1 \rangle \cdots \langle X_{t_n}, f_n \rangle] \\
& \leq \prod_{i=1}^n \langle \mu, P_{t_i} f_i \rangle + C_1^{(n)} \sum_{i=1}^n \prod_{j \neq i} \langle \mu, P_{t_j} f_j \rangle \\
& \quad + C_2^{(n)} \sum_{i_1 \neq i_2} \prod_{j \neq i_1, i_2} \langle \mu, P_{t_j} f_j \rangle + \cdots + C_{n-1}^{(n)} \sum_{j=1}^n \langle \mu, P_{t_j} f_j \rangle + C_n^{(n)},
\end{aligned}$$

where $C_k^{(n)}$, $k = 1, \dots, n$ are positive constants, depending on $(n, \{\|f_i\|_\infty\}_{i \leq n})$.

We introduce a non-negative operator Q as $Qf = Af^2 - 2fAf$ for $f \in D_g$, which is well-defined by (i) of Assumption 2 and plays an important role to investigate the exponents of Hölder (right) continuity. The non-negativity follows from $(P_t f^2 - f^2) - 2f(P_t f - f) \geq (P_t f)^2 - 2fP_t f + f^2 = (P_t f - f)^2 \geq 0$.

THEOREM 1 (Th. 2.3 and Cor. 2.1 in [2]). *Let $(w(t), P_x)$ be a discontinuous Markov process in $\mathbf{D}([0, \zeta(w)) \rightarrow S)$ with transition semigroup (P_t) satisfying Assumption 1 and 2. Let $\mu \in \mathcal{M}_{g_0}$. The following holds with \mathbf{P}_μ -probability one.*

- (i) $\{\langle X_t, g_0 \rangle\}$ is $((1 - \gamma)/2 - \varepsilon)$ -Hölder right continuous at $t = 0$ for sufficiently small $\varepsilon > 0$, where the constant $0 \leq \gamma < 1$ is in (iv) of Assumption 2.
- (ii) If $\langle \mu, g_1 \rangle < \infty$, in particular, if $g_1(x) = g_0(x)$ then $\{\langle X_t, g_0 \rangle\}$ is $(1/2 - \varepsilon)$ -Hölder right continuous at $t = 0$ for sufficiently small $\varepsilon > 0$.
- (iii) For each fixed $t_0 > 0$, $\{\langle X_t, g_0 \rangle\}$ is $(1/2 - \varepsilon)$ -Hölder right continuous at $t = t_0$ for sufficiently small $\varepsilon > 0$.

2. Sampling Replacement Markov Particle Systems

Let $\mu = \sum_n \delta_{x_n} \in \mathcal{M}_{g_0}$. Let (Y_t, \mathbf{P}_μ^Y) be a sampling replacement Markov particle system associated with the motion process $(w(t), P_x)$, sampling replacement rate $\lambda > 0$ and sampling replacement probability $q(d(m, n)) = \sum_{k, l} p_{k, l} \delta_{(k, l)}(d(m, n))$ on \mathbf{N}^2 , where $p_{k, l} \geq 0$, $p_{k, k} = 0$ and $\sum p_{k, l} = 1$. Each particle first moves independently each other. After a λ -exponential random time, two particles are selected randomly, for example, m -th and n -th particles are selected with probability $p_{m, n}$, and at that time the m -th particle jumps to the place of the n -th particle. Then the m -th particle moves independently. And these operations are continued. We denote each particle by $w_n^*(t)$ such that $w_n^*(0) = x_n$, and hence $Y_t = \sum_n \delta_{w_n^*(t)}$. Note that if (P_t) is non-conservative, then it is possible that the dead particles come life again.

Recall (X_t, \mathbf{P}_μ) is the independent Markov particle system with the motion process $(w(t), P_x)$. For $f \in D_g$, set $L_t^Y(\mu) = \mathbf{E}_\mu^Y[\exp -\langle Y_t, f \rangle]$ and $L_t(\mu) = \mathbf{E}_\mu[\exp -\langle X_t, f \rangle]$. Then

$$L_t(\mu) = e^{-\langle \mu, V_t f \rangle} \quad \text{with } V_t f(x) = -\log E_x[e^{-f(w(t))}] = -\log(1 - P_t(1 - e^{-f})).$$

It is easy to see that $L_t^Y(\mu)$ satisfies the following equation:

$$L_t^Y(\mu) = e^{-\lambda t} L_t(\mu) + \lambda \int_0^t ds e^{-\lambda s} \int_{\mathbf{N}^2} q(d(m, n)) \mathbf{P}_s(\Theta_{m, n} L_{t-s})(\mu),$$

where (\mathbf{P}_t) is the transition semigroup of (X_t, \mathbf{P}_μ) and $\Theta_{m, n}$ is an operator such that it makes the m -th particle jump to the place of n -th particle of $\mu = \sum \delta_k \in \mathcal{M}_{g_0}$ on a class of all functions $F(\mu)$ and it is defined by $\Theta_{m, n} F(\mu) = F(\mu^{m, n})$ with $\mu^{m, n} = \mu - \delta_{x_m} + \delta_{x_n}$. Note that $\mathbf{P}_s \Theta_{m, n} = \Theta_{m, n} \mathbf{P}_s$ holds. The solution is given as

$$(2.1) \quad L_t^Y(\mu) = \mathbf{T}_t e^{-\langle \cdot, V_t f \rangle}(\mu) \quad \text{with } \mathbf{T}_t = \sum_k e^{-\lambda t} \frac{(\lambda t)^k}{k!} \left(\int_{\mathbf{N}^2} q(d(m, n)) \Theta_{m, n} \right)^k,$$

where \mathbf{T}_t is an operator on a class of functions $F(\mu)$ with polynomial growth of $\langle \mu, f_1 \rangle, \langle \mu, f_2 \rangle, \dots, \langle \mu, f_n \rangle$ ($f_i \in D_g$) and

$$\left(\int_{\mathbf{N}^2} q(d(m, n)) \Theta_{m, n} \right)^k F(\mu) = \left(\int_{\mathbf{N}^2} q(d(m, n)) \Theta_{m, n} \right)^{k-1} \sum_{m, n \in \mathbf{N}} p_{m, n} F(\mu^{m, n}).$$

The generator \mathcal{L}^Y of this particle system is given by the following: for $f \in C_c^\infty$,

$$\begin{aligned} \mathcal{L}^Y e^{-\langle \cdot, f \rangle}(\mu) &= \mathcal{L}_0 e^{-\langle \cdot, f \rangle}(\mu) + \lambda \int (e^{-\langle \mu^{m, n}, f \rangle} - e^{-\langle \mu, f \rangle}) q(d(m, n)) \\ &= - \left\{ \langle \mu, Af - \Gamma f \rangle + \lambda \int \langle \delta_{x_n} - \delta_{x_m}, f \rangle q(d(m, n)) \right. \\ &\quad \left. - \lambda \int [e^{-\langle \delta_{x_n} - \delta_{x_m}, f \rangle} - 1 + \langle \delta_{x_n} - \delta_{x_m}, f \rangle] q(d(m, n)) \right\} e^{-\langle \mu, f \rangle} \end{aligned}$$

(more general formula of $\mathcal{L}^Y F(\mu)$ is given in §5). We have the following result. Recall that we denote the particles of Y_t by $w_n^*(t)$, i.e., $Y_t = \sum \delta_{w_n^*(t)}$. Note that $w_n^*(t)$ moves like as $w_n(t)$ during the jump times.

THEOREM 2 (Semi-martingale Representation of Y_t). *Under Assumption 1 and 2 for (P_t) , if $\mu \in \mathcal{M}_{g_0}$, then (Y_t, \mathbf{P}_μ^Y) is an \mathcal{M}_{g_0} -valued Markov process with sample paths in $\mathbf{D}([0, \infty) \rightarrow \mathcal{M}_{g_0})$ satisfying the following:*

(i) $\{\langle Y_t, g_0 \rangle\}$ has the same exponent of Hölder right continuity as in Theorem 1.

