TSUKUBA J. MATH.
Vol. 37 No. 1 (2013), 1–11

ON THE FOURIER COEFFICIENTS OF HILBERT MODULAR FORMS OF HALF-INTEGRAL WEIGHT OVER ARBITRARY ALGEBRAIC NUMBER FIELDS

By

Hisashi Kojima

Abstract. In Theorem 2.5 in previous paper [4], we determined the Fourier coefficients of the image of Shimura correspondence of modular forms f of half integral weight over arbitrary algebraic number fields in terms of those of f. It seems that there is a gap in the proof. We give a correct proof of Theorem 2.5 in [4]. Moreover, we deduce useful formulas between the product of Fourier coefficients of f and the central value of quadratic twisted L-series associated with the image of Shimura correspondence of f.

Introduction

Shimura [7] proved that the square of Fourier coefficients of a holomorphic Hilbert modular form of half-integral weight over a totally real number field gives essentially the critical value of the zeta function of the corresponding form of integral weight, which generalizes a previous result of Waldspurger [9] in the elliptic modular case. In [3] and [4], we extended Shimura [6] and [7] in the case of Hilbert modular forms of half-integral weight over arbitrary algebraic number fields. It seems that there is a gap in the proof of Theorem 2.5 in [4].

The purpose of this note is to deduce another useful formula between the product of Fourier coefficients of a modular form f of half-integral weight over an arbitrary algebraic number field and the central value of quadratic twisted L-series associated with the image of Shimura correspondence of f. In the last section, we shall give a correct proof of Theorem 2.5 in [4].

Received February 21, 2012.
Revised October 17, 2012.

§ 1. Fourier Coefficients of Modular Forms of Half-Integral Weight

Our notation follows closely that of [2], [4], [5] and [7]. Let $\varepsilon \in G_{\mathbf{A}}$ (resp. C'') be the element (resp. set) given in [4, pp. 29–30]. Take $\alpha \in G \cap U\varepsilon^{-1}$, for U a sufficiently small open subgroup of C''. Let f be an element of $\mathscr{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{b}, \mathfrak{b}'; \psi)$, where $\mathscr{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{b}, \mathfrak{b}'; \psi)$ is the space given in [4, p. 31] and [5, (2)]. Then define the inversion f^* of f by

$$f^* = \psi(\delta) f \|_{m+(1/2)u_{r_1}} \alpha. \tag{1.1}$$

Here δ and $f\|_{m+(1/2)u_{r_1}} \alpha$ are given in [4, p. 30] and [4, (1.16)]. We see that f^* belongs to $\mathscr{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{b}', \mathfrak{b}; \bar{\psi})$ (cf. [2, (4.19)]). Take a $f \in \mathscr{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{b}, \mathfrak{b}'; \psi)$. Let τ be an element of $F^\times$ such that $\tau \gg 0$, $\tau\mathfrak{b} = \mathfrak{q}^2\mathfrak{r}$ with a fractional ideal $\mathfrak{q}$ and a square free integral ideal $\mathfrak{r}$. From [4, Lemma 1.2], we find an element $h \in \mathscr{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{o}, \mathfrak{r}\mathfrak{b}\mathfrak{b}'; \varphi)$ such that

$$\mu_h(\xi, \mathfrak{m}) = \mu_f(\tau\xi, (\mathfrak{q}\mathfrak{r})^{-1}\mathfrak{m}) \tag{1.2}$$

for every $\xi \in F^\times$ and fractional ideal $\mathfrak{m}$ in F, where $\varphi = \psi\varepsilon_\tau$ with the Hecke character ε_τ associated with the quadratic extension $F(\sqrt{\tau})/F$. Let D be the set given in [4, (1.9)]. Define a function $g_{\tau,\lambda}(\mathfrak{w}) = \Psi_{\tau,\lambda}(f)(\mathfrak{w})$ on D by

$$Cg_{\tau,\lambda}(\mathfrak{w}) = \int_{\Gamma_{\mathfrak{r}\mathfrak{c}} \backslash D} h(\mathfrak{z}) \Theta(\mathfrak{z}, \mathfrak{w}; \eta_\lambda) \Im(z)^{m+(1/2)u_{r_1}} w^3 \, d\mathfrak{z} \tag{1.3}$$

for every $\mathfrak{w} \in D$, where $C = i^{\{m\}} 2^{1+r_1-r_2+\{m\}} (1/\sqrt{2\pi})^{r_2} \varphi_a(1/2) N(\mathfrak{r}\mathfrak{c})$, $\Gamma_{\mathfrak{r}\mathfrak{c}}$ and $\Theta(\mathfrak{z}, \mathfrak{w}; \eta_\lambda)$ are given in [4, p. 39]. We deduced the following theorem [4, (2.33)].

THEOREM 0.1. *Let f be an element of $\mathscr{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{b}, \mathfrak{b}'; \psi)$. Then*

$$\begin{aligned} \Psi_{\tau,\lambda}(f)(\mathfrak{w}) = N(t_\lambda/\mathfrak{r}) \sum_{\mathfrak{m}} \sum_{l \in t_\lambda \mathfrak{r}^{-1}\mathfrak{m}} & N(\mathfrak{m}) l^{m-1} |l|^{-1} \varphi_a(l) \varphi^*(l\mathfrak{r}/t_\lambda \mathfrak{m}) \\ & \times \mu_f(\tau, (\mathfrak{r}\mathfrak{q})^{-1}\mathfrak{m}) e_s(l\Re(z)) e_c(lu) \prod_{i=1}^{r_1} c(\mathrm{sgn}(l^{(i)})) \\ & \times \exp(-2\pi l \Im(z)) v K_{2v}(4\pi |l| v), \end{aligned} \tag{1.4}$$

where $\mathfrak{m}$ runs over all integral ideals, l runs over $t_\lambda \mathfrak{r}^{-1}\mathfrak{m}$ under the condition $(lt_\lambda^{-1}\mathfrak{m}^{-1}\mathfrak{r}, \mathfrak{r}\mathfrak{c}) = 1$, $\mathfrak{w} = (z_1, \dots, z_{r_1}, \mathfrak{z}_{r_1+1}, \dots, \mathfrak{z}_{r_1+r_2})$, $z = (z_1, \dots, z_{r_1})$, $\mathfrak{z}_{r_1+i} = u_{r_1+i} + jv_{r_1+i}$ $(1 \le i \le r_2)$, $u = (u_{r_1+1}, \dots, u_{r_1+r_2})$, $v = (v_{r_1+1}, \dots, v_{r_1+r_2})$, $l^{m-1} = \prod_{i=1}^{r_1} (l^{(i)})^{m_i-1}$ and $|l| = \prod_{i=1}^{r_2} |l^{(r_1+i)}|$.

