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ON A CLASSIFICATION OF 3-SIMPLE PREHOMOGENEOUS VECTOR SPACES WITH TWO IRREDUCIBLE COMPONENTS

By

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Abstract. In this paper, we give some results about a classification of reductive prehomogeneous vector spaces with two irreducible components. In particular, we give the complete classification of 3-simple prehomogeneous vector spaces with two irreducible components. We consider everything over the complex number field **C**.

Introduction

Let G be a linear algebraic group and ρ its rational representation on a finite dimensional vector space V, all defined over the complex number field $\mathbf{C}$. If there exists a Zariski-dense G-orbit, we call the triplet (G, ρ, V) a prehomogeneous vector space (abbrev. PV). For basic properties of PVs, see [K1].

For a classification of PVs, we may assume that G is connected since (G, ρ, V) is a PV if and only if $(G^\circ, \rho|_{G^\circ}, V)$ is a PV, where G° is the connected component of G. Since non-reductive linear algebraic groups are not classified, it is reasonable to assume that G is reductive.

For any PV $(G, \rho, V_1 \oplus \cdots \oplus V_l)$, the triplet $(GL(1)^l \times G, \tilde{\rho}, V_1 \oplus \cdots \oplus V_l)$ is also a PV, where $\tilde{\rho}$ is the composition of ρ and the independent scalar multiplications $GL(1)^l$. Hence the first problem is to classify PVs with full scalar multiplications. Let G_{ss} be a connected semisimple linear algebraic group and let ρ_i $(i = 1, \ldots, l)$ be some irreducible rational representation of G_{ss} on a finite dimensional vector space V_i. Let ρ be the representation of $G = GL(1)^l \times G_{ss}$ on $V = V_1 \oplus \cdots \oplus V_l$ which is the composition of $\rho_1 \oplus \cdots \oplus \rho_l$ and the independent scalar multiplications $GL(1)^l$ on each irreducible component V_i $(i = 1, \ldots, l)$. In

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this paper, we call such a triplet (G, ρ, V) a reductive PV if the triplet (G, ρ, V) is a PV. If the triplet (G, ρ, V) is a reductive PV, then each irreducible component $(GL(1) \times G_{ss}, \Lambda_1 \otimes \rho_i, V_i)$ $(i = 1, \ldots, l)$ is also a reductive PV. However the converse does not hold. Since G_{ss} is semisimple, we may assume that G_{ss} is a product of simple linear algebraic groups (See Definition 2.1). When G_{ss} is a product of k simple linear algebraic groups, such a reductive PV is called a k-simple PV. We call a reductive PV with $l = 1$ an irreducible PV.

The classification problem of all reductive PVs is still very difficult and only the cases for $l = 1$ or $k = 1, 2$ are completely solved (See [SK], [K2], [KKIY] and [KKTI]). The cases for $l = 2$ or $k = 3$ are solved only for the case of nontrivial PVs (See [Ka1], [KUY] and Definition 2.2). A classification for $l \geq 3$ or $k \geq 4$ is not known at all. In general, a classification related with trivial PVs is very difficult so far (See Definition 2.2).

In this paper, we give some results for the case $l = 2$ related with trivial PVs and as their application, we give the complete classification of the case $l = 2$ and $k = 3$ including the case of trivial PVs.

Since any reductive PV is castling equivalent to a reduced reductive PV, the classification of all reduced reductive PVs implies that of all reductive PVs (See Definitions 2.7 and 2.8). Hence it is not necessary to consider necessary and sufficient conditions for the prehomogeneity of non-reduced triplets (See Theorems 3.1 and 6.2). Any 3-simple PV with two irreducible components is castling equivalent to either a simple PV with two irreducible components or a 2-simple PV with two irreducible components or a reduced 3-simple PV with two irreducible components. Since all simple PVs and 2-simple PVs are completely classified (See [K2], [KKIY] and [KKTI]), we can complete the classification of 3-simple PVs with two irreducible components by giving the complete list of reduced 3-simple PVs with two irreducible components.

In 1988, Shin-ichi Kasai classified all reductive PVs with two irreducible components, at least one of which is not castling equivalent to a non-regular trivial PV (See [Ka1]). He also showed that the case for $l = 2$ will be completely solved if we can classify all reductive PVs of the following form $T(H, \rho, \sigma, n, l, j)^{(*)}$ (See [Ka1, the triplet (9) in Theorem 2.19 and §3] and Definition 2.1). Especially, if we can classify all 3-simple PVs of the form $T(H, \rho, \sigma, n, l, j)^{(*)}$ (see definition below), then we can classify all 3-simple PVs with two irreducible components. However, in [Ka1], there is no result about a classification of all reductive PVs of the form $T(H, \rho, \sigma, n, l, j)^{(*)}$. Furthermore, 3-simple PVs of the form $T(H, \rho, \sigma, n, l, j)^{(*)}$ are not classified in [KUY]. Now we give the definition of $T(H, \rho, \sigma, n, l, j)^{(*)}$.

Let H be a connected semisimple linear algebraic group, ρ an m-dimensional irreducible rational representation of H and σ an r-dimensional irreducible rational representation of H with $m \geq 2$ and $|\ker \rho \cap \ker \sigma| < \infty$, where $|\ker \rho \cap \ker \sigma|$ denotes the cardinality of $\ker \rho \cap \ker \sigma$. For each m, we define a sequence $\{c_i\}_{i \geq -1}$ by $c_{-1} = -1$, $c_0 = 0$ and $c_i = mc_{i-1} - c_{i-2}$ $(i \geq 1)$. Let $n \geq 1$, $l \geq 1$ and $j \geq 0$ be integers with $n > ml$ and $c_{j+1}n - c_j l > r$. We define a triplet $T(H, \rho, \sigma, n, l, j)$ (resp. $T(H, \rho, \sigma, n, l, j)^*$) as $(H \times GL(c_j n - c_{j-1} l) \times GL(c_{j+1} n - c_j l), \rho \otimes \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 \otimes \Lambda_1, V(m) \otimes V(c_j n - c_{j-1} l) \otimes V(c_{j+1} n - c_j l) + V(r) \otimes V(c_{j+1} n - c_j l))$ (resp. $(H \times GL(c_j n - c_{j-1} l) \times GL(c_{j+1} n - c_j l), \rho \otimes \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 \otimes \Lambda_1^*, V(m) \otimes V(c_j n - c_{j-1} l) \otimes V(c_{j+1} n - c_j l) + V(r) \otimes V(c_{j+1} n - c_j l)^*))$. Here $T(H, \rho, \sigma, n, l, j)^{(*)}$ stands for $T(H, \rho, \sigma, n, l, j)$ or $T(H, \rho, \sigma, n, l, j)^*$.