(ii) If the motion process $(w(t), P_x)$ has generator A of the form as in (1.1), then for $f \in D_g$,

$$\begin{aligned} \langle Y_t, f \rangle &= \langle Y_0, f \rangle + \int_0^t \left\{ \langle Y_s, Af \rangle + \lambda \int \langle \delta_{w_n^*(s)} - \delta_{w_m^*(s)}, f \rangle q(d(m, n)) \right\} ds \\ &\quad + M_t^c(f) + M_t^d(f), \end{aligned}$$

where

$M_t^c(f)$ is a continuous L^2 -martingale

with quadratic variation $\langle\langle M^c(f) \rangle\rangle_t = \int_0^t \langle Y_s, Q^c f \rangle ds = 2 \int_0^t \langle Y_s, \Gamma^c f \rangle ds$ and

$$M_t^d(f) = \int_0^t \int_{\mathcal{M}_{g_0}^\pm} \langle \mu, f \rangle \tilde{N}(ds, d\mu) \text{ is a purely discontinuous martingale}$$

where $\tilde{N} = N - \hat{N}$ is the martingale measure with

$$N(ds, d\mu) = \sum_{u: \Delta Y_u \neq 0} \delta_{(u, \Delta Y_u)}(ds, d\mu): \text{ the jump measure of } \{Y_t\},$$

$$\begin{aligned} \hat{N}(ds, d\mu) &= ds \left\{ \int Y_s(dx) \left(\int v(x, dy) \delta_{(\delta_y, -\delta_x)} + k(x) \delta_{-\delta_x} \right) \right. \\ &\quad \left. + \lambda \int q(d(m, n)) \delta_{(\delta_{w_n^*(s)} - \delta_{w_m^*(s)})} \right\} (d\mu): \text{ the compensator of } N. \end{aligned}$$

PROOF. The proof is the same as the independent case (Proof of Theorem 2.4 in [2]). However, we need some computations. First the Markov property can be shown by mathematical induction. For $t_1 < t_2$, $f_1, f_2 \in C_c^\infty$, let $L_{t_1, t_2}^{f_1, f_2}(\mu) = \mathbf{E}_\mu^Y[\exp(-\langle Y_{t_1}, f_1 \rangle - \langle Y_{t_2}, f_2 \rangle)]$. Recall that $L_t^Y(\mu) = L_t^f(\mu) = \mathbf{E}_\mu^Y[\exp -\langle Y_t, f \rangle]$ satisfies (2.1) and the solution is given as $L_t^f(\mu) = \mathbf{T}_t \mathbf{P}_t[\exp -\langle \cdot, f \rangle](\mu)$. Hence it is easy to see that $L_{t_1, t_2}^{f_1, f_2}(\mu)$ satisfies the following equation:

$$\begin{aligned} L_{t_1, t_2}^{f_1, f_2}(\mu) &= e^{-\lambda t_1} \mathbf{P}_{t_1}(e^{-\langle \cdot, f_1 \rangle} L_{t_2 - t_1}^{f_2})(\mu) \\ &\quad + \lambda \int_0^{t_1} ds e^{-\lambda s} \int q(d(m, n)) \mathbf{P}_s(\Theta_{m, n} L_{t_1 - s, t_2 - s}^{f_1, f_2})(\mu). \end{aligned}$$

The solution is given as

$$\begin{aligned} L_{t_1, t_2}^{f_1, f_2}(\mu) &= \mathbf{T}_{t_1} \mathbf{P}_{t_1}(e^{-\langle \cdot, f_1 \rangle} L_{t_2 - t_1}^{f_2})(\mu) \\ &= \mathbf{E}_\mu^Y [e^{-\langle Y_{t_1}, f_1 \rangle} \mathbf{E}_{Y_{t_1}}^Y [e^{-\langle Y_{t_2 - t_1}, f_2 \rangle}]]. \end{aligned}$$

Therefore by induction, for every $n \in \mathbf{N}$, if $t_1 < t_2 < \dots < t_n$, $f_1, \dots, f_n \in C_c^\infty$, then it holds that

$$\begin{aligned} &\mathbf{E}_\mu^Y [\exp(-\langle Y_{t_1}, f_1 \rangle - \dots - \langle Y_{t_n}, f_n \rangle)] \\ &= \mathbf{E}_\mu^Y [e^{-\langle Y_{t_1}, f_1 \rangle} \mathbf{E}_{Y_{t_1}}^Y [e^{-\langle Y_{t_2 - t_1}, f_2 \rangle} \dots \mathbf{E}_{Y_{t_{n-1}}}^Y [e^{-\langle Y_{t_n - t_{n-1}}, f_n \rangle} \dots]]]. \end{aligned}$$

Next we shall show that (Y_t, \mathbf{P}_μ^Y) satisfies a moment inequality of the same type as in Proposition 1.

PROPOSITION 2. *Let $T > 0$ and $n \in \mathbf{N}$. For every $0 \leq t_1 \leq \dots \leq t_n \leq T$ and $f_i \in D_g^+$, $i = 1, 2, \dots, n$,*

$$\begin{aligned} &\mathbf{E}_\mu^Y [\langle Y_{t_1}, f_1 \rangle \dots \langle Y_{t_n}, f_n \rangle] \\ &\leq \prod_{i=1}^n \langle \mu, P_{t_i} f_i \rangle + C_{1,T}^{(n)} \sum_{i=1}^n \prod_{j \neq i} \langle \mu, P_{t_j} f_j \rangle \\ &\quad + C_{2,T}^{(n)} \sum_{i_1 \neq i_2} \prod_{j \neq i_1, i_2} \langle \mu, P_{t_j} f_j \rangle + \dots + C_{n-1,T}^{(n)} \sum_{j=1}^n \langle \mu, P_{t_j} f_j \rangle + C_{n,T}^{(n)}, \end{aligned}$$

where $C_{k,T}^{(n)}$, $k = 1, \dots, n$ are positive constants depending on $(n, T, \{\|f_i\|_\infty\}_{i \leq n})$.

PROOF. For simplicity, we use notations $f_m = f(x_m)$ and $\|\cdot\| = \|\cdot\|_\infty$. Since $\Theta_{m,\ell} \langle \mu, f \rangle = \langle \mu, f \rangle + f_\ell - f_m \leq \langle \mu, f \rangle + \|f\|$, we have for every $k \in \mathbf{N}$,

$$\left(\int_{\mathbf{N}^2} q(d(m, \ell)) \Theta_{m,\ell} \right)^k (\langle \mu, f_1 \rangle \dots \langle \mu, f_n \rangle) \leq (\langle \mu, f_1 \rangle + k\|f_1\|) \dots (\langle \mu, f_n \rangle + k\|f_n\|).$$

Moreover if we denote by $M(j; \lambda t)$ the j -th moment of λt -Poisson distribution, then

$$\begin{aligned} &\mathbf{T}_t(\langle \cdot, f_1 \rangle \dots \langle \cdot, f_n \rangle)(\mu) \\ &= \sum_{k \geq 0} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \left(\int_{\mathbf{N}^2} q(d(m, \ell)) \Theta_{m,\ell} \right)^k (\langle \cdot, f_1 \rangle \dots \langle \cdot, f_n \rangle)(\mu) \\ &\leq \sum_{j=0}^n M(n-j; \lambda t) \sum_{\{i_1, \dots, i_j\} \in \{1, \dots, n\}} \langle \mu, f_{i_1} \rangle \dots \langle \mu, f_{i_j} \rangle \|f_{i_{j+1}}\| \dots \|f_{i_n}\|. \end{aligned}$$

Therefore by applying Proposition 1 and the above inequality to

$$\mathbf{E}_\mu^Y[\langle Y_{t_1}, f_1 \rangle \cdots \langle Y_{t_n}, f_n \rangle] = \mathbf{T}_t(\mathbf{E}[\langle X_{t_1}, f_1 \rangle \cdots \langle X_{t_n}, f_n \rangle])(\mu),$$

we can get the desired inequality. \blacksquare

Thus the proof can be proceeded by the same way as in the independent case (see §4 in [2]). In that way we can get the following result. For $\mu = \sum_n \delta_{x_n} \in \mathcal{M}_{g_0}$, $f \in C_c^\infty$, let

$$\Psi(\mu; f) = \langle \mu, Af - \Gamma f \rangle - \lambda \int [e^{-\langle \delta_{sm} - \delta_{sn}, f \rangle} - 1] q(d(m, n)).$$

THEOREM 3. For $f \in C_c^\infty$,

$$e^{-\langle Y_t, f \rangle} - e^{-\langle Y_0, f \rangle} - \int_0^t \mathcal{L}^Y e^{-\langle \cdot, f \rangle}(Y_s) ds$$

is a \mathbf{P}_μ^Y -martingale. Moreover

$$H_t(f) = \exp \left[-\langle Y_t, f \rangle + \int_0^t \Psi(Y_s; f) ds \right]$$

is also a \mathbf{P}_μ^Y -martingale.