We shall give a correct proof of Theorem 0.1, that is, Theorem 2.5 in [4] in Section 2.

We showed the following in [4, pp. 47–48].

THEOREM 0.2. *Let f be an element of $\mathscr{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{b},\mathfrak{b}';\psi)$. Suppose that f is a common eigenform of T_v for each $v \in h$, i.e.,*

$$(1.5)\qquad f|T_v = \chi(v)N_v^{-1}f \quad \textit{for each } v \in h.$$

Then there exists the normalized eigenform $\boldsymbol{g}$ belonging to $\mathscr{S}_{2m,\tilde{\omega}}(2^{-1}\mathfrak{c},\psi^2)$ attached to χ such that

$$(1.6)\qquad \mu_f(\tau,\mathfrak{q}^{-1})\boldsymbol{g} = (g_{\tau,1},\ldots,g_{\tau,\kappa}),$$

where $\tilde{\omega} = (0,\ldots,0,4\omega_{r_1+1}+3,\ldots,4\omega_{r_1+r_2}+3)$ *with* $\omega = (0,\ldots,0,\omega_{r_1+1},\ldots,\omega_{r_1+r_2})$.

Let $\boldsymbol{g}$ be the above element of $\mathscr{S}_{2m,\tilde{\omega}}(2^{-1}\mathfrak{c},\psi^2)$ in Theorem 0.2. Take the matrix $\pi = \begin{pmatrix} 0 & -1 \\ \delta^2 s & 0 \end{pmatrix}$ with $s \in F_f^\times$ such that $s\mathfrak{o} = 2^{-1}\mathfrak{c}$. Define

$$(1.7)\qquad (J_{2^{-1}\mathfrak{c}}\boldsymbol{g})(p) = \psi^2(\det p)^{-1}\boldsymbol{g}(p\pi) \quad \text{for every } p \in \tilde{G}_{\mathbf{A}}$$

Then $J_{2^{-1}\mathfrak{c}}\boldsymbol{g}$ belongs to $\mathscr{S}_{2m,\tilde{\omega}}(2^{-1}\mathfrak{c},\psi^{-2})$. We put $\boldsymbol{g}^* = J_{2^{-1}\mathfrak{c}}(\boldsymbol{g}) = (g'_\lambda)$.

Here we assume the following condition.

(1.8) (i) $\psi_a(x) = (\operatorname{sgn} x_s)^m |x_s|^{i\lambda}|x_c|^{2i\mu}$ $(x \in F_a^\times)$, where $(\operatorname{sgn}(x_s))^m = \prod_{i=1}^{r_1}\operatorname{sgn}(x_i)^{m_i}$, $|x_s|^{i\lambda} = \prod_{i=1}^{r_1}|x_i|^{\sqrt{-1}\lambda i}$ $(x_s = (x_1,\ldots,x_{r_1}) \in F_s^\times)$, $|x_c|^{2i\mu} = \prod_{i=1}^{r_2}|x_{r_1+i}|^{2\sqrt{-1}\mu_{r_1+i}}$ $(x_c = (x_{r_1+1},\ldots,x_{r_1+r_2}) \in F_c)$, $(\lambda_1,\ldots,\lambda_{r_1},\mu_{r_1+1},\ldots,\mu_{r_1+r_2}) \in \mathbf{R}^{r_1+r_2}$ and $\sum_{i=1}^{r_1}\lambda_i + \sum_{i=1}^{r_2}\mu_{r_1+i} = 0$.

(ii) $\mathfrak{r}$ divide $\mathfrak{h}$, where $\mathfrak{h}$ is the conductor of φ.

(iii) If v is a common prime of 2 and $\mathfrak{r}$, then φ_v satisfies either
(a) $(\mathfrak{rc})_v = \mathfrak{h}_v = 4\mathfrak{r}_v$ and $\varphi_v(1+4x) = \varphi_v(1+4x^2)$ for every $x \in \mathfrak{o}_v$, or
(b) $(\mathfrak{rc})_v \neq \mathfrak{h}_v \subset 4\mathfrak{r}_v$.

(iv) If $f' \in \mathscr{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{b},\mathfrak{b}';\psi)$ and $f'|T_v = N_v^{-1}\chi(v)f'$ for every $v \nmid \mathfrak{h}^{-1}\mathfrak{rc}$, then f' is a constant times f.

We shall deduce the following theorem.