Since H is semisimple and ρ and σ are irreducible rational representations of H, we may assume that the triplet $T(H, \rho, \sigma, n, l, j)^{(*)}$ is equal to $(H_1 \times GL(c_j n - c_{j-1} l) \times GL(c_{j+1} n - c_j l) \times H_2 \times H_3, \rho_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \rho_2 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1^{(*)} \otimes \sigma_2 \otimes \sigma_3, V(m_1) \otimes V(c_j n - c_{j-1} l) \otimes V(c_{j+1} n - c_j l) \otimes V(m_2) + V(c_{j+1} n - c_j l)^{(*)} \otimes V(r_2) \otimes V(r_3))$, where H_i $(i = 1, 2, 3)$ is a connected semisimple linear algebraic group with $H = H_1 \times H_2 \times H_3$ and ρ_a $(a = 1, 2)$ (resp. σ_b $(b = 2, 3)$) is an m_a-dimensional (resp. an r_b-dimensional) irreducible rational representation of H_a with $\rho = \rho_1 \otimes \rho_2 \otimes 1$ (resp. H_b with $\sigma = 1 \otimes \sigma_2 \otimes \sigma_3$).

If the triplet (G, ρ, V) with $l \geq 2$ is a reductive PV, then each $(GL(1)^2 \times G_{ss}, \Lambda_1 \otimes 1 \otimes \rho_i + 1 \otimes \Lambda_1 \otimes \rho_j, V_i \oplus V_j)$ $(1 \leq i \neq j \leq l)$ is also a reductive PV with two irreducible components. Therefore, by the result of [Ka1], the complete classification of all reductive PVs when each irreducible component is castling equivalent to a non-trivial reduced irreducible PV (resp. when each irreducible component is a regular PV) is given in [Ka2] (resp. [Ka3]). Thus a classification of all reductive PVs with two irreducible components is important to that of all reductive PVs.

In this paper, we give some results about a classification of all reductive PVs of the form $T(H, \rho, \sigma, n, l, j)^{(*)}$. Especially, we give the complete classification of all reductive PVs of the form $T(H, \rho, \sigma, n, l, 0)^{(*)}$ and the complete classification of all 3-simple PVs with two irreducible components.

This paper consists of seven sections.

In Section 1, we give some correction to [Ka1].

In Section 2, we give some preliminaries for later use.

In Section 3, we classify all reductive PVs of the form $T(H, \rho, \sigma, n, l, 0)^{(*)}$.

In Section 4, we calculate generic isotropy subgroups.

In Section 5, we give some results about a classification of all reductive PVs of the form $T(H, \rho, \sigma, n, l, j)^{(*)}$ with $j \geq 1$ using results of Section 4.

In Section 6, we classify all 3-simple *PVs* of the form $T(H, \rho, \sigma, n, l, j)^{(*)}$ with $j \geq 1$ using results of Section 5.

In Section 7, we give the complete list of indecomposable (See Definition 7.1) reduced 3-simple *PVs* with two irreducible components which are neither trivial *PVs* nor *PVs* of trivial type (See Definition 7.2).

In general, we denote by ρ^* the contragredient representation of a rational representation ρ. We denote by $V(n)$ an n-dimensional vector space in general. If $V(n)$ and $V(n)^*$ appear at the same time, $V(n)^*$ denotes the dual space of $V(n)$. We use $+$ instead of $\oplus$ if $\otimes$ and $\oplus$ appear at the same time. For $x_1, \dots, x_l \in M(n,m)$, we define $\langle x_1, \dots, x_l \rangle$ (resp. $\langle x_1, \dots, x_l \rangle^\perp$) as $\{\lambda_1 x_1 + \cdots + \lambda_l x_l \mid \lambda_1, \dots, \lambda_l \in \mathbf{C}\}$ (resp. $\{y \in M(n,m) \mid \mathrm{Tr}^t\, x_i y = 0 \text{ for } 1 \leq i \leq l\}$). For a rational representation ρ, $\rho^{(*)}$ stands for ρ or ρ^*. For positive integers n and m, we denote by $0_{(n,m)}$ the n by m zero matrix. For $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq n} \in GL(n)$ and $B \in GL(m)$, we define $A \otimes B$ as $(a_{ij}B)_{1 \leq i \leq n, 1 \leq j \leq n} \in GL(nm)$. For $F_i \in M(m_i, l)$ $(i = 1, 2, \dots, n)$, we define $(F_1, F_2, \dots, F_n)'$ as $\begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix} \in M(m_1 + m_2 + \cdots + m_n, l)$.

1. Some Correction to a Paper by S. Kasai

In this section, we shall give some correction to [Ka1].