PROOF. By the same way as in the proof of Theorem 4.1 in [2] we have if $s < t$, then

$$\begin{aligned} \partial_t \mathbf{E}_\mu^Y[e^{-\langle Y_t, f \rangle} | \mathcal{F}_s] &= \partial_t \mathbf{T}_{t-s}(e^{-\langle \cdot, V_{t-s} f \rangle})(Y_s) \\ &= \partial_{u=0+} \mathbf{T}_{t-s+u}(e^{-\langle \cdot, V_{t-s+u} f \rangle})(Y_s) \\ &= \partial_{u=0+} \mathbf{E}_\mu^Y[\mathbf{T}_u e^{-\langle \cdot, V_u f \rangle}(Y_t) | \mathcal{F}_s] \\ &= \mathbf{E}_\mu^Y[\partial_{u=0+} \mathbf{T}_u e^{-\langle \cdot, V_u f \rangle}(Y_t) | \mathcal{F}_s] \\ &= \mathbf{E}_\mu^Y[\mathcal{L}^Y e^{-\langle \cdot, f \rangle}(Y_t) | \mathcal{F}_s]. \end{aligned} \quad \blacksquare$$

By using the above results it is not difficult to prove the semi-martingale representation of Y_t as of X_t in [2]. In fact, for $f \in C_c^\infty$, $\langle Y_t, f \rangle$ is a special semi-martingale, thus,

$$\langle Y_t, f \rangle = \langle Y_0, f \rangle + C_t(f) + M_t^c(f) + \tilde{N}_t(f) + N_t(f),$$

where $C_t(f)$ is a continuous process of locally bounded variation, $M_t^c(f)$ is a continuous L^2 -martingale with quadratic variation $\langle\langle M^c(f) \rangle\rangle_t$, and

$$\begin{aligned}\tilde{N}_t(f) &= \int_0^t \int_{\mathcal{M}^\pm} \langle \mu, f \rangle I(\|\mu\| < 1) \tilde{N}(ds, d\mu), \\ N_t(f) &= \int_0^t \int_{\mathcal{M}^\pm} \langle \mu, f \rangle I(\|\mu\| \geq 1) N(ds, d\mu)\end{aligned}$$

with the jump measure N of $\{Y_t\}$, its compensator \hat{N} and $\tilde{N} = N - \hat{N}$.

If we set

$$\begin{aligned}B_t(f) &= C_t(f) + \int_0^t \int_{\{\|\mu\| \geq 1\}} \langle \mu, f \rangle \hat{N}(ds, d\mu) \\ &\quad + \lambda \int_0^t ds \int \langle \delta_{w_n(s)} - \delta_{w_m(s)}, f \rangle q(d(m, n)),\end{aligned}$$

then by applying Ito's formula for $Z_t(f)$ we can get

$$\begin{aligned}-dB_t(f) &+ \frac{1}{2} d\langle\langle M^c(f) \rangle\rangle_t + \int [e^{-\langle \mu, f \rangle} - 1 + \langle \mu, f \rangle] \hat{N}(dt, d\mu) \\ &= -\Psi(Y_t; f) dt \\ &= \left\{ -\langle Y_t, Af \rangle + \langle Y_t, \Gamma f \rangle + \lambda \int (e^{-\langle \delta_{w_n^*}(t) - \delta_{w_m^*}(t), f \rangle} - 1) q(d(m, n)) \right\} dt \\ &= \left\{ -\left[\langle Y_t, Af \rangle + \lambda \int \langle \delta_{w_n^*}(t) - \delta_{w_m^*}(t), f \rangle q(d(m, n)) \right] + \langle Y_t, \Gamma^c f \rangle \right. \\ &\quad \left. + \langle Y_t, \Gamma^d f \rangle + \lambda \int [e^{-\langle \delta_{w_n^*}(t) - \delta_{w_m^*}(t), f \rangle} - 1 + \langle \delta_{w_n^*}(t) - \delta_{w_m^*}(t), f \rangle] q(d(m, n)) \right\} dt\end{aligned}$$

Thus we have

$$\begin{aligned}B_t(f) &= \int_0^t \langle Y_s, Af \rangle ds + \lambda \int \langle \delta_{w_n^*}(t) - \delta_{w_m^*}(t), f \rangle q(d(m, n)), \\ \langle\langle M^c(f) \rangle\rangle_t &= 2 \int_0^t \langle Y_s, \Gamma^c f \rangle ds = \int_0^t \langle Y_s, \mathcal{Q}^c f \rangle ds\end{aligned}$$

and

$$\begin{aligned} \hat{N}(ds, d\mu) = ds \left\{ \int Y_s(dx) \left(\int v(x, dy) \delta_{(\delta_y - \delta_x)} + k(x) \delta_{-\delta_x} \right) \right. \\ \left. + \lambda \int q(d(m, n)) \delta_{(\delta_{w_n^*(s)} - \delta_{w_m^*(s)})} \right\} (d\mu). \end{aligned}$$

Therefore the proof is completed. ■

3. Martingale Problems for \mathcal{L}^Y

The following assumption is needed to prove the well-posedness of martingale problems.

ASSUMPTION 3. For each $f \in (C_c^\infty)^+$, $AV_t f = -A \log(1 - P_t(1 - e^{-f}))$ is well-defined and $AV_t f$ is continuous in t under the norm $\|\cdot\|_{g_1}$, i.e.,

$$\|(AV_t f - AV_{t_0} f)/g_1\|_\infty \rightarrow 0 \quad (t \rightarrow t_0).$$

In the following we suppose that the generator A of the motion process has the form of (1.1).

For $\eta \in \mathcal{M}_{g_0}$, let $F(\eta) = \Phi(\langle \eta, f_1 \rangle, \dots, \langle \eta, f_n \rangle) \in \mathcal{D}_0 \stackrel{\text{def}}{\iff} \Phi(x) \in C^\infty(\mathbf{R}^n)$ is a polynomial growth function with polynomial growth derivatives of all orders and $f_i \in D_g$, $i = 1, \dots, n$. For this $F(\eta)$, the generator \mathcal{L}_0 of X_t will be extended to the following form:

$$\begin{aligned} \mathcal{L}_0 F(\eta) = & \sum_{i=1}^n \partial_i \Phi(\langle \eta, f_1 \rangle, \dots, \langle \eta, f_n \rangle) \langle \eta, A f_i \rangle \\ & + \frac{1}{2} \sum_{i,j=1}^n \partial_{ij}^2 \Phi(\langle \eta, f_1 \rangle, \dots, \langle \eta, f_n \rangle) \langle \eta, Q^c(f_i, f_j) \rangle \\ & + \int_S \left\{ \int_{S \setminus \{x\}} v(x, dy) \left[\Phi(\langle \eta, f_1 \rangle + f_1(y) - f_1(x), \dots, \langle \eta, f_n \rangle \right. \right. \\ & \qquad \qquad \qquad \left. \left. + f_n(y) - f_n(x) - \Phi(\langle \eta, f_1 \rangle, \dots, \langle \eta, f_n \rangle) \right. \right. \\ & \qquad \qquad \qquad \left. \left. - \sum_{i=1}^n \partial_i \Phi(\langle \eta, f_1 \rangle, \dots, \langle \eta, f_n \rangle) (f_i(y) - f_i(x)) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + k(x) \left[\Phi(\langle \eta, f_1 \rangle - f_1(x), \dots, \langle \eta, f_n \rangle - f_n(x)) \right. \\
& \quad \left. - \Phi(\langle \eta, f_1 \rangle, \dots, \langle \eta, f_n \rangle) \right. \\
& \quad \left. + \sum_{i=1}^n \partial_i \Phi(\langle \eta, f_1 \rangle, \dots, \langle \eta, f_n \rangle) f_i(x) \right] \eta(dx),
\end{aligned}$$

where

$$Q^c(f, g)(x) = \sum_{i,j} a^{ij}(x) \partial_i f(x) \partial_j g(x).$$

For $F(\eta) \in \mathcal{D}_0$, the generator \mathcal{L}^Y of Y_t will be extended to

$$\mathcal{L}^Y F(\eta) = \mathcal{L}_0 F(\eta) + \lambda \int (\Theta_{m,n} F(\eta) - F(\eta)) q(d(m, n)).$$

THEOREM 4 (Martingale Problem for $(\mathcal{L}^Y, \mathcal{D}_0, \mu)$). *Under Assumption 1, 2 and 3, suppose that the generator A is given as in (1.1). Let $\mu \in \mathcal{M}_{g_0}$.*