THEOREM 1. *Let $f \in \mathscr{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{b},\mathfrak{b}';\psi)$ be an eigenform of all Hecke operators T_v satisfying $f|T_v = N_v^{-1}\chi(v)f$. Suppose that f, $\mathfrak{r}$, $\mathfrak{h}$, $\mathfrak{c}$, ψ and φ satisfy the condition* (1.8), *and $\boldsymbol{g}$ and $\boldsymbol{g}^*$ are the elements in Theorem* 0.2. *Then*

$$(1.9)\qquad \overline{\mu(\tau, \mathfrak{q}^{-1}; f, \psi)}\mu(\tau, \mathfrak{q}^{-1}\mathfrak{b}; f^*, \bar{\psi})\langle \boldsymbol{g}, \boldsymbol{g}\rangle/\langle f, f\rangle$$

$$= Q \sum_{\mathfrak{o}\supset\mathfrak{t}\supset\mathfrak{i}} \mu(\mathfrak{t})\overline{\varphi^*}(\mathfrak{t})N(\mathfrak{t})^{-1}D(0, \boldsymbol{g}^*, \varphi, \mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}),$$

where $D(0, \boldsymbol{g}^*, \varphi, \mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c})$ *is given in* [4, p. 37], $Q = 2^{(r_1/2)-\{m\}+3r_2-1}\pi^{-\{m\}}\cdot |\tau_c|^2\tau_s^m\psi_a(\tau)^{-1}N(\mathfrak{h})^{-1}\overline{r(\varphi)}h_F[\mathfrak{o}_+^\times : (\mathfrak{o}^\times)^2]\Gamma'(m)\Gamma'(v+1/2)\Gamma'(-v+1/2)$, $|\tau_c|^2 = \prod_{i=1}^{r_2}|\tau^{(r_1+i)}|^2$, $\tau_s^m = \prod_{i=1}^{r_1}(\tau^{(i)})^{m_i}$, $\mathfrak{i} = \prod_{\mathfrak{p}}\mathfrak{p}$ $(\mathfrak{p}|\mathfrak{r}\mathfrak{c}, \mathfrak{p}\nmid\mathfrak{h})$ *and* $\Gamma'(m)\Gamma'(v+1/2) \times \Gamma'(-v+1/2)$ *is given in* [4].

Let η be an element in [4, p. 38]. Put $\tilde{h}(\mathfrak{z}) = \langle\Theta(\mathfrak{z}, p; \eta), \boldsymbol{g}(p)\rangle$, where $\Theta(\mathfrak{z}, p; \eta)$ is the function given in [4, (2.4)] and $\boldsymbol{g}$ is the function given in Theorem 0.2. By [7, Proposition 5.8] and [2, Theorem 5.2 and the arguments in p. 440], we have

$$(1.10)\qquad \tilde{h}(\mathfrak{z}) = Ah(\mathfrak{z})$$

with a constant A under the assumption (1.8), where $h(\mathfrak{z})$ is the function given in (1.2). Since $\langle h, h\rangle = \tau_s^{m+(1/2)u_{r_1}}|\tau_c|^3N(\mathfrak{q}\mathfrak{r})^{-1}\langle f, f\rangle$ and

$$(1.11)\qquad C\mu_f(\tau, \mathfrak{q}^{-1})\boldsymbol{g}(p) = \int_\Phi \Theta(\mathfrak{z}, p; \eta)h(\mathfrak{z})y^{m+(1/2)u_{r_1}}w^3\, d\mathfrak{z},$$

we obtain

$$(1.12)\qquad A = i^{-\{m\}}2^{1+r_1-r_2+\{m\}}(1/\sqrt{2\pi})^{r_2}\varphi_a(1/2)\tau_s^{-(m+(1/2)u_{r_1})}|\tau_c|^{-3}$$

$$\times \frac{N(\mathfrak{q}\mathfrak{r}^2\mathfrak{c})\langle \boldsymbol{g}, \boldsymbol{g}\rangle\overline{\mu_f(\tau, \mathfrak{q}^{-1})}}{\mathrm{vol}(\Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}]\backslash D)\langle f, f\rangle}$$

with Φ, h, C as in [4, p. 39]. As shown at [7, p. 540], $Ah(\mathfrak{z}) = \langle\Theta(\mathfrak{z}, p; \eta), \boldsymbol{g}(p)\rangle$ implies that

$$(1.13)\qquad Ah^*(\mathfrak{z}) = \langle\Theta(\mathfrak{z}, p; \sigma), \boldsymbol{g}^*(p)\rangle = \sum_\lambda \langle\Theta(\mathfrak{z}, \mathfrak{w}; \sigma_\lambda), g'_\lambda(\mathfrak{w})\rangle,$$

where σ_λ (resp. $\Theta(\mathfrak{z}, p; \sigma)$) is the symbol given in [7, (6.2)] (resp. [4, (2.4)]).

Given a function f on D and $\alpha = \begin{pmatrix} * & * \\ c_\alpha & d_\alpha \end{pmatrix}$ in G, we put

$$(1.14)\qquad f\|_m\alpha(\mathfrak{z}) = (c_\alpha z + d_\alpha)^{-m}f(\alpha(\mathfrak{z})),$$

where $\mathfrak{z} = (z_1, \ldots, z_{r_1}, \mathfrak{z}_{r_1+1}, \ldots, \mathfrak{z}_{r_1+r_2})$, $z = (z_1, \ldots, z_{r_1})$ and $\mathfrak{z}_{r_1+i} = z_{r_1+i} + jw_{r_1+i}$. Let $\Gamma = \Gamma[\mathfrak{x}, \mathfrak{h}]$ (cf. [4, p. 29]). We put

$$(1.15)\quad E(\mathfrak{z},s;\Gamma)=\sum_{\alpha\in R}\varphi_a(d_\alpha)\varphi^*(d_\alpha\mathfrak{A}_\alpha^{-1})N(\mathfrak{A}_\alpha)^{2s}y^{su_{r_1}+(i\lambda-m)/2}w^{2su_{r_2}+i\mu}\|_m\alpha$$

$$C(\mathfrak{z},s;\Gamma)=L_{\mathfrak{x}\mathfrak{h}}(2s,\varphi)E(\mathfrak{z},s;\Gamma)$$

Here R is a set of representatives for $P\backslash(G\cap P_{\mathbf{A}}D[\mathfrak{x},\mathfrak{h}])$, for $\alpha\in R$, we define $\mathfrak{A}_\alpha$ by writing $\alpha=pw$ with $p\in P_{\mathbf{A}}$ and $w\in D(\mathfrak{x},\mathfrak{h})$, and setting $\mathfrak{A}_\alpha=d_p\mathfrak{o}$. We put

$$(1.16)\qquad L_{\mathfrak{x}\mathfrak{h}}(s,\varphi)=\sum_{\mathfrak{m}+\mathfrak{x}\mathfrak{h}=\mathfrak{o}}\varphi^*(\mathfrak{m})N(\mathfrak{m})^{-s}.$$

We obtain the following proposition.