CORRECTION 1.1. *Let G be a reductive linear algebraic group and ρ_i $(i = 1, 2)$ a rational representation of G on a finite dimensional vector space V_i. Assume that a triplet (G, ρ_1, V_1) is a regular PV. Then the triplet $P_1 := (G, \rho_1 \oplus \rho_2, V_1 \oplus V_2)$ is a PV if and only if $P_2 := (G, \rho_1 \oplus \rho_2^*, V_1 \oplus V_2^*)$ is a PV. However, in a classification of PVs, we consider that these two PVs P_1 and P_2 are different in general. Hence Theorem* 3.25 *in* [Ka1] *should be corrected as follows: A triplet* (9) *with $j = 0$ and $N = 0$ (or $m'm'' = mL + N$), namely, $T := (GL(1)^2 \times H \times (SL(m_1 m_2) \times H_2) \times H', \rho \otimes (\Lambda_1^{(*)} \otimes \rho_2) \otimes 1 + 1 \otimes (\Lambda_1 \otimes \tau'') \otimes \tau', V(m_1) \otimes V(m_1 m_2)^{(*)} \otimes V(m_2) + V(m_1 m_2) \otimes V(m'') \otimes V(m'))$ with $m_1 m_2 \geq m'm''$ and $m_1 m_2 \neq m'$, is a PV if and only if it satisfies one of the following conditions*:

(1) *$H_2 = \{1\}$ and $(GL(1) \times H \times H', \Lambda_1 \otimes \rho \otimes \tau', V(m_1) \otimes V(m'))$ with $m_1 > m'$ is an irreducible PV. In this case, T is a PV of the form* (3.52) *in* [Ka1].

(2) *$H_2 \neq \{1\}$ and $(GL(1) \times H \times H_2 \times H', \Lambda_1 \otimes \rho \otimes (\rho_2^{(*)*} \otimes \tau'') \otimes \tau', V(m_1) \otimes (V(m_2)^{(*)*} \otimes V(m'')) \otimes V(m'))$ with $m_1 m_2 \geq m'm''$ is castling equivalent*

to a regular trivial PV of the form $(H_2 \times G \times GL(N), (\rho_2^{(*)*} \otimes \tau'') \otimes \sigma \otimes \Lambda_1, (V(m_2)^{(*)*} \otimes V(m'')) \otimes V(k) \otimes V(N))$ *with* $N = m_2 m'' k$. *In this case,* T *is a regular PV with* 2 *basic relative invariants.*

(3) $H_2 \neq \{1\}$ *and* $(GL(1) \times H \times H_2 \times H', \Lambda_1 \otimes \rho \otimes (\rho_2^{(*)*} \otimes \tau'') \otimes \tau', V(m_1) \otimes (V(m_2)^{(*)*} \otimes V(m'')) \otimes V(m'))$ *with* $m_1 m_2 \geq m' m''$ *is castling equivalent to a non-regular trivial PV of the form* $(H_2 \times G \times GL(N), (\rho_2^{(*)*} \otimes \tau'') \otimes \sigma \otimes \Lambda_1, (V(m_2)^{(*)*} \otimes V(m'')) \otimes V(k) \otimes V(N))$ *with* $N > m_2 m'' k$. *In this case,* T *is a non-regular PV with* 1 *basic relative invariant.*

Proof. Note that T is isomorphic to $T' := (H \times (GL(m_1 m_2) \times H_2) \times H' \times GL(1), \rho \otimes (\Lambda_1^{(*)} \otimes \rho_2) \otimes 1 \otimes 1 + 1 \otimes (\Lambda_1 \otimes \tau'') \otimes \tau' \otimes \Lambda_1, V(m_1) \otimes V(m_1 m_2)^{(*)} \otimes V(m_2) + V(m_1 m_2) \otimes V(m'') \otimes V(m'))$ (See Definition 2.1). Since a generic isotropy subgroup of $(H \times H_2 \times GL(m_1 m_2), \rho \otimes \rho_2 \otimes \Lambda_1^{(*)}, V(m_1) \otimes V(m_2) \otimes V(m_1 m_2)^{(*)})$ is $\{(h, h_2, (\rho \otimes \rho_2)^{(*)*}(h, h_2)) \mid (h, h_2) \in H \times H_2\}$, by Proposition 2.4, we see that T is a PV if and only if $T'' := (GL(1) \times H \times H_2 \times H', \Lambda_1 \otimes \rho^{(*)*} \otimes (\rho_2^{(*)*} \otimes \tau'') \otimes \tau', V(m_1)^{(*)*} \otimes (V(m_2)^{(*)*} \otimes V(m'')) \otimes V(m'))$ is a PV. Since H is semisimple, T'' is isomorphic to $(GL(1) \times H \times H_2 \times H', \Lambda_1 \otimes \rho \otimes (\rho_2^{(*)*} \otimes \tau'') \otimes \tau', V(m_1) \otimes (V(m_2)^{(*)*} \otimes V(m'')) \otimes V(m'))$. By Propositions 3.2 and 3.3, we obtain our assertion. □

Correction 1.2. *Theorem* 3.22 *in* [Ka1] *should be corrected as follows*: *A triplet* (9) *with* $N = 0$, $m = m_1 = 2$ *and* $m_2 = 1$, *namely*, $(GL(1)^2 \times SL(2) \times SL((j+1)L) \times SL((j+2)L) \times H', \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1^{(*)} \otimes \tau')$ $((j+2)L > \deg \tau')$ *is a PV if and only if* $(SL(2) \times GL(L) \times H', (j+1)\Lambda_1 \otimes \Lambda_1 \otimes \tau')$ *with* $(j+2)L > \deg \tau'$, *is an irreducible PV. Especially,* $(SL(2) \times GL(L) \times H', (j+1)\Lambda_1 \otimes \Lambda_1 \otimes \tau')$ *with* $(j+2)L > \deg \tau'$ *and* $j \geq 1$ *is an irreducible PV if and only if it is castling equivalent to one of* $(GL(2), 2\Lambda_1)$, $(GL(2), 3\Lambda_1)$, $(Sp(t) \times GL(2), \Lambda_1 \otimes 2\Lambda_1)$ $(t \geq 2)$ *and a trivial PV.*