(i) $\mathbf{P}_\mu^Y(Y_0 = \mu) = 1$ holds and for each $F(\mu) = \Phi(\langle \mu, f_1 \rangle, \dots, \langle \mu, f_n \rangle) \in \mathcal{D}_0$,

$$M_t^F = F(Y_t) - F(Y_0) - \int_0^t \mathcal{L}^Y F(Y_s) ds \text{ is } \mathbf{P}_\mu^Y\text{-martingale.}$$

(ii) *If there is a probability measure \mathbf{Q}_μ on $\mathbf{D} = \mathbf{D}([0, \infty) \rightarrow \mathcal{M}_{g_0})$ such that the canonical process $\tilde{Y}_t(\omega) = \omega(t)$ ($\omega \in \mathbf{D}$) satisfies the same conditions as (Y_t, \mathbf{P}_μ^Y) in (i) and*

$$\int_0^t \langle \tilde{Y}_s, g_1 \rangle ds < \infty \quad \mathbf{Q}_\mu\text{-a.s. for all } t \geq 0,$$

then $\mathbf{Q}_\mu = \mathbf{P}_\mu^Y \circ Y^{-1}$ on \mathbf{D} , that is, martingale problem for $(\mathcal{L}^Y, \mathcal{D}_0, \mu)$ on \mathbf{D} is well-posed.

PROOF. The proof is essentially the same as the independent case (see §5 in [2]). However, the computations are more complicated, so we give the outline of the proof. (i) is easily obtained. We show (ii). We always fix $f \in C_c^\infty$, $T > 0$, and set $\Psi_t^T(\eta) = \mathbf{T}_{T-t}(\exp -\langle \cdot, V_{T-t} f \rangle)(\eta)$ ($0 \leq t \leq T, \eta \in \mathcal{M}_{g_0}$). It is no difficult to show that $\{\Psi_t^T(\tilde{Y}_t)\}_{t \leq T}$ is a \mathbf{Q}_μ -martingale. In fact, by using Ito's formula

$$\begin{aligned}
d(\exp -\langle \tilde{Y}_t, f \rangle) &= -\langle \tilde{Y}_t, Af - \Gamma f \rangle e^{-\langle \tilde{Y}_t, f \rangle} dt \\
&\quad + \lambda \int q(d(m, n))(\Theta_{m, n} - I) e^{-\langle \cdot, f \rangle}(\tilde{Y}_t) dt + d(\mathbf{Q}_\mu\text{-martingale}).
\end{aligned}$$

Since \mathbf{T}_t is a bounded operator, we have (set $v_t = v_t^T = V_{T-t}f$ again)

$$\begin{aligned}
d(\Psi_t^T(\tilde{Y}_t)) &= \mathbf{T}_{T-t} \left(-\lambda \int q(d(m, n))(\Theta_{m, n} - I) e^{-\langle \cdot, v_t \rangle} - \langle \cdot, \partial_t v_t \rangle e^{-\langle \cdot, v_t \rangle} \right. \\
&\quad \left. - \langle \cdot, Av_t - \Gamma v_t \rangle e^{-\langle \cdot, v_t \rangle} + \lambda \int q(d(m, n))(\Theta_{m, n} - I) e^{-\langle \cdot, v_t \rangle} \right) (\tilde{Y}_t) dt \\
&\quad + d(\mathbf{Q}_\mu\text{-martingale}) \\
&= \mathbf{T}_{T-t}(-\langle \cdot, \partial_t v_t + Av_t - \Gamma v_t \rangle e^{-\langle \cdot, v_t \rangle})(\tilde{Y}_t) + d(\mathbf{Q}_\mu\text{-martingale}) \\
&= d(\mathbf{Q}_\mu\text{-martingale})
\end{aligned}$$

Hence for $0 \leq s < t \leq T$, we have

$$\mathbf{Q}_\mu[\Psi_t^T(\tilde{Y}_t) \mid \mathcal{F}_s] = \Psi_s^T(\tilde{Y}_s)$$

and set $T = t$, then

$$\mathbf{Q}_\mu[e^{-\langle \tilde{Y}_t, f \rangle} \mid \mathcal{F}_s] = \mathbf{T}_{t-s} e^{-\langle \cdot, V_{t-s}f \rangle}(\tilde{Y}_s).$$

Therefore $\mathbf{P}_\mu = \mathbf{Q}_\mu$ on \mathbf{D} . ■

4. Multi-Dimensional Absorbing Stable Motions on a Half Space

In §3 of [2] as a motion process we considered absorbing Brownian motion and absorbing stable motion on $(0, \infty)$ and discussed the Hölder (right) continuities of $\{X_t\}$. It is possible to consider absorbing motions on $H = \mathbf{R}^{d-1} \times (0, \infty)$ and we can get the same results as in Theorem 3.1 and in Corollary 3.1 of [2]. For the absorbing Brownian motion, it is not so difficult and essentially done in [1]. So in this section we only discuss the absorbing stable motion on H .

For a function f on H , let \tilde{f} be an extension of f to on \mathbf{R}^d defined as

$$\tilde{f}(x) = \begin{cases} f(x) & (x_d > 0), \\ f(\tilde{x}, 0+) = 0 & (x_d = 0), \\ -f(\tilde{x}, -x_d) & (x_d < 0), \end{cases}$$

where $x = (\tilde{x}, x_d) \in H$. Note that if $x \in H$, then $\bar{f}(x) = f(x)$. The generator $A^- \equiv A^{-,\alpha}$ of absorbing α -stable motion $(w^-(t), P_x^-) \equiv (w^{-,\alpha}(t), P_x^{-,\alpha})$ on H is given as $A^{-,\alpha}f(x) = A^\alpha \bar{f}(x)$; (A^- is the same as L^- in §4 of [1], however, in which we have some miss-prints)

$$\begin{aligned}
(4.1) \quad A^{-,\alpha}f(x) &= c \int_{\mathbf{R}^d \setminus \{0\}} [\bar{f}(x+y) - \bar{f}(x) - \nabla \bar{f}(x) \cdot y I(|y| < 1)] \frac{dy}{|y|^{d+\alpha}} \\
&= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{-x_d}^{x_d} [f(x+y) - f(x) - \nabla f(x) \cdot y I(|y| < 1)] \frac{dy_d}{|y|^{d+\alpha}} \\
&\quad + c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{x_d}^{\infty} [f(x+y) - f(\tilde{x} + \tilde{y}, y_d - x_d) - 2f(x)] \frac{dy_d}{|y|^{d+\alpha}}
\end{aligned}$$

with some positive constant c , where in the last term the integral corresponding to $\nabla f(x) \cdot y$ is equal to zero by the symmetric property (of course, it is integrable). We can also write that if $0 < \alpha < 1$, then

$$\begin{aligned}
A^{-,\alpha}f(x) &= c \int_{\mathbf{R}^d \setminus \{x\}} [\bar{f}(y) - \bar{f}(x)] \frac{dy}{|y-x|^{d+\alpha}} \\
&= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \left\{ \int_0^\infty [f(y) - f(x)] K(x, y) dy_d \right. \\
&\quad \left. - 2f(x) \int_0^\infty \frac{dy_d}{|(\tilde{y} - \tilde{x}, y_d + x_d)|^{d+\alpha}} \right\}, \\
&= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_0^\infty [f(y) - f(x)] K(x, y) dy_d - f(x)k(x),
\end{aligned}$$

and that if $1 \leq \alpha < 2$, then

$$\begin{aligned}
A^{-,\alpha}f(x) &= c \int_{\mathbf{R}^d \setminus \{x\}} [\bar{f}(y) - \bar{f}(x) - \nabla \bar{f}(x) \cdot (y-x) I(|y-x| < 1)] \frac{dy}{|y-x|^{d+\alpha}} \\
&= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \left\{ \int_0^\infty [f(y) - f(x) - \nabla f(x) \cdot (y-x) I(|y-x| < 1)] K(x, y) dy_d \right. \\
&\quad + \int_0^\infty [-2f(x) - \nabla f(x) \cdot (y-x) I(|y-x| < 1) \\
&\quad \quad \left. - \nabla f(x) \cdot (\tilde{y} - \tilde{x}, -y_d - x_d) I(|(\tilde{y} - \tilde{x}, y_d + x_d)| < 1)] \right. \\
&\quad \left. \times \frac{dy_d}{|(\tilde{y} - \tilde{x}, y_d + x_d)|^{d+\alpha}} \right\}
\end{aligned}$$