Proposition 2. *Let* $\Gamma=\Gamma[2^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c},2\mathfrak{h}]$ *and let* $\vartheta(\mathfrak{z})$ *be the function in* [4, (4.1)]. *Then*

$$(1.17)\quad \int_{\Gamma\backslash D}h^*(\mathfrak{z})\vartheta(\mathfrak{z})E(\mathfrak{z},\bar{s}+1/2;\Gamma)y^{m+(1/2)u_{r_1}}w^2\,d\mathfrak{z}$$
$$=D_F^{-1/2}2^{1-r_1}|\tau_c|\tau^{(1/2)u_{r_1}}\psi_a(\tau)(2\pi)^{-2su_{r_2}-u_{r_2}+i\mu}2^{-2su_{r_2}+i\mu-(1/2)u_{r_2}}$$
$$\times\sqrt{\pi}^{r_2}(2\pi)^{-su_{r_1}+(1/2)i\lambda-(1/2)\{m\}}\Gamma'(s+(m-i\lambda)/2)$$
$$\times\Gamma'(2su_{r_2}-i\mu+(1/2)u_{r_2}-v)\Gamma'(2su_{r_2}-i\mu+(1/2)u_{r_2}+v)$$
$$\times\Gamma'(2su_{r_2}-i\mu+u_{r_2})^{-1}\sum_{\mathfrak{m}}\mu_{f^*}(\tau,\mathfrak{q}^{-1}\mathfrak{b}\mathfrak{m})N(\mathfrak{m})^{-2s}.$$

By (1.13) and Proposition 2, we see that A times the integral in (1.17) is equal to

$$(1.18)\quad \sum_\lambda\left\langle\int_{\Gamma\backslash D}\vartheta(\mathfrak{z})\Theta(\mathfrak{z},\mathfrak{w};\sigma_\lambda)E(\mathfrak{z},\bar{s}+1/2;\Gamma)y^{m+(1/2)u_{r_1}}w^2\,d\mathfrak{z},g'_\lambda(\mathfrak{w})\right\rangle.$$

By the same method as that of [7, pp. 543–544], we have the following equation (cf. [4, (4.19)]).

$$(1.19)\quad AN(\mathfrak{q}\mathfrak{r})^{-1}2^{-r_1/2-2sr_1-\{m\}}2^{-4su_{r_2}-(3/2)u_{r_2}}|\tau_c|\tau_s^{(1/2)u_{r_1}}\psi_a(\tau)\pi^{r_1/2}\pi^{r_2/2}$$
$$\times\Gamma'(2su_{r_2}-i\mu+(1/2)u_{r_2}-v)\Gamma'(2su_{r_2}-i\mu+(1/2)u_{r_2}+v)$$
$$\times\Gamma'(2su_{r_2}-i\mu+u_{r_2})^{-1}\Gamma'(s+(m-i\lambda)/2)\Gamma'(s+(1+m-\lambda i)/2)^{-1}$$

$$\times \Gamma'(2su_{r_2} - i\mu + u_{r_2})^{-1} 2^{i\lambda} 2^{2i\mu} \sum_{\mathfrak{m}} \mu_{f^*}(\tau, \mathfrak{q}^{-1}\mathfrak{b}\mathfrak{m}) N(\mathfrak{m})^{-2s}$$

$$= \sum_{\lambda} \left\langle \sum_{\beta \in B} \bar{\varphi}^*(\mathfrak{A}_\beta) N(\mathfrak{A}_\beta)^{2\bar{s}+1} S_{\beta\lambda}(\mathfrak{w}, \bar{s}), g'_\lambda(\mathfrak{w}) \right\rangle,$$

where B is determined by $G \cap P_{\mathbf{A}} D[2^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}, 2\mathfrak{h}] = \coprod_{\beta \in B} P\beta\Gamma$. The ideals $\mathfrak{A}_\beta$ are as in (1.15), and run through a set of representatives for the ideal class group of F. Here

$$S_{\beta\lambda}(\mathfrak{w}, s) = \sum_{\xi, b} \sigma_\lambda(\gamma\xi) \mu_\beta(b) [\xi, \mathfrak{w}]^{-m} |[\xi, \mathfrak{w}]/\eta(\mathfrak{w})|^{-2su_{r_1} - u_{r_1} + m - i\lambda} \times \left| \left[\xi + b \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathfrak{w} \right] \Big/ \eta(\mathfrak{w}) \right|^{-2(2su_{r_2} + u_{r_2} + i\mu)}, \tag{1.20}$$

where $[*, *]$ and $\eta(*)$ are symbols given in [4, (2.3)] and the sum is over the pairs $(\xi, b) \in V \times \mathfrak{A}_\beta/\mathfrak{o}^\times$ such that $\xi \neq 0$ and $\det \xi = -b^2$ with $V = \{\xi \in M_2(F) \mid \mathrm{tr}\, \xi = 0\}$. Furthermore, we have chosen $\gamma \in F_f^\times$ such that $\gamma\mathfrak{o} = \mathfrak{A}_\beta$ and $\gamma_v = 1$ for $v | \mathfrak{r}\mathfrak{c}$. By [7, 7.14a, 7.14b], we have the following.