Correction 1.3. *In* §4 *Table in* [Ka1], *the following PVs were missed.*

(1) $T_1 := (GL(1)^2 \times H \times (SL(m_1 m_2) \times H_2) \times H', \rho \otimes (\Lambda_1^{(*)} \otimes \rho_2) \otimes 1 + 1 \otimes (\Lambda_1 \otimes \tau'') \otimes \tau', V(m_1) \otimes V(m_1 m_2)^{(*)} \otimes V(m_2) + V(m_1 m_2) \otimes V(m'') \otimes V(m'))$ *with* $m_1 m_2 \geq m' m''$ *and* $m_1 m_2 \neq m'$ *where* $H_2 \neq \{1\}$ *and* $(GL(1) \times H \times H_2 \times H', \Lambda_1 \otimes \rho \otimes (\rho_2^{(*)*} \otimes \tau'') \otimes \tau', V(m_1) \otimes (V(m_2)^{(*)*} \otimes V(m'')) \otimes V(m'))$ *is castling equivalent to a regular trivial PV of the form* $(H_2 \times G \times GL(N), (\rho_2^{(*)*} \otimes \tau'') \otimes \sigma \otimes \Lambda_1, (V(m_2)^{(*)*} \otimes V(m'')) \otimes V(k) \otimes V(N))$ *with* $N = m_2 m'' k$. T_1 *is a regular PV with* 2 *basic relative invariants.*

(2) $T_2 := (GL(1)^2 \times G \times SL(2) \times G', \sigma \otimes \Lambda_1 \otimes 1 + 1 \otimes \tau \otimes \rho)$ *where* (G, σ) *is one of* $(Sp(n), \Lambda_1)$ $(n \geq 2)$ *and* (Spin(10), *a half-spin rep.*); *and*

$(G' \times GL(2), \rho \otimes \tau)$ is an arbitrary non-regular irreducible PV. T_2 is a non-regular PV and the number of basic relative invariants of it is 2 (*resp.* 1) *if* $(G' \times GL(2), \rho \otimes \tau)$ *is castling equivalent to* $(Sp(m) \times GL(2), \Lambda_1 \otimes 2\Lambda_1)$ $(m \geq 2)$ (*resp. if otherwise*).

(3) $T_3 := (GL(1)^2 \times H \times (SL(m_1m_2) \times H_2) \times H', \rho \otimes (\Lambda_1^{(*)} \otimes \rho_2) \otimes 1 + 1 \otimes (\Lambda_1 \otimes \tau'') \otimes \tau', V(m_1) \otimes V(m_1m_2)^{(*)} \otimes V(m_2) + V(m_1m_2) \otimes V(m'') \otimes V(m'))$ *with* $m_1m_2 \geq m'm''$ *and* $m_1m_2 \neq m'$ *where* $H_2 \neq \{1\}$ *and* $(GL(1) \times H \times H_2 \times H', \Lambda_1 \otimes \rho \otimes (\rho_2^{(*)*} \otimes \tau'') \otimes \tau', V(m_1) \otimes (V(m_2)^{(*)*} \otimes V(m'')) \otimes V(m'))$ *is castling equivalent to a non-regular trivial PV of the form* $(H_2 \times G \times GL(N), (\rho_2^{(*)*} \otimes \tau'') \otimes \sigma \otimes \Lambda_1, (V(m_2)^{(*)*} \otimes V(m'')) \otimes V(k) \otimes V(N))$ *with* $N > m_2m''k$. T_3 *is a non-regular PV with* 1 *basic relative invariant.*

(4) $T_4 := (GL(1)^2 \times SL(2) \times SL(2L) \times SL(3L) \times G', \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1^{(*)} \otimes \tau')$ $(3L > \deg \tau')$ *where* $(SL(2) \times GL(L) \times G', 2\Lambda_1 \otimes \Lambda_1 \otimes \tau')$ *is castling equivalent to* $(Sp(n) \times GL(2), \Lambda_1 \otimes 2\Lambda_1)$ $(n \geq 2)$. T_4 *is a non-regular PV with* 2 *basic relative invariants.*

For (1), (3) *and* (4) (*resp.* (2)), *see Correction* 1.1 *and Correction* 1.2 (*resp. Theorem* 3.9 *in* [Ka1]).

CORRECTION 1.4. II)-(17) *in* §4 *Table of* [Ka1] *should be corrected as follows*: $(GL(1)^2 \times Sp(n) \times SL(m) \times SL(2), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^{(*)} \otimes 2\Lambda_1)$ $(2n > m \geq 3)$, $N = 2$ *for* $m = even$, $\Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes 2\Lambda_1$ *with* $m = odd$, $\Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes 2\Lambda_1$ *with* $m = 3$; $N = 1$ *for* $\Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes 2\Lambda_1$ *with* $m = odd$ *and* $m \geq 5$.

2. Preliminaries

DEFINITION 2.1. *Two triplets* (G_i, ρ_i, V_i) $(i = 1, 2)$ *are said to be isomorphic if there exist an isomorphism* $\sigma : \rho_1(G_1) \to \rho_2(G_2)$ *of groups and an isomorphism* $\tau : V_1 \to V_2$ *of vector spaces such that* $\tau(\rho_1(g_1)(x_1)) = \sigma(\rho_1(g_1))(\tau(x_1))$ *for* $g_1 \in G_1$, $x_1 \in V_1$. *In this case, we write* $(G_1, \rho_1, V_1) \cong (G_2, \rho_2, V_2)$ *and we identify* (G_1, ρ_1, V_1) *with* (G_2, ρ_2, V_2). *Note that for any connected linear algebraic group* G *and a surjective homomorphism* $\sigma : \tilde{G} \to G$, *we have* $(G, \rho, V) \cong (\tilde{G}, \rho \circ \sigma, V)$. *Hence we may assume that* G *is simply connected if necessary.*

DEFINITION 2.2. *Let* G *be a linear algebraic group and let* ρ *be a rational representation of* G *on a finite dimensional vector space* V. *For a positive integer* n *satisfying* $n \geq \dim V$, *the triplet* $(G \times GL(n), \rho \otimes \Lambda_1, V \otimes V(n))$ *is always* *a PV. We call such a triplet a trivial PV. It is regular if and only if* $n = \dim V$.