$$\begin{aligned}
&= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_0^\infty [f(y) - f(x) - \nabla f(x) \cdot (y - x)I(|y - x| < 1)]K(x, y) dy_d \\
&\quad - f(x)k(x) + \nabla f(x) \cdot c(x),
\end{aligned}$$

where

$$\begin{aligned}
K(x, y) &= \frac{I(y \neq x)}{|y - x|^{d+\alpha}} - \frac{1}{|(\tilde{y} - \tilde{x}, y_d + x_d)|^{d+\alpha}}, \\
k(x) &= k(x_d) = 2c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{x_d}^\infty \frac{dy_d}{|y|^{d+\alpha}}
\end{aligned}$$

and

$$\begin{aligned}
c(x) &= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_0^\infty [-(\tilde{y} - \tilde{x}, -y_d - x_d)I(|(\tilde{y} - \tilde{x}, y_d + x_d)| < 1) \\
&\quad - (y - x)I(|y - x| < 1)] \frac{dy_d}{|(\tilde{y} - \tilde{x}, y_d + x_d)|^{d+\alpha}} \\
&= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_0^\infty [(\tilde{y}, y_d + x_d)I(|(\tilde{y}, y_d + x_d)| < 1) \\
&\quad - (\tilde{y}, y_d - x_d)I(|(\tilde{y}, y_d - x_d)| < 1)] \frac{dy_d}{|(\tilde{y}, y_d + x_d)|^{d+\alpha}}.
\end{aligned}$$

Let $h_0(v)$ be a C^∞ -function on $(0, \infty)$ such that $0 < h_0 \leq 1$ on $(0, \infty)$, $h_0(v) = v$ for $v \in (0, 1/2]$ and $h_0(v) = 1$ for $v \geq 1$. Let $d < p < d + \alpha$. Set $g_p(x) = (1 + |x|^2)^{-p/2}$ and $g_{p,0}(x) := g_p(x)h_0(x_d)$ for $x \in H$. Let $f \in C_p \stackrel{\text{def}}{\iff} f \in C(\mathbf{R}^d)|_H$; $\|f/g_p\|_\infty < \infty$. $f \in C_{p,0} \stackrel{\text{def}}{\iff} f \in C(\mathbf{R}^d)|_H$; $\|f/g_{p,0}\| < \infty$. Moreover set

$$f \in C_{p,0}^3 \stackrel{\text{def}}{\iff} f \in C_b^3(\mathbf{R}^d)|_H;$$

$$\text{for } i, j \neq d, f, \partial_d^2 f, \partial_i f, \partial_{ij}^2 f \in C_{p,0} \text{ and } \partial_d f, \partial_{id}^2 f \in C_p.$$

Then we can take $D_g = C_{p,0}^3$.

Moreover for each $0 < \alpha < 2$, $Q^- f \equiv Q^{-,\alpha} f = Af^2 - fAf$ is given by the following formula:

$$\begin{aligned}
(4.2) \quad Q^{-,\alpha} f(x) &= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{-x_d}^{x_d} [f(x + y) - f(x)]^2 \frac{dy_d}{|y|^{d+\alpha}} \\
&\quad + c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{x_d}^\infty \{f(x + y) - f(\tilde{y} + \tilde{x}, y_d - x_d)\}
\end{aligned}$$

$$\begin{aligned} & \{f(x+y) + f(\tilde{y} + \tilde{x}, y_d - x_d) - 2f(x)\} + 2f(x)^2 \frac{dy_d}{|y|^{d+\alpha}} \\ &= c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_0^\infty [f(y) - f(x)]K(x, y) dy_d + f(x)^2 k(x). \end{aligned}$$

THEOREM 5. *Let $d \geq 1$, $d < p < d + \alpha$, $\mu \in \mathcal{M}_{g_{p,0}}$ and let the motion process be absorbing α -stable motion on H with $0 < \alpha < 2$. Let $\varepsilon > 0$ denote an arbitrary small number.*

- (i) *For (X_t, \mathbf{P}_μ) the following holds.*
 - (a) *Under \mathbf{P}_μ , $\{\langle X_t, g_{p,0} \rangle\}$ is $(1/(2(\alpha \vee 1)) - \varepsilon)$ -Hölder right continuous at $t = 0$. Moreover in case of $1 < \alpha < 2$, if $\langle \mu, g_1 \rangle < \infty$ with $g_1(x) = g_p(x)h_0(x_d)^{2-\alpha}$, then $\{\langle X_t, g_{p,0} \rangle\}$ is $(1/2 - \varepsilon)$ -Hölder right continuous at $t = 0$.*
 - (b) *If $t_0 > 0$, then under \mathbf{P}_μ , $\{\langle X_t, g_{p,0} \rangle\}$ is $(1/2 - \varepsilon)$ -Hölder right continuous at $t = t_0$ for every $0 < \alpha < 2$.*
- (ii) *For (Y_t, \mathbf{P}_μ^Y) the same results hold as above.*

PROOF. Let $d \geq 2$. The proof is proceeded in the same way as the case of $d = 1$. It suffices to check that the conditions in Assumption 1 and 2 are fulfilled with $g_0 = g_{p,0}$ and with suitable $g_1 \in C^\infty$, $0 \leq \gamma < 1$ as follows. Let $h_1 \in C^\infty$; $0 < h_1 \leq 1$, $h_1(v) = v \log(1/v)$ for $v \in (0, 1/e]$ and $h_1(v) = 1$ for $v \geq 1$.

- (i) If $0 < \alpha < 1$, then $g_1(x) = g_{p,0}(x)$, $\gamma = 0$.
- (ii) If $\alpha = 1$, then $g_1(x) = g_p(x)h_1(x_d)$, $\gamma = \delta$ for any small $0 < \delta < 1$.
- (iii) If $1 < \alpha < 2$, then $g_1(x) = g_p(x)h_0(x_d)^{2-\alpha}$, $\gamma = 1 - 1/\alpha$.

Note that as $x_d \downarrow 0$,

$$g_1(x) \sim x_d \quad (0 < \alpha < 1), \quad \sim x_d \log(1/x_d) \quad (\alpha = 1), \quad \sim x_d^{2-\alpha} \quad (1 < \alpha < 2).$$

For simplicity of the notations we omit the superscript “ α ” as $P_t^{-,\alpha} = P_t^-$, $A^{-,\alpha} = A^-$. We shall show the following. Since they imply $\|g_{p,0}P_t^-g_1\|_\infty \leq Ct^{-\gamma}$, we can get the $((1-\gamma)/2 - \varepsilon)$ -Hölder right continuity.

- (C1) $C_{p,0}^3 \subset \mathcal{D}(A^-)$, $P_t^-C_c^\infty \subset C_{p,0}^3$ for every $t \geq 0$, $\sup_{t \geq 0, 0 < x_d \leq 1} |x_d^{-1}P_t^-g_{p,0}(x)| < \infty$ and $A^-C_{p,0}^3 \subset C_{p,0}^3$ (these imply Assumption 1 and that $C_{p,0}^3$ is a core).

- (C2) For every $f \in C_{p,0}^3$, $\partial_t P_t^- f^2(x) = A^- P_t^- f^2(x) = P_t^- A^- f^2(x)$ ($x \in H$), $A^- f^2 \in C_b$ and $\|g_1^{-1} Q^- f\|_\infty < \infty$ (these imply (i) of Assumption 2).
- (C3) For each $0 < \beta \leq 1$, $\sup_{t \geq 0} P_t^-(y_d^\beta)(x) \leq 2(1 + \beta)x_d^\beta$ for all $x \in H$ (this implies (ii) of Assumption 2).
- (C4) For each $0 < \beta \leq 1$, $\sup_{0 < x_d \leq 1} x_d^{-1} P_t^-(y_d^\beta)(x) \leq C_\beta t^{-(1-\beta)/\alpha}$ with a constant $C_\beta > 0$ depending only on β (this implies (iii), (iv) of Assumption 2).