PROPOSITION 3. *Let q range through a set of representatives for $2^{-1}t_\lambda\mathfrak{r}\mathfrak{c}\mathfrak{h}/t_\lambda\mathfrak{r}\mathfrak{c}\mathfrak{h}$ and let $\Gamma^\lambda = \Gamma[2t_\lambda^{-1}\mathfrak{h}^{-1}, t_\lambda\mathfrak{r}\mathfrak{c}\mathfrak{h}]$. Then there exist functions $T_{\beta\lambda}(\mathfrak{w}, s)$ such that*

$$S_{\beta\lambda}(\mathfrak{w}, s) = (-1)^{\{m\}} 2^{r_1} \sum_q T_{\beta\lambda}(\mathfrak{w}, s) \|_{2m} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \tag{1.21}$$

$$\sum_{\beta \in B} \bar{\varphi}^*(\mathfrak{A}_\beta) N(\mathfrak{A}_\beta)^{2s} T_{\beta\lambda}(\mathfrak{w}, s - 1/2) = N(2^{-1}t_\lambda\mathfrak{r}\mathfrak{c}\mathfrak{h})^{2s} C(\mathfrak{w}, s; \Gamma^\lambda) E(\mathfrak{w}, s; \Gamma^\lambda)$$

By Proposition 3, we find that the expression of (1.19) is equal to the value at $s = t$ of

$$(-1)^{\{m\}} 2^{2d} \sum_\lambda \langle N(2^{-1}t_\lambda\mathfrak{r}\mathfrak{c}\mathfrak{h})^{\bar{s}+\bar{t}+1} C(\mathfrak{z}, \bar{s} + 1/2; \Gamma^\lambda) E(\mathfrak{z}, \bar{t} + 1/2; \Gamma^\lambda), g'_\lambda(\mathfrak{z}) \rangle. \tag{1.22}$$

The equality (1.22) becomes

$$(-1)^{\{m\}} 2^d \operatorname{vol}(\Gamma[2\mathfrak{h}^{-1}, 2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{h}] \backslash D)^{-1} \sum_\lambda N(2^{-1}t_\lambda\mathfrak{r}\mathfrak{c}\mathfrak{h})^{s+t+1} \times \int_{\Gamma^\lambda \backslash D} \overline{C(\mathfrak{z}, \bar{s} + 1/2; \Gamma^\lambda) E(\mathfrak{z}, \bar{t} + 1/2; \Gamma^\lambda)} g'_\lambda(\mathfrak{z}) y^{2m} \, d\mathfrak{z} \tag{1.23}$$

The integral appeared in (1.23) is equal to

$$(1.24)\qquad \int_{\Gamma^\lambda\backslash D} \overline{C(\mathfrak{z},\bar{s}+1/2;\Gamma^\lambda)E(\mathfrak{z},\bar{t}+1/2;\Gamma^\lambda)}g'_\lambda(\mathfrak{z})y^{2m}\,d\mathfrak{z}$$
$$= \sum_{\alpha\in A}\bar{\varphi}_a(d_\alpha)\bar{\varphi}^*(d_\alpha\mathfrak{A}_\alpha^{-1})N(\mathfrak{A}_\alpha)^{2t+1}$$
$$\times \int_{\Psi_\lambda^\alpha} g_\lambda^\alpha(\mathfrak{z})\overline{C_\lambda^\alpha(\mathfrak{z})}\,y^{tu+(u+3m-i\lambda)/2}w^{2(t+1/2)u_{r_2}-i\mu}\,d\mathfrak{z},$$

where $\Psi_\lambda^\alpha = P\cap\alpha\Gamma^\lambda\alpha^{-1}\backslash D$, $g_\lambda^\alpha = g'_\lambda\|_{2m}\alpha^{-1}$ and $C_\lambda^\alpha(\mathfrak{z}) = C(\mathfrak{z},\bar{s}+1/2;\Gamma^\lambda)\|_m\alpha^{-1}$. By [7, Lemma 3.8], we have

$$(1.25)\quad g_\lambda^\alpha(\mathfrak{z}) = \varphi_a(d_\alpha)^2\varphi^*(d_\alpha\mathfrak{A}_\alpha^{-1})^2\sum_{0\neq\xi\in t_\lambda\mathfrak{A}_\alpha^2,\,\xi\gg0} c(\xi t_\lambda^{-1}\mathfrak{A}_\alpha^{-2},\operatorname{sgn}(\xi);\boldsymbol{g}^*)|\xi|^{m-i\lambda}|\xi|^{1-2i\mu}$$
$$\times e_s(\xi\Re(z))e_c(\xi u)\exp(-2\pi\xi\Im(z))wK_{2\nu}(4\pi|\xi|w)$$

and

$$\varphi_a(d_\alpha)\varphi^*(d_\alpha\mathfrak{A}_\alpha^{-1})N(\mathfrak{A}_\alpha)^{-2s-1}y^{-su-(u-m-i\lambda)/2}w^{-2(s+1/2)u_{r_2}+i\mu}\overline{C}_\lambda^\alpha(\mathfrak{z})$$
$$= L_{\mathfrak{r}\mathfrak{c}}(2s+1,\bar{\varphi}) + 2^{r_2}D_F^{-1/2}\overline{\gamma(\varphi)}N(\mathfrak{h})^{-1}\sum_{\mathfrak{o}\supset\mathfrak{t}\supset2\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}}\mu(\mathfrak{t})\bar{\varphi}^*(\mathfrak{t})N(\mathfrak{t})^{-2s-1}$$
$$\times\sum_{\mathfrak{y}}N(\mathfrak{y})^{2s}\sum_{h,b}\varphi_a(b)N(b)^{-2s}\varphi_a(h)\varphi^*(h\mathfrak{h}\mathfrak{d}\mathfrak{y})e_s(-bh\Re(z))$$
$$\times\xi(y,w,bh;\bar{s}u+(u+m+i\lambda)/2,\bar{s}u+(u-m+i\lambda)/2,2(\bar{s}+1/2)+i\mu),$$

where $c(\mathfrak{m},\sigma;\boldsymbol{g}^*)$ is given in [4, (1.36) and (1.37)] and [2, p. 409] for a fractional ideal $\mathfrak{m}$ and a signature $\sigma\in\{\pm1\}^{r_1}$, and $\xi(y,w,bh;\bar{s}u+(u+m+i\lambda)/2, \bar{s}u+(u-m+i\lambda)/2,2(\bar{s}+1/2)+i\mu)$ is given in [4, (3.21)].