Lemma 2.3. *Assume that an algebraic group G acts on both of two irreducible algebraic varieties W and W′. Let* $\varphi : W \to W'$ *be a morphism satisfying*

(I) $\varphi(gw) = g\varphi(w)$ $(g \in G, w \in W)$,

(II) $\overline{\varphi(W)} = W'$.

Then the following assertions (1) *and* (2) *are equivalent*:

(1) $W = \overline{G \cdot w}$ *for some* $w \in W$; *that is, W is G-prehomogeneous.*

(2) (a) $W' = \overline{G \cdot w'}$ *for some* $w' \in W'$.

(b) *For the above point* $w' \in W'$ *in* (a), *there exists a point* $w \in \varphi^{-1}(w')$ *such that* $\varphi^{-1}(w') = \overline{G_{w'} \cdot w}$, *where* $G_{w'} = \{g \in G \mid gw' = w'\}$ *is the isotropy subgroup of G at* w'.

Note that a generic isotropy subgroup of (1) *is isomorphic to that of* (2)(b) *since* $(G_{w'})_w = G_w$.

Proof. For the proof, see Proposition 7.6 in [K1]. □

Proposition 2.4. *The following assertions are equivalent.*

(1) $(G, \rho_1 \oplus \rho_2, V_1 \oplus V_2)$ *is a PV.*

(2) (G, ρ_1, V_1) *is a PV and* $(H, \rho_2|_H, V_2)$ *is also a PV, where H denotes the generic isotropy subgroup of* (G, ρ_1, V_1).

Proof. By Lemma 2.3, we obtain our assertion. □

Theorem 2.5. *Let G be a linear algebraic group and* ρ *an m-dimensional rational representation of G on a finite dimensional vector space V. For a positive integer n satisfying* $n < m = \dim V$, *the following assertions are equivalent.*

(1) $(G \times GL(n), \rho \otimes \Lambda_1, V \otimes V(n))$ *is a PV.*

(2) $(G \times GL(m-n), \rho^* \otimes \Lambda_1, V^* \otimes V(m-n))$ *is a PV.*

Furthermore, the generic isotropy subgroups of (1) *and* (2) *are isomorphic. We say that two triplets* (1) *and* (2) *are castling transforms of each other. In this paper, we call the triplet* (1) (*resp.* (2)) *the castling transform at* $GL(m-n)$ (*resp.* $GL(n)$) *of the triplet* (2) (*resp.* (1)).

Proof. For the proof, see Theorem 7.3 in [K1]. □

Theorem 2.6. *Let G be a linear algebraic group and let* ρ (*resp.* σ) *be an m-dimensional* (*resp. an r-dimensional*) *rational representation of G on a finite* *dimensional vector space* V (*resp.* W). *For a positive integer* n *satisfying* $n < m = \dim V$, *the following assertions are equivalent.*

(1) $(G \times GL(n), \rho \otimes \Lambda_1 + \sigma \otimes 1, V \otimes V(n) + W)$ *is a* PV.

(2) $(G \times GL(m-n), \rho^* \otimes \Lambda_1 + \sigma \otimes 1, V^* \otimes V(m-n) + W)$ *is a* PV.

Furthermore, the generic isotropy subgroups of (1) *and* (2) *are isomorphic. We say that two triplets* (1) *and* (2) *are castling transforms of each other. In this paper, we call the triplet* (1) (*resp.* (2)) *the castling transform at* $GL(m-n)$ (*resp.* $GL(n)$) *of the triplet* (2) (*resp.* (1)).

PROOF. By Proposition 2.4 and Theorem 2.5, we obtain our assertion. □

DEFINITION 2.7. *Two triplets* (G, ρ, V) *and* (G', ρ', V') *are said to be castling equivalent if one is obtained from the other by a finite number of successive castling transformations.*

DEFINITION 2.8. *A triplet* (G, ρ, V) *is said to be reduced if* $\dim V' \geq \dim V$ *holds for any castling transform* (G', ρ', V') *of* (G, ρ, V).

DEFINITION 2.9. *Let* G *be a connected semisimple linear algebraic group and* $\rho : G \to GL(V)$ *an irreducible rational representation. If a triplet* $(G \times GL(1), \rho \otimes \Lambda_1, V \otimes V(1))$ *is non-reduced, then there exists a unique castling transformation which makes the dimension of the space smaller. If we use only such castling transformations, the number of castling transformations to reach a reduced triplet is uniquely determined. We call this number the reducing number of* $(G \times GL(1), \rho \otimes \Lambda_1, V \otimes V(1))$ (*cf. p.* 799 *in* [KTK]). *If a triplet* $(G \times GL(1), \rho \otimes \Lambda_1, V \otimes V(1))$ *is reduced, then its reducing number is* 0.

REMARK 2.10. The two triplets $(H \times GL(c_j n - c_{j-1} l) \times GL(c_{j+1} n - c_j l), \rho \otimes \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(c_j n - c_{j-1} l) \otimes V(c_{j+1} n - c_j l))$ and $(H \times GL(c_j n - c_{j-1} l) \times GL(c_{j+1} n - c_j l), \sigma \otimes 1 \otimes \Lambda_1^{(*)}, V(r) \otimes V(c_{j+1} n - c_j l)^{(*)})$ are irreducible components of the triplet $T(H, \rho, \sigma, n, l, j)^{(*)}$. The triplet $(H \times GL(c_j n - c_{j-1} l) \times GL(c_{j+1} n - c_j l), \rho \otimes \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(c_j n - c_{j-1} l) \otimes V(c_{j+1} n - c_j l))$ is obtained from the non-regular trivial PV $(H \times GL(l) \times GL(n), \rho \otimes \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(l) \otimes V(n))$ $(m \geq 2,\ n > ml)$ by applying a castling transformation j times. The triplet $(H \times GL(c_j n - c_{j-1} l) \times GL(c_{j+1} n - c_j l), \sigma \otimes 1 \otimes \Lambda_1^{(*)}, V(r) \otimes V(c_{j+1} n - c_j l)^{(*)})$ $(c_{j+1} n - c_j l > r)$ is a non-regular trivial PV.