Note that we take $\gamma = (1 - \beta)/\alpha$ in Assumption 2. More exactly, if $0 < \alpha < 1$, then $\beta = 1$, i.e., $\gamma = 0$. If $\alpha = 1$, then $\beta = 1 - \delta$ for any small $0 < \delta < 1$, i.e., $\gamma = \delta$. If $1 < \alpha < 2$, then $\beta = 2 - \alpha$, i.e., $\gamma = 1 - 1/\alpha$. (C3) and (C4) can be shown in a way similar to the case of $d = 1$; (B3) and (B4) in [2] by using the following. For the density $p^\alpha(x)$ of the rotation invariant α -stable motion on \mathbf{R}^d starting from 0, $p_t^\alpha(x) = t^{-d/\alpha} p_1^\alpha(t^{-1/\alpha}x)$ and $p_1^\alpha(x) \leq C(1 \wedge |x|^{-d-\alpha})$. The transition density $p_t^-(x, y) \equiv p_t^{-\alpha}(x, y)$ of absorbing α -stable motion on H is given as

$$p_t^-(x, y) = p_t^\alpha(y - x) - p_t^\alpha(\tilde{y} - \tilde{x}, y_d + x_d) = - \int_{-x_d}^{x_d} \partial_d p_t^\alpha(\tilde{y} - \tilde{x}, y_d + v) dv.$$

We also use the following result.

$$\int_H z_d^{\beta-1} p_1^\alpha(\tilde{z}, z_d + u) dz \text{ is bounded in } u \in \mathbf{R}.$$

From these results we can get (C3), (C4).

In each (C1), (C2), the claims except the last one can be shown by the same way as in $d = 1$. In order to show the last claims of (C1), (C2), it is enough to prove that for each $f \in C_{p,0}^3$, there is a constant $C > 0$ such that

$$|A^- f(x)| \leq Cx_d \text{ for } 0 < x_d \leq 1/2 \text{ and } Q^- f(x) \leq Cg_1(x) \text{ for all } x \in H.$$

Let $0 < x_d \leq 1/2$. For A^- we use the formula (4.1). In the following we decompose as $A^- f = (J_{1,1} + J_{1,2}) + (J_2 + J_3)$ and we shall show each term has order of x_d^2 , x_d , x_d , x_d , respectively. The main calculus is of $J_{1,2}$ ($1 < \alpha < 2$) and J_3 . In the first term of (4.1) we divide the integral area to $\{|y| \geq 1\}$, $\{|y| < 1\}$ and denote the corresponding terms by $J_{1,1}(x)$, $J_{1,2}(x)$ respectively. In the following we use the same symbols C' , C'' as any positive finite constants which are independent of x . First note that if $|y| \geq 1$ and $|y_d| \leq x_d \leq 1/2$, then $|\tilde{y}|^2 \geq 1 - x_d^2 \geq 3/4 =: b$. By $|f(x)| \leq Cx_d$,

$$\begin{aligned}
|J_{1,1}(x)| &= \left| c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{-x_d}^{x_d} [f(x+y) - f(x)] I(|y| \geq 1) \frac{dy_d}{|y|^{d+\alpha}} \right| \\
&\leq 2c \int_{|\tilde{y}| \geq b} d\tilde{y} \int_0^{x_d} C(2x_d + y_d) \frac{dy_d}{|y|^{d+\alpha}} \\
&\leq 2cC \int_{|\tilde{y}| \geq b} \frac{d\tilde{y}}{|\tilde{y}|^{d+\alpha}} \int_0^{x_d} (2x_d + y_d) dy_d \\
&\leq 2cC \int_b^\infty \frac{dr}{r^{2+\alpha}} \int_0^{x_d} (2x_d + y_d) dy_d \\
&= \frac{5cC}{1+\alpha} b^{-1-\alpha} x_d^2 = C' x_d^2.
\end{aligned}$$

Next note that if $|y| < 1$, then $|\tilde{y}| < 1$ and that for some $\theta \in (0, 1)$,

$$|f(x+y) - f(x) - \nabla f(x) \cdot y| = \frac{1}{2} |f^{(2)}(x + \theta y) \cdot y^2| \leq \frac{1}{2} \|f^{(2)}\|_\infty |y|^2.$$

If $0 < \alpha < 1$, then $|y|^{d-2+\alpha} \geq |\tilde{y}|^{d-2+\alpha}$ by $d-2+\alpha > 0$, and

$$\begin{aligned}
|J_{1,2}(x)| &= \left| c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{-x_d}^{x_d} [f(x+y) - f(x) - \nabla f(x) \cdot y] I(|y| < 1) \frac{dy_d}{|y|^{d+\alpha}} \right| \\
&\leq c \int_{|\tilde{y}| < 1} d\tilde{y} \int_{-x_d}^{x_d} |f(x+y) - f(x) - \nabla f(x) \cdot y| \frac{dy_d}{|y|^{d+\alpha}} \\
&\leq c \int_{|\tilde{y}| < 1} d\tilde{y} \int_0^{x_d} \|f^{(2)}\|_\infty |y|^2 \frac{dy_d}{|y|^{d+\alpha}} \\
&\leq c \int_{|\tilde{y}| < 1} \frac{d\tilde{y}}{|\tilde{y}|^{d+\alpha-2}} \|f^{(2)}\|_\infty x_d \\
&\leq c \int_0^1 r^{-\alpha} dr \|f^{(2)}\|_\infty x_d \\
&= \frac{c \|f^{(2)}\|_\infty}{1-\alpha} x_d = C' x_d.
\end{aligned}$$

On the other hand if $1 \leq \alpha < 2$, then by using

$$f(x+y) - f(x) - \nabla f(x) \cdot y = \frac{1}{2} f^{(2)}(x) \cdot y^2 + \frac{1}{6} f^{(3)}(x + \theta y) \cdot y^3$$

with some $\theta \in (0, 1)$, and the corresponding integral to $\sum_{i=1}^{d-1} \partial_{id}^2 f(x) y_i y_d$ is equal to zero by symmetric property in y_d , we have

$$|J_{1,2}(x)| \leq c \int_{|\tilde{y}| < 1} d\tilde{y} \int_0^{x_d} \left\{ \sum_{i,j=1}^{d-1} |\partial_{ij}^2 f(x)| |y_i| |y_j| + |\partial_d^2 f(x)| y_d^2 + \frac{1}{3} \|f^{(3)}\| |y|^3 \right\} \frac{dy_d}{|y|^{d+\alpha}}.$$

Let $0 < \epsilon < 2 - \alpha$ and set $\alpha_\epsilon := \alpha + \epsilon \in (1, 2)$, then $|y|^{d+\alpha-2} \geq |\tilde{y}|^{d-1-\epsilon} |y_d|^{-1+\alpha_\epsilon}$. By $|\partial_{ij}^2 f(x)|, |\partial_d^2 f(x)| \leq Cx_d$ for $i, j \neq d$, corresponding terms to $f^{(2)}$ are less than or equal to

$$Cx_d \int_{|\tilde{y}| < 1} \frac{d\tilde{y}}{|\tilde{y}|^{d-1-\epsilon}} \int_0^{x_d} \frac{dy_d}{y_d^{-1+\alpha_\epsilon}} = Cx_d \int_0^1 r^{\epsilon-1} dr \frac{x_d^{2-\alpha_\epsilon}}{2-\alpha_\epsilon} = \frac{C}{(2-\alpha_\epsilon)\epsilon} x_d^{3-\alpha_\epsilon}.$$

For the last term, by $d \geq 2$, $\alpha \geq 1$, i.e., $d + \alpha - 3 \geq 0$, we have $|y|^{d+\alpha-3} \geq |\tilde{y}|^{d+\alpha-3}$. Hence the last term is less than or equal to

$$\int_{|\tilde{y}| < 1} \frac{d\tilde{y}}{|\tilde{y}|^{d+\alpha-3}} \|f^{(3)}\|_\infty x_d = \int_0^1 r^{1-\alpha} dr \|f^{(3)}\|_\infty x_d = \frac{\|f^{(3)}\|_\infty}{2-\alpha} x_d.$$

These estimates imply $|J_{1,2}(x)| \leq C'x_d$. In the second term of (4.1) we also divide the integral area to $\{|y| \geq 1\}$, $\{|y| < 1\}$ and denote the corresponding terms by $J_2(x)$, $J_3(x)$ respectively. For $J_2(x)$, by

$$|f(x+y) - f(\tilde{y} + \tilde{x}, y_d - x_d)| \leq 2x_d \|\partial_d f\|_\infty$$

and $|f(x)| \leq Cx_d$, we have

$$\begin{aligned} |J_2(x)| &= \left| c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{x_d}^\infty [f(x+y) - f(\tilde{x} + \tilde{y}, y_d - x_d) - 2f(x)] I(|y| \geq 1) \frac{dy_d}{|y|^{d+\alpha}} \right| \\ &\leq c \int_H 2(\|\partial_d f\|_\infty + C)x_d I(|y| \geq 1) \frac{dy}{|y|^{d+\alpha}} \\ &\leq C'x_d \int_{|y| \geq 1} \frac{dy}{|y|^{d+\alpha}} = C''x_d. \end{aligned}$$