We note the formula (cf. [1, p. 334]),

$$(1.26)\quad \int_0^\infty y^lK_{s'}(y)K_{s''}(y)\,dy = 2^{l-2}\frac{\Gamma\left(\frac{l+s'+s''+1}{2}\right)\Gamma\left(\frac{l-s'+s''+1}{2}\right)\Gamma\left(\frac{l+s'-s''+1}{2}\right)\Gamma\left(\frac{l-s'-s''+1}{2}\right)}{\Gamma(l+1)}$$
$$(\Re(l+1)>|\Re(s')|+|\Re(s'')|)$$

By the same method as that of [4, p. 59], we see that the integral (1.24) is equal to

$$\overline{\gamma(\varphi)}2^{r_2}N(2t_\lambda\mathfrak{h}^{-1}\mathfrak{d}^{-1})\sum_{\alpha}N(\mathfrak{A}_\alpha)^{2s+2t}\sum_{\mathfrak{t},\mathfrak{y}}\mu(\mathfrak{t})\bar{\varphi}^*(\mathfrak{t})N(\mathfrak{t})^{-2s-1}N(\mathfrak{y})^{2s} \tag{1.27}$$

$$\times \sum_{a\in F^\times\backslash(\mathfrak{o}^\times)^2, a\gg 0}\ \sum_{bh=a} c(at_\lambda^{-1}\mathfrak{A}_\alpha^{-2}, \operatorname{sgn} a; \boldsymbol{g}^*)N(a)^{su-tu}N(b)^{-2su}$$

$$\times \varphi^*(ab^{-1}\mathfrak{h}\mathfrak{d}\mathfrak{y})(2\pi)^{r_2}2^{2tu_{r_2}-i\mu-2u_{r_2}}\frac{(4\pi)^{-u_{r_2}+(2s-2t)u_{r_2}}}{2^{2su_2-i\mu}}$$

$$\times \Gamma'(2u_{r_2}s+i\mu+u_{r_2})^{-1}\Gamma'((t-s)u_{r_2}+v+(1/2)u_{r_2})$$

$$\times \Gamma'((t-s)u_{r_2}-v+(1/2)u_{r_2})\Gamma'(2tu_{r_2}-i\mu+u_{r_2})^{-1}$$

$$\times \Gamma'((t+s)u_{r_2}+v+(1/2)u_{r_2}-i\mu)$$

$$\times \Gamma'((t+s)u_{r_2}-i\mu-v+(1/2)u_{r_2})M(s,t),$$

where

$$M(s,t)=\int_{y\gg 0}\exp(-2\pi y)\overline{\xi(y,1;\bar{s}u_{r_1}+(u_{r_1}+m+i\lambda)/2,\bar{s}u_{r_1}+(u_{r_1}-m+i\lambda)/2)}$$

$$\times\ y^{su_{r_1}+tu_{r_1}+m-i\lambda-u_{r_1}}\,dy\quad (\text{cf. [4, p. 59]}).$$

Here $\xi(y,1;\alpha,\beta)$ is the function in [7, p. 530]. Therefore we find that the equality (1.23) is equal to

$$(-1)^{\{m\}}2^d\operatorname{vol}(\Gamma[2\mathfrak{d}^{-1},2\mathfrak{r}\mathfrak{c}\mathfrak{d}]\backslash D)^{-1}h_F2^{r_2}\overline{r(\varphi)}[\mathfrak{o}_+^\times:(\mathfrak{o}^\times)^2] \tag{1.28}$$

$$\times N(\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c})N(2\mathfrak{o})^{-s-t}N(\mathfrak{d}\mathfrak{h})^{t-s}\sum_{\mathfrak{o}\supset\mathfrak{t}\supset 2\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}}\mu(\mathfrak{t})\bar{\varphi}^*(\mathfrak{t})N(\mathfrak{t})^{t-s-1}$$

$$\times\sum_{\mathfrak{m},\mathfrak{n}}c(\mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{m}\mathfrak{n},u;\boldsymbol{g}^*)N(\mathfrak{n})^{-s-t}\varphi^*(\mathfrak{m})N(\mathfrak{m})^{s-t}M(s,t)(2\pi)^{r_2}$$

$$\times 2^{2tu_{r_2}-i\mu-2u_{r_2}}\frac{(4\pi)^{-u_{r_2}+(2s-2t)u_{r_2}}}{2^{2su_{r_2}-i\mu}}\Gamma'(2su_{r_2}-i\mu+u_{r_2})^{-1}$$

$$\times \Gamma'((t-s)u_{r_2}+v+(1/2)u_{r_2})\Gamma'((t-s)u_{r_2}-v+(1/2)u_{r_2})$$

$$\times \Gamma'(2tu_{r_2}-i\mu+u_{r_2})^{-1}\Gamma'((t+s)u_{r_2}+v+(1/2)u_{r_2}-i\mu)$$

$$\times \Gamma'((t+s)u_{r_2}-v+(1/2)u_{r_2}-i\mu),$$

where $u=(1,\dots,1)$. Put $Y_{\mathfrak{t}}(s,t)=\sum_{\mathfrak{m},\mathfrak{n}\subset\mathfrak{o}}c(\mathfrak{t}^{-1}\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{m}\mathfrak{n},u;\boldsymbol{g}^*)N(\mathfrak{n})^{-s-t}\cdot\varphi^*(\mathfrak{m})N(\mathfrak{m})^{s-t}$.

We note that

$$\lim_{s\to+\infty} Y_{\mathfrak{t}}(s,s) = D(0,\boldsymbol{g}^*,\varphi,\mathfrak{t}^{-1}\mathfrak{h}\mathfrak{r}\mathfrak{c})$$

and

$$M(s,s) = i^{\{m\}}2^{-2r_1s-\{m\}+i\lambda}\Gamma'(m)(2\pi)^{r_1-\{m\}}(2\pi)^{-(1/2)u_{r_1}}$$
$$\times 2^{-(1/2)u_{r_1}+2su_{r_1}+\{m\}-i\lambda}\Gamma'(s+(m-i\lambda)/2)\Gamma'(s+(1+m-i\lambda)/2)^{-1}$$

(cf. [7, (4.18)]).