Theorem 2.11. *Let G be a connected linear algebraic group and let ρ_i $(i = 1, 2)$ be an m_i-dimensional rational representation of G on a finite dimensional vector space V_i. For a positive integer n satisfying $n \geq \max\{m_1, m_2\}$, the following assertions are equivalent.*

(1) $(G \times GL(n), \rho_1 \otimes \Lambda_1 + \rho_2 \otimes \Lambda_1^*, V_1 \otimes V(n) + V_2 \otimes V(n)^*)$ *is a PV.*

(2) $(G, \rho_1 \otimes \rho_2, V_1 \otimes V_2)$ *is a PV.*

Furthermore, if $n > \max\{m_1, m_2\}$, then a PV (1) is regular if and only if $m_1 = m_2$, and the number of basic relative invariants of a PV (1) is equal to that of a PV (2).

Proof. For the proof, see [K1, Theorem 7.8] and [KKTI, Propositions 1.18, 1.20 and 1.22]. □

Lemma 2.12. *Let K be a connected semisimple linear algebraic group and $\tau : K \to GL(W)$ an irreducible rational representation with* $\dim W \geq 3$. *Assume that $(K \times GL(1), \tau \otimes \Lambda_1, W \otimes V(1))$ is reduced. Triplets (G_i, ρ_i, V_i) $(i \geq 0)$ are defined inductively by the following method*:

(1) $(G_0, \rho_0, V_0) := (K \times GL(1), \tau \otimes \Lambda_1, W \otimes V(1))$.

(2) (G_i, ρ_i, V_i) *is a castling transform of $(G_{i-1}, \rho_{i-1}, V_{i-1})$ with* $\dim V_i > \dim V_{i-1}$ *for $i \geq 1$.*

We put $n_i := m_{i-1} - l_{i-1}$ $(i \geq 1)$, where $(G_{i-1}, \rho_{i-1}, V_{i-1}) \cong (\tilde{G}_{i-1} \times GL(l_{i-1}), \tilde{\rho}_{i-1} \otimes \Lambda_1, V(m_{i-1}) \otimes V(l_{i-1}))$ and $(G_i, \rho_i, V_i) \cong (\tilde{G}_{i-1} \times GL(m_{i-1} - l_{i-1}), \tilde{\rho}_{i-1} \otimes \Lambda_1, V(m_{i-1}) \otimes V(m_{i-1} - l_{i-1}))$. Then we have $n_i < n_{i+1}$ for $i \geq 1$.

Proof. We fix an index $i \geq 1$. We may assume that a triplet (G_i, ρ_i, V_i) is of the following form: $(G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))$ with $2n > m > n \geq 2$. Then we see that $(G_{i-1}, \rho_{i-1}, V_{i-1}) \cong (G \times GL(m-n), \rho \otimes \Lambda_1, V(m) \otimes V(m-n))$ and $n_i = n$. For a triplet $(G_{i+1}, \rho_{i+1}, V_{i+1})$, there exist a triplet $(H, \sigma, V(k))$ with $k \geq 2$ and a positive integer l such that $(G, \rho, V(m)) \cong (H \times SL(l), \sigma \otimes \Lambda_1, V(k) \otimes V(l))$, $(G_{i+1}, \rho_{i+1}, V_{i+1}) \cong (H \times GL(kn - l) \times SL(n), \sigma \otimes \Lambda_1 \otimes \Lambda_1, V(k) \otimes V(kn - l) \otimes V(n))$ and $kn > 2l$. Then we see that $n_{i+1} = kn - l$. Since $kn - l - n = (k-1)n - l \geq kn/2 - l > 0$, we obtain our assertion. □

Lemma 2.13. *Assume that $(G_s \times GL(1), \rho_s \otimes \Lambda_1, V(t) \otimes V(1))$ $(t \geq 3)$ is a non-trivial reduced irreducible simple PV, that is, one of* I)-(2), (3), (4), (5), (6), (7), (14), (15) *with* $m = 1$, (16), (19), (22), (23), (24), (25), (27), (29), III)-(3), (5) *with* $2m + 1 = 1$, (6) *in* §7 *of* [SK]. *Then a non-reduced irreducible PV (G, ρ, V) which is castling equivalent to $(G_s \times GL(1), \rho_s \otimes \Lambda_1, V(t) \otimes V(1))$ is of the following form*: $(GL(1) \times G_s \times SL(a_1) \times \cdots \times SL(a_r), \Lambda_1 \otimes \rho_s \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes$

$V(t) \otimes V(a_1) \otimes \cdots \otimes V(a_r))$ for some $r \geq 1$, $a_i < a_{i+1}$ $(1 \leq i \leq r-1)$, $a_1 \geq t-1$ and $a_2 \geq t(t-1)-1$.

Proof. Let k be the reducing number of (G, ρ, V). We shall prove this by induction on the reducing number k. If $k = 1$, then $(G, \rho, V) \cong (GL(1) \times G_s \times SL(t-1), \Lambda_1 \otimes \rho_s \otimes \Lambda_1, V(1) \otimes V(t) \otimes V(t-1))$. If $k = 2$, then $(G, \rho, V) \cong (GL(1) \times G_s \times SL(t-1) \times SL(t(t-1)-1), \Lambda_1 \otimes \rho_s \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(t) \otimes V(t-1) \otimes V(t(t-1)-1))$. Therefore we see that our assertion holds for $k = 1, 2$. Now assume that our assertion holds for $k = n$ $(n \geq 1)$ and show that our assertion holds for $k = n+1$. Assume that the reducing number of (G, ρ, V) is $n+1$. Let a triplet (H, σ, W) be a castling transform of (G, ρ, V) with $\dim V > \dim W$. Since the reducing number of (H, σ, W) is n, by the assumption of induction, (H, σ, W) is of the following form: $(GL(1) \times G_s \times SL(a_1) \times \cdots \times SL(a_r), \Lambda_1 \otimes \rho_s \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(t) \otimes V(a_1) \otimes \cdots \otimes V(a_r))$ where $r \geq 1$, $a_i < a_{i+1}$ $(1 \leq i \leq r-1)$, $a_1 \geq t-1$ and $a_2 \geq t(t-1)-1$. For the above triplet, the castling transform at $GL(a_i)$ $(1 \leq i \leq r-1)$ (*resp.* $GL(1)$) has larger dimension since $ta_1 \cdots a_{i-1}a_{i+1} \cdots a_r - a_i \geq ta_r - a_i > 2a_r - a_r = a_r > a_i$ (*resp.* $ta_1 \cdots a_r - 1 > t-1 > 1$). Hence the castling transform at $GL(a_r)$ has smaller dimension. By Lemma 2.12, we see that our assertion holds for $k = n+1$. Hence our assertion holds by induction on k. □