For $J_3(x) = c \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{x_d}^\infty [f(x+y) - f(\tilde{y} + \tilde{x}, y_d - x_d) - 2f(x)] I(|y| < 1) \frac{dy_d}{|y|^{d+\alpha}}$,

$$\begin{aligned} &f(x+y) - f(\tilde{x} + \tilde{y}, y_d - x_d) - 2f(x) \\ &= [f(x+y) - f(\tilde{x} + \tilde{y}, y_d - x_d) - 2f(\tilde{x} + \tilde{y}, x_d)] - 2[f(\tilde{x} + \tilde{y}, x_d) - f(x)]. \end{aligned}$$

For the first term, by the same way as in case of $d = 1$ (in $J_3(x)$) corresponding to the variable y_d , we have

$$\begin{aligned} & |f(x + y) - f(\tilde{x} + \tilde{y}, y_d - x_d) - 2f(x)| \\ & \leq 2\|\partial_d^3 f\|_\infty x_d(y_d^2 + x_d^2) + C(y_d + x_d)x_d y_d + 2Cx_d^3. \end{aligned}$$

For the second term, note that

$$f(\tilde{x} + \tilde{y}, x_d) - f(x) = \sum_{i=1}^{d-1} \partial_i f(\tilde{x}, x_d) y_i + \frac{1}{2} \sum_{i,j=1}^{d-1} \partial_{ij}^2 f(\tilde{x} + \theta\tilde{y}, x_d) y_i y_j.$$

and $|\partial_{ij}^2 f(\tilde{x} + \theta\tilde{y}, x_d)| \leq Cx_d$ for $i, j \leq d$. Moreover note that by the symmetric property in y_i we have

$$\begin{aligned} & \int_{\mathbf{R}^{d-1}} d\tilde{y} \int_{x_d}^{\infty} \sum_{i=1}^{d-1} \partial_i f(\tilde{x}, x_d) y_i I(|y| < 1) \frac{dy_d}{|y|^{d+\alpha}} \\ & = \sum_{i=1}^{d-1} \partial_i f(\tilde{x}, x_d) \int_{x_d}^1 dy_d \int_{|\tilde{y}| < \sqrt{1-y_d^2}} \frac{y_i d\tilde{y}}{|y|^{d+\alpha}} = 0. \end{aligned}$$

Let $0 < \epsilon < 2 - \alpha$ and set $\alpha_\epsilon := \alpha + \epsilon \in (0, 2)$, then $|y|^{d+\alpha} \geq |\tilde{y}|^{d-1-\epsilon} |y_d|^{1+\alpha_\epsilon}$ by $d \geq 2$. Thus we can get the following: by $x_d \leq y_d$,

$$\begin{aligned} |J_3(x)| & \leq c \int_{|\tilde{y}| < 1} d\tilde{y} \int_{x_d}^1 [2\|\partial_d^3 f\|_\infty x_d(y_d^2 + x_d^2) + C(y_d + x_d)x_d y_d + 2Cx_d^3] \frac{dy_d}{|y|^{d+\alpha}} \\ & \quad + c \int_{|\tilde{y}| < 1} d\tilde{y} \int_{x_d}^1 \frac{1}{2} \sum_{i,j=1}^{d-1} |\partial_{ij}^2 f(\tilde{x} + \theta\tilde{y}, x_d) y_i y_j| \frac{dy_d}{y_d^{d+\alpha}} \\ & \leq C' x_d \int_{|\tilde{y}| < 1} \frac{d\tilde{y}}{|\tilde{y}|^{d-1-\epsilon}} \int_{x_d}^1 (y_d^2 + x_d y_d + x_d^2) \frac{dy_d}{y_d^{1+\alpha_\epsilon}} \\ & \quad + c \int_{|\tilde{y}| < 1} \frac{1}{2} Cx_d |\tilde{y}|^2 d\tilde{y} \int_0^1 \frac{dy_d}{|y|^{d+\alpha}} \\ & \leq 3C' x_d \frac{1}{\epsilon} \int_0^1 y_d^{1-\alpha_\epsilon} dy_d + \frac{cC}{2} x_d \int_{|\tilde{y}| < 1} d\tilde{y} |\tilde{y}|^2 \int_0^1 \frac{dy_d}{|y|^{d+\alpha}}. \end{aligned}$$

For the second term if $0 < \alpha < 1$, then by $|y|^{d+\alpha} \geq |\tilde{y}|^d y_d^\alpha$,

$$\int_{|\tilde{y}| < 1} d\tilde{y} |\tilde{y}|^2 \int_0^1 \frac{dy_d}{|y|^{d+\alpha}} \leq \int_{|\tilde{y}| < 1} \frac{|\tilde{y}|^2}{|\tilde{y}|^d} d\tilde{y} \int_0^1 \frac{dy_d}{y_d^\alpha} = \frac{1}{1-\alpha},$$

or if $1 \leq \alpha < 2$, then by $|y|^{d+\alpha} \geq |\tilde{y}|^{d+1-\epsilon} y_d^{\alpha-1}$,

$$\int_{|\tilde{y}|<1} d\tilde{y} |\tilde{y}|^2 \int_0^1 \frac{dy_d}{|y|^{d+\alpha}} \leq \int_{|\tilde{y}|<1} \frac{|\tilde{y}|^2}{|\tilde{y}|^{d+1-\epsilon}} d\tilde{y} \int_0^1 \frac{dy_d}{y_d^{\alpha-1}} = \frac{1}{\epsilon(2-\alpha_\epsilon)}.$$

Therefore

$$\begin{aligned} |J_3(x)| &\leq \begin{cases} \left(\frac{3C'}{\epsilon(2-\alpha_\epsilon)} + \frac{cC}{2(1-\alpha)} \right) x_d & (0 < \alpha < 1) \\ \frac{6C'+cC}{2\epsilon(2-\alpha_\epsilon)} x_d & (1 \leq \alpha < 2) \end{cases} \\ &= C'' x_d. \end{aligned}$$

Therefore we have $|A^-f(x)| \leq C'' x_d$.

Next in order to show $Q^-f(x) \leq Cg_1(x)$, it suffices to prove that there is a constant $C > 0$ such that for $0 < x_d \leq 1$, if $0 < \alpha < 1$, then $Q^-f(x) \leq Cx_d$, if $\alpha = 1$, then $Q^-f(x) \leq Cx_d \log(1/x_d)$ if $1 < \alpha < 2$, then $Q^-f(x) \leq Cx_d^{2-\alpha}$. We use the first formula of (4.2). In the following we decompose as $Q^-f = (R_1 + R_2) + (S_1 + S_2)$ and we shall show each R_1, R_2, S_1 has order of $x_d^3, x_d^{2-\alpha}, x_d$ respectively, and the main parts is S_2 . In the first term of the right hand side of (4.2), we divide the integral area of \mathbf{R}^{d-1} to $\{|\tilde{y}| \geq 1\}, \{|\tilde{y}| < 1\}$ and denote the corresponding terms by $R_1(x), R_2(x)$ respectively. By $f(x) \leq Cx_d$, we have

$$\begin{aligned} R_1(x) &= c \int_{|\tilde{y}| \geq 1} d\tilde{y} \int_{-x_d}^{x_d} [f(x+y) - f(x)]^2 \frac{dy_d}{|y|^{d+\alpha}} \\ &\leq 2c \int_{|\tilde{y}| \geq 1} \frac{d\tilde{y}}{|\tilde{y}|^{d+\alpha}} \int_0^{x_d} C^2 (2x_d + y_d)^2 dy_d \\ &= Cx_d^3 \end{aligned}$$

For R_2 , by $|\partial_i f(x)| \leq Cx_d$ if $i \neq d$,

$$\begin{aligned} R_2(x) &= 2c \int_{|\tilde{y}|<1} d\tilde{y} \int_0^{x_d} [f(x+y) - f(x)]^2 \frac{dy_d}{|y|^{d+\alpha}} \\ &\leq 2c \int_{|\tilde{y}|<1} d\tilde{y} \int_0^{x_d} \left[C(x_d + y_d) \sum_{i=1}^{d-1} y_i + \|\partial_d f\|_\infty y_d \right]^2 \frac{dy_d}{|y|^{d+\alpha}} \\ &\leq C \int_{|\tilde{y}|<1} d\tilde{y} \int_0^{x_d} [(x_d^2 + y_d^2) |\tilde{y}|^2 + y_d^2] \frac{dy_d}{|y|^{d+\alpha}} \end{aligned}$$