Therefore, by (1.12), (1.19) and (1.28), we have

$$\begin{aligned}(1.29)\quad & i^{-\{m\}}2^{1+r_1-r_2+\{m\}}(1/\sqrt{2\pi})^{r_2}\varphi_a(1/2)\tau_s^{-(m+(1/2)u_{r_1})}|\tau_c|^{-3}N(\mathfrak{q}\mathfrak{r})^{-1}\langle \boldsymbol{g},\boldsymbol{g}\rangle \\ & \times \overline{\mu_f(\tau,\mathfrak{q}^{-1})}\ \mathrm{vol}(\Gamma[2\mathfrak{d}^{-1},2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}]\backslash D)^{-1}\langle f,f\rangle^{-1}N(\mathfrak{q}\mathfrak{r}^2\mathfrak{c}) \\ & \times 2^{-r_1/2-\{m\}}2^{-(3/2)u_{r_2}}|\tau_c|\tau_s^{(1/2)u_{r_1}}\psi_a(\tau)\pi^{r_1/2}\pi^{r_2/2}2^{i\lambda}2^{2i\mu} \\ & \times \sum_{\mathfrak{m}}\mu_{f^*}(\tau,\mathfrak{q}^{-1}\mathfrak{b}\mathfrak{m})N(\mathfrak{m})^{-s} \\ & = (-1)^{\{m\}}2^d\ \mathrm{vol}(\Gamma[2\mathfrak{d}^{-1},2^{-1}\mathfrak{r}\mathfrak{c}\mathfrak{d}]\backslash D)^{-1}h_F2^{r_2}\overline{\gamma(\varphi)}[\mathfrak{o}_+^\times:(\mathfrak{o}^\times)^2] \\ & \times N(\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c})\sum_{\mathfrak{o}\supset\mathfrak{t}\supset 2\mathfrak{h}^{-1}\mathfrak{r}\mathfrak{c}}\mu(\mathfrak{t})\bar{\varphi}^*(\mathfrak{t})N(\mathfrak{t})^{-1}Y_{\mathfrak{t}}(s,s)(2\pi)^{r_2}2^{-2u_{r_2}}(4\pi)^{-u_{r_2}} \\ & \times \Gamma'(\nu+1/2)\Gamma'(-\nu+1/2)i^{\{m\}}2^{-\{m\}+i\lambda}\Gamma'(m) \\ & \times (2\pi)^{r_1-\{m\}}(2\pi)^{-(1/2)u_{r_1}}2^{-(1/2)u_{r_1}+\{m\}-i\lambda}.\end{aligned}$$

Letting s tend to $+\infty$ we deduce our Theorem 1.

§ 2. A Correct Proof of Theorem 0.1

We use the notation in [4] and [5]. The changes of [4] are as follows:

(1) [4, (2.15)] should read

$$e_c(-\xi u) = \prod_{i=1}^{r_2} e[-2\Re(\xi^{(r_1+i)}u_{r_1+i})],\quad e_c(\xi^2 z/2) = \prod_{i=1}^{r_2} e[\Re((\xi^{(r_1+i)})^2 z_{r_1+i})].$$

(2) [4, (2.24)] should read

This proposition implies that

$$\Im(y(\beta^{-1}(\mathfrak{z})))^{-n}\overline{\vartheta_{m-n}(\beta^{-1}(\mathfrak{z}), tu, l(\beta\gamma))}\,\overline{\varphi_{\mathfrak{rc}}(td(\beta\gamma)/2)}$$
$$\times e_s(\sqrt{-1}(\tilde{r}t)^2\Im(\beta\gamma(\beta^{-1}(\mathfrak{z})))^{-1}/4)\exp(-\pi(|t|v)^2 w(\beta\gamma(\beta^{-1}(\mathfrak{z})))^{-1})$$
$$\times \Im(\beta^{-1}(\mathfrak{z}))^{m+(1/2)u_{r_1}} t^n j(\beta\gamma, \beta^{-1}(\mathfrak{z}))^n w(\beta^{-1}(\mathfrak{z}))^2 h(\beta^{-1}(\mathfrak{z}))$$
$$= (y'/j(\gamma^{-1}\beta^{-1}, \mathfrak{z}')\overline{j(\gamma^{-1}\beta^{-1}, \mathfrak{z}')})^{-n} t^n \overline{\tilde{J}_{m-n}(\gamma^{-1}\beta^{-1}, \mathfrak{z}')}$$
$$\times \overline{\tilde{\vartheta}_{m-n}(\mathfrak{z}', tu)\varphi_{\mathfrak{rc}}(td(\beta\gamma)/2)}\, j(\gamma^{-1}\beta^{-1}, \mathfrak{z}')^{-n}$$
$$\times e_s(\sqrt{-1}(\tilde{r}t)^2\Im(\mathfrak{z}')^{-1}/4)\exp(-\pi(|t|v)^2 w(\mathfrak{z}')^{-1}) h(\gamma^{-1}\beta^{-1}(\mathfrak{z}'))$$
$$\times \Im(\gamma^{-1}\beta^{-1}(\mathfrak{z}'))^{m+(1/2)u_{r_1}} w(\gamma^{-1}\beta^{-1}(\mathfrak{z}'))^2,$$

(3) The line 11 in [4, p. 44]:

$$\varphi_{\mathfrak{rc}}(td(\beta\gamma)/2)\varphi_{\mathfrak{rc}}(a_{\gamma^{-1}}) = \overline{\varphi_{\mathfrak{rc}}(td_\beta/2)}.$$

should read

$$\overline{\varphi_{\mathfrak{rc}}(td(\beta\gamma)/2)}\varphi_{\mathfrak{rc}}(a_{\gamma^{-1}}) = \overline{\varphi_{\mathfrak{rc}}(td_\beta/2)}.$$