Lemma 2.14. *Let $(G_s \times GL(t), \rho_s \otimes \Lambda_1, V(n) \otimes V(t))$ $(n \geq 2t \geq 4)$ be a reduced irreducible PV which is one of* I)-(8), (9), (10), (11), (13), (15) *with* $m \geq 2$, (17), (18), (20), (21), (26), (28), III)-(4), (5) *with* $2m+1 \geq 3$ *in* §7 *of* [SK]. *Then a non-reduced irreducible PV which is castling equivalent to $(G_s \times GL(t), \rho_s \otimes \Lambda_1, V(n) \otimes V(t))$ is of the following form: $(GL(1) \times G_s \times SL(a_1) \times \cdots \times SL(a_r), \Lambda_1 \otimes \rho_s \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(n) \otimes V(a_1) \otimes \cdots \otimes V(a_r))$ for some $r \geq 1$, $a_i < a_{i+1}$ $(1 \leq i \leq r-1)$, $a_1 \geq t$ and $a_2 \geq nt-1$. Here, if $a_1 < n$, then $a_1 = t$ or $a_1 = n-t$.*

Proof. Similarly to the proof of Lemma 2.13, we obtain our assertion. □

Lemma 2.15. *A non-reduced irreducible PV which is castling equivalent to $(Sp(n) \times GL(2), \Lambda_1 \otimes 2\Lambda_1, V(2n) \otimes V(3))$ $(n \geq 2)$ is of the following form: $(GL(1) \times Sp(n) \times SL(2) \times SL(a_1) \times \cdots \times SL(a_r), \Lambda_1 \otimes \Lambda_1 \otimes 2\Lambda_1 \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(2n) \otimes V(3) \otimes V(a_1) \otimes \cdots \otimes V(a_r))$ for some $r \geq 1$, $a_i < a_{i+1}$ $(1 \leq i \leq r-1)$ and $a_1 \geq 6n-1$.*

Proof. Similarly to the proof of Lemma 2.13, we obtain our assertion. □

Lemma 2.16. *An irreducible PV which is castling equivalent to* $(SL(2) \times SL(2) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(2) \otimes V(2) \otimes V(2))$ *is one of the following form*:

(1) $(GL(1) \times SL(2) \times SL(2) \times SL(a_1) \times \cdots \times SL(a_r), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(2) \otimes V(2) \otimes V(a_1) \otimes \cdots \otimes V(a_r))$ *for some* $r \geq 1$, $a_i < a_{i+1}$ $(1 \leq i \leq r-1)$, $a_1 \geq 2$ *and* $a_2 \geq 7$.

(2) $(GL(1) \times SL(b_1) \times \cdots \times SL(b_r), \Lambda_1 \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(b_1) \otimes \cdots \otimes V(b_r))$ *for some* $r \geq 4$, $b_i < b_{i+1}$ $(1 \leq i \leq r-1)$ *and* $b_1 \geq 2$.

Proof. Similarly to the proof of Lemma 2.13, we obtain our assertion. □

Lemma 2.17. *An irreducible PV which is castling equivalent to* $(SL(3) \times SL(3) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3) \otimes V(2))$ *is one of the following form*:

(1) $(GL(1) \times SL(3) \times SL(3) \times SL(a_1) \times \cdots \times SL(a_r), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(3) \otimes V(3) \otimes V(a_1) \otimes \cdots \otimes V(a_r))$ *for some* $r \geq 1$, $a_i < a_{i+1}$ $(1 \leq i \leq r-1)$, $a_1 \geq 2$ *and* $a_2 \geq 17$.

(2) $(GL(1) \times SL(b_1) \times \cdots \times SL(b_r), \Lambda_1 \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(b_1) \otimes \cdots \otimes V(b_r))$ *for some* $r \geq 3$, $b_i < b_{i+1}$ $(1 \leq i \leq r-1)$ *and* $b_1 \geq 2$.

Proof. Similarly to the proof of Lemma 2.13, we obtain our assertion. □

Lemma 2.18. *An irreducible PV which is castling equivalent to* $(SL(n) \times GL(n), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(n))$ $(n \geq 2)$ *is one of the following form*:

(1) $(GL(1) \times SL(n) \times SL(n) \times SL(a_1) \times \cdots \times SL(a_r), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(n) \otimes V(n) \otimes V(a_1) \otimes \cdots \otimes V(a_r))$ *for some* $r \geq 0$, $a_i < a_{i+1}$ $(1 \leq i \leq r-1)$ *and* $a_1 \geq n^2 - 1$.

(2) $(GL(1) \times SL(b_1) \times \cdots \times SL(b_r), \Lambda_1 \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(b_1) \otimes \cdots \otimes V(b_r))$ *for some* $r \geq 3$, $b_i < b_{i+1}$ $(1 \leq i \leq r-1)$ *and* $b_1 \geq n$.