In the above we first consider the last term (which is the main term), i.e.,

$$\int_{|\tilde{y}|<1} d\tilde{y} \int_0^{x_d} y_d^2 \frac{dy_d}{|y|^{d+\alpha}} = \int_0^{x_d} dy_d y_d^2 \left(\int_{|\tilde{y}|<y_d} + \int_{y_d \leq |\tilde{y}|<1} \right) \frac{d\tilde{y}}{|y|^{d+\alpha}} =: R_{2,1}(x) + R_{2,2}(x).$$

For $R_{2,1}$, let $\alpha_\epsilon = \alpha + \epsilon < 2$ be the same as before. By $|y|^{d+\alpha} \geq |\tilde{y}|^{d-1-\epsilon} |y_d|^{1+\alpha_\epsilon}$,

$$\int_{|\tilde{y}|<y_d} \frac{d\tilde{y}}{|\tilde{y}|^{d-1-\epsilon}} = \int_0^{y_d} r^{\epsilon-1} dr = \frac{y_d^\epsilon}{\epsilon}.$$

Hence

$$R_{2,1}(x) \leq \int_0^{x_d} \frac{y_d^2}{y_d^{1+\alpha+\epsilon}} \frac{y_d^\epsilon}{\epsilon} dy_d = \frac{1}{\epsilon} \int_0^{x_d} y_d^{1-\alpha} dy_d = \frac{1}{(2-\alpha)\epsilon} x_d^{2-\alpha}.$$

For $R_{2,2}$, by $|y|^{d+\alpha} \geq |\tilde{y}|^{d+\alpha}$ and

$$\int_{y_d \leq |\tilde{y}|<1} \frac{d\tilde{y}}{|\tilde{y}|^{d+\alpha}} = \int_{y_d}^1 \frac{dr}{r^{2+\alpha}} = \frac{1}{1+\alpha} (y_d^{-1-\alpha} - 1) \leq \frac{1}{1+\alpha} y_d^{-1-\alpha}.$$

Hence

$$R_{2,2}(x) \leq \int_0^{x_d} y_d^2 \frac{1}{1+\alpha} y_d^{-1-\alpha} dy_d = \frac{1}{1+\alpha} \int_0^{x_d} y_d^{1-\alpha} dy_d = \frac{1}{(1+\alpha)(2-\alpha)} x_d^{2-\alpha}.$$

Furthermore we can show more easily that the other terms of R_2 are $o(x_d^2)$. In fact, by $|y|^{d+\alpha} \geq |\tilde{y}|^{d-1+\alpha_\epsilon} |y_d|^{1-\epsilon}$,

$$\int_{|\tilde{y}|<1} d\tilde{y} \int_0^{x_d} (x_d^2 + y_d^2) |\tilde{y}|^2 \frac{dy_d}{|y|^{d+\alpha}} \leq \int_{|\tilde{y}|<1} \frac{|\tilde{y}|^2}{|\tilde{y}|^{d-1+\alpha_\epsilon}} d\tilde{y} \int_0^{x_d} (x_d^2 + y_d^2) \frac{dy_d}{y_d^{1-\epsilon}} = Cx_d^{2+\epsilon}.$$

Therefore we have $R_2(x) \leq Cx_d^{2-\alpha}$ for all $0 < \alpha < 2$.

In the second term of the right hand side of (4.2), we divide the integral area to $\{|y| \geq 1\}$, $\{|y| < 1\}$ and denote the corresponding terms by $S_1(x)$, $S_2(x)$ respectively. For S_1 , by

$$(4.3) \quad |f(x+y) - f(\tilde{y} + \tilde{x}, y_d - x_d)| \leq 2x_d \|\partial_d f\|_\infty,$$

we have

$$S_1(x) \leq (2x_d \|\partial_d f\|_\infty \cdot 3\|f\|_\infty + 2Cx_d^2) \int_{|y| \geq 1} \frac{dy_d}{|y|^{d+\alpha}} \leq Cx_d.$$

For S_2 , by (4.3) and by $|f(x)| \leq Cx_d$,

$$\begin{aligned}
& |\{f(x+y) - f(\tilde{y} + \tilde{x}, y_d - x_d)\}\{f(x+y) + f(\tilde{y} + \tilde{x}, y_d - x_d) - 2f(x)\} + 2f(x)^2| \\
& \leq 2\|\partial_d f\|_\infty x_d \cdot C(x_d + y_d) + 2Cx_d^2 \\
& \leq Cx_d(x_d + y_d).
\end{aligned}$$

Hence, noting that $\{|y| < 1\} \subset \{|\tilde{y}| < 1\} \times \{|y_d| < 1\}$,

$$|S_2(x)| \leq Cx_d \int_{|\tilde{y}| < 1} d\tilde{y} \int_{x_d}^1 (x_d + y_d) \frac{dy_d}{|y|^{d+\alpha}} \leq 2Cx_d \int_{|\tilde{y}| < 1} d\tilde{y} \int_{x_d}^1 y_d \frac{dy_d}{|y|^{d+\alpha}}.$$

By the same way as in R_2 , we can show the desired estimate as follows. Let

$$\int_{x_d}^1 dy_d y_d \left(\int_{|\tilde{y}| < y_d} + \int_{y_d \leq |\tilde{y}| < 1} \right) \frac{d\tilde{y}}{|y|^{d+\alpha}} =: (S_{2,1}(x) + S_{2,2}(x)).$$

Then $|S_2(x)| \leq Cx_d(S_{2,1}(x) + S_{2,2}(x))$. Let $0 < \epsilon < 2 - \alpha$. By $|y|^{d+\alpha} \geq |\tilde{y}|^{d-1-\epsilon} |y_d|^{1+\alpha_\epsilon}$,

$$S_{2,1}(x) \leq \int_{x_d}^1 \frac{dy_d}{y_d^{1+\alpha_\epsilon}} y_d \int_{|\tilde{y}| < y_d} \frac{d\tilde{y}}{|\tilde{y}|^{d-1+\epsilon}} = \int_{x_d}^1 \frac{dy_d}{y_d^{\alpha+\epsilon}} \frac{y_d^\epsilon}{\epsilon} = \frac{1}{\epsilon} \int_{x_d}^1 \frac{dy_d}{y_d^\alpha}.$$

That is, if $0 < \alpha < 1$, then $S_{2,1}(x) \leq C$, if $\alpha = 1$, then $S_{2,1}(x) \leq C \log(1/x_d)$, if $0 < \alpha < 1$, then $S_{2,1}(x) \leq Cx_d^{1-\alpha}$. Moreover for $S_{2,2}$, as in $R_{2,2}$, by $\int_{y_d \leq |\tilde{y}| < 1} d\tilde{y}/|\tilde{y}|^{d+\alpha} \leq y_d^{-1-\alpha}/(1+\alpha)$,

$$S_{2,2}(x) \leq \int_{x_d}^1 dy_d y_d \int_{y_d \leq |\tilde{y}| < 1} \frac{d\tilde{y}}{|\tilde{y}|^{d+\alpha}} \leq \int_{x_d}^1 y_d \frac{1}{1+\alpha} y_d^{-1-\alpha} dy_d = \frac{1}{1+\alpha} \int_{x_d}^1 y_d^{-\alpha} dy_d.$$

Thus $S_{2,2}$ satisfies the same estimates as $S_{2,1}$. By $|S_2(x)| \leq Cx_d(S_{2,1}(x) + S_{2,2}(x))$, we have if $0 < \alpha < 1$, then $S_2(x) \leq Cx_d$, if $\alpha = 1$, then $S_2(x) \leq Cx_d \log(1/x_d)$ if $1 < \alpha < 2$, then $S_2(x) \leq Cx_d^{2-\alpha}$. These imply our desired result. \blacksquare

By $P_t^- C_c^\infty \subset C_{p,0}^3$, the following result for martingale problem is obtained by the same way as in $d = 1$.

THEOREM 6. *Let $\mu \in \mathcal{M}_{g_p,0}$. The martingale problems for $(\mathcal{L}_0, \mathcal{D}_0, \mu)$, $(\mathcal{L}^Y, \mathcal{D}_0, \mu)$ associated with absorbing stable motion on H are well-posed.*

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