(4) [4, (2.25)] should read

$$(y')^{-n}\overline{\varphi_{\mathfrak{rc}}(td_\beta/2)\tilde{\vartheta}_{m-n}(\mathfrak{z}', tu)} J_m(\beta, \beta^{-1}(\mathfrak{z}')) t^n h(\beta^{-1}(\mathfrak{z}'))$$
$$\times e_s(\sqrt{-1}(\tilde{r}t)^2\Im(\mathfrak{z}')^{-1}/4)(w')^2\exp(-\pi(|t|v)^2 w(\mathfrak{z}')^{-1})\Im(\mathfrak{z}')^{m+(1/2)u_{r_1}}.$$

(5) The element l in [4, (2.33)] runs over $t_\lambda \mathfrak{r}^{-1}\mathfrak{m}$ under the condition that $(l\mathfrak{m}^{-1}\mathfrak{r}/t_\lambda, \mathfrak{rc}) = 1$.

We sketch a correct proof of Theorem 0.1. Let f be an element of $\mathscr{S}_{m+(1/2)u_{r_1},\omega}(\mathfrak{b}, \mathfrak{b}'; \psi)$. Since f is holomorphic with respect to $z_1, \ldots, z_{r_1}$, the function $g_{\tau,\lambda}(\mathfrak{w})$ in [4, (2.11)] is holomorphic with respect to $z_1', \ldots, z_{r_1}'$, where $\mathfrak{w} = (z_1', \ldots, z_{r_1}', \mathfrak{z}_{r_1+1}', \ldots, \mathfrak{z}_{r_1+r_2}')$ (cf. [2, p. 406], [4, (2.14)] and [5, (2)]). To determine the Fourier coefficients of $g_{\tau,\lambda}(\mathfrak{w})$, it is sufficient to calculate $g_{\tau,\lambda}(\mathfrak{w})$ for $z_1' = iy_1', \ldots, z_{r_1}' = iy_{r_1}'$ ($y_1' > 0, \ldots, y_{r_1}' > 0$). We put $h_1 = 0, \ldots, h_{r_1} = 0$ in [4, (2.15) and (2.16)]. By [6, pp. 772–777], [6, pp. 783–785], [8, pp. 1015–1024], [8, Theorem 1.2] and [8, Proposition 1.3], we can prove the proposition 2.3 in [4] in the case of $(h_1, \ldots, h_{r_1}) = (0, \ldots, 0)$. We note [5, (6), (7), (8) and (9)]. By the same method as that of [4], we deduce

$$\Psi_{\tau,\lambda}(f)(\mathfrak{w}) = N(t_\lambda/\mathfrak{r}) \sum_{\mathfrak{m}} \sum_{l \in t_\lambda \mathfrak{r}^{-1}\mathfrak{m}} N(\mathfrak{m}) l^{m-1} |l|^{-1} \varphi_a(l) \varphi^*(l\mathfrak{r}/t_\lambda \mathfrak{m}) \mu_f(\tau, (\mathfrak{r}\mathfrak{q})^{-1}\mathfrak{m})$$

$$\times e_c(lu) \prod_{i=1}^{r_1} c(\operatorname{sgn}(l^{(i)})) \exp(-2\pi l \Im(z)) v K_{2\nu}(4\pi |l| v),$$

for $\mathfrak{w} = (iy_1', \dots, iy_{r_1}', \mathfrak{z}_{r_1+1}', \dots, \mathfrak{z}_{r_1+r_2}')$, where $\mathfrak{m}$ runs over all integral ideals, l runs over $t_\lambda \mathfrak{r}^{-1}\mathfrak{m}$ under the condition $(l\mathfrak{m}^{-1}\mathfrak{r}/t_\lambda, \mathfrak{r}\mathfrak{c}) = 1$, $\mathfrak{z}_{r_1+i}' = u_{r_1+i}' + jv_{r_1+i}'$, $z = (iy_1', \dots, iy_{r_1}')$ $u = (u_{r_1+1}', \dots, u_{r_1+r_2}')$, $v = (v_{r_1+1}', \dots, v_{r_1+r_2}')$, $l^{m-1} = \prod_{i=1}^{r_1} (l^{(i)})^{m_i - 1}$ and $|l| = \prod_{i=1}^{r_2} |l^{(r_1+i)}|$. Therefore we deduce Theorem 0.1.

References

[1] A. Erdélyi, et al., Tables of Integral Transforms, Vol. 1, McGraw-Hill, New York, 1954.

[2] K. Khuri-Makdisi, On the Fourier coefficients of nonholomorphic Hilbert modular forms of half-integral weight, Duke Math. J. **84** (1996), 399–452.

[3] H. Kojima, On the Fourier coefficients of Maass wave forms of half integral weight over an imaginary quadratic field, J. Reine Angew. Math. **526** (2000), 155–179.

[4] H. Kojima, On the Fourier coefficients of Hilbert-Maass wave forms of half integral weight over arbitrary algebraic number fields, J. of Number Theory **107** (2004), 25–62.

[5] H. Kojima, Corrigendum to "On the Fourier coefficients of Hilbert-Maass wave forms of half integral weight over arbitrary algebraic number fields" [J. Number Theory 107 (1) (2004), 25–62], J. of Number Theory **130** (2010), 1252–1253.

[6] G. Shimura, On Hilbert modular forms of half-integral weight, Duke Math. J. **55** (1987), 765–838.

[7] G. Shimura, On the Fourier coefficients of Hilbert modular forms of half-integral weight, Duke Math. J. **71** (1993), 501–557.

[8] G. Shimura, On the transformation formulas of theta series, Amer. J. Math. **115** (1993), 1011–1052.

[9] J.-L. Waldspurger, Sur les coefficients de Fourier des formes modulaires de poids demi-entier, J. Math. Pures Appl. **60** (1981), 375–484.

Department of Mathematics
Graduate school of Science and Engineering
Saitama University, Saitama, 338-8570
Japan
E-mail: hkojima@rimath.saitama-u.ac.jp