Proof. Similarly to the proof of Lemma 2.13, we obtain our assertion. □

Lemma 2.19. *An irreducible PV which is castling equivalent to* $(SL(n) \times GL(m), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(m))$ $(n \geq 2m \geq 2)$ *is of the following form*: $(GL(1) \times$

$SL(a_1) \times \cdots \times SL(a_r), \Lambda_1 \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(a_1) \otimes \cdots \otimes V(a_r))$ *for some* $r \geq 1$ *and* $a_i < a_{i+1}$ $(1 \leq i \leq r-1)$.

PROOF. Similarly to the proof of Lemma 2.13, we obtain our assertion. □

LEMMA 2.20. *Let* G_{ss} *be a connected semisimple linear algebraic group and* ρ_{ss} *a t-dimensional irreducible rational representation of* G_{ss} *with* $(G_{ss}, \rho_{ss}, V(t)) \not\cong (SL(t), \Lambda_1, V(t))$. *An irreducible PV which is castling equivalent to* $(GL(n) \times G_{ss}, \Lambda_1 \otimes \rho_{ss}, V(n) \otimes V(t))$ $(n \geq t \geq 3)$ *is of the following form*: $(GL(1) \times K \times SL(a_1) \times \cdots \times SL(a_r), \Lambda_1 \otimes \sigma \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(k) \otimes V(a_1) \otimes \cdots \otimes V(a_r))$ *for some* $r \geq 1$, $a_i < a_{i+1}$ $(1 \leq i \leq r-1)$, $a_1 \geq n$. *Here, for* K *and* σ, *there exist a connected semisimple linear algebraic group* L *and an irreducible rational representation* τ *of* L *such that* $G_{ss} = K \times L$ *and* $\rho_{ss} = \sigma \otimes \tau$.

PROOF. Similarly to the proof of Lemma 2.13, we obtain our assertion. □

LEMMA 2.21. *Let* n, m *and* l *be positive integers.*

We put $G := GL(n) \times GL(m) \times GL(l)$, $V := M(n,m) \overbrace{\oplus \cdots \oplus}^{l} M(n,m)$ *and* $W := M(l,m) \overbrace{\oplus \cdots \oplus}^{n} M(l,m)$.

Let $\rho : G \to GL(V)$ *be the representation which is defined by* $V \ni X = (X_1, \ldots, X_l) \mapsto \rho(g)(X) = (AX_1{}^tB, \ldots, AX_l{}^tB){}^tC \in V$ *for* $g = (A,B,C) \in GL(n) \times GL(m) \times GL(l)$.

Let $\sigma : G \to GL(W)$ *be the representation which is defined by* $W \ni Y = (Y_1, \ldots, Y_n) \mapsto \sigma(g)(Y) = (CY_1{}^tB, \ldots, CY_n{}^tB){}^tA \in W$ *for* $g = (A,B,C) \in GL(n) \times GL(m) \times GL(l)$.

Let $\Phi : V \to W$ *be the isomorphism of vector spaces which is defined by* $\Phi(X) = (\Phi_1(X), \ldots, \Phi_n(X)) \in W$, *where* $X = (X_1, \ldots, X_l) \in V$, $X_i = (x_{i1}, \ldots, x_{in})'$, $x_{ij} \in M(1,m)$ *for* $1 \leq i \leq l$, $1 \leq j \leq n$, $\Phi_j(X) = (x_{1j}, \ldots, x_{lj})' \in M(l,m)$ *for* $1 \leq j \leq n$.

Then we have $\Phi(\rho(g)(X)) = \sigma(g)(\Phi(X))$ *for* $g \in G$, $X \in V$.

PROOF. By direct calculation, we obtain our assertion. □

LEMMA 2.22. *Let* n, m *and* l *be positive integers with* $m \geq 2$ *and* $n > ml$. *For each* m, *we define a sequence* $\{c_i\}_{i \geq -1}$ *by* $c_{-1} = -1$, $c_0 = 0$ *and* $c_i = mc_{i-1} - c_{i-2}$ $(i \geq 1)$. *Then we have* $c_{j+1}n - c_jl > c_{j-1}n - c_{j-2}l + n$ *for* $j \geq 1$.

Proof. Since $c_{i+1}n - c_i l = m(c_i n - c_{i-1} l) - (c_{i-1} n - c_{i-2} l)$ for $i \geq 1$, we have $(c_{i+1}n - c_i l) - (c_i n - c_{i-1} l) \geq (c_i n - c_{i-1} l) - (c_{i-1} n - c_{i-2} l)$ for $i \geq 1$. Then we see that $(c_{j+1}n - c_j l) - (c_{j-1}n - c_{j-2} l) = (c_{j+1}n - c_j l) - (c_j n - c_{j-1} l) + (c_j n - c_{j-1} l) - (c_{j-1} n - c_{j-2} l) \geq 2(c_1 n - c_0 l) - 2(c_0 n - c_{-1} l) = 2n - 2l > n$ for $j \geq 1$. Thus we obtain our assertion. □

3. A Classification of All Reductive PVs of the Form $T(H, \rho, \sigma, n, l, 0)^{(*)}$

In this section, we shall classify all reductive PVs of the form $T(H, \rho, \sigma, n, l, 0)^{(*)}$. First we shall consider the triplet $T(H, \rho, \sigma, n, l, 0)$.

Theorem 3.1. *If $n \geq ml + r$, then the triplet $T(H, \rho, \sigma, n, l, 0)$ is a trivial PV. If $n < ml + r$, then its castling transform at $GL(n)$ has smaller dimension.*

Proof. The first assertion is obvious. Since $n > ml$ and $n > r$, we obtain the second assertion. □

Theorem 3.1 means that it is not necessary to consider the triplet $T(H, \rho, \sigma, n, l, 0)$ in classification of reductive PVs with two irreducible components.

Next we shall consider the triplet $T(H, \rho, \sigma, n, l, 0)^*$. By Theorem 2.11, the triplet $T(H, \rho, \sigma, n, l, 0)^*$ is a PV if and only if a triplet $(H \times GL(l), (\rho \otimes \sigma) \otimes \Lambda_1, (V(m) \otimes V(r)) \otimes V(l)) = (H_1 \times H_2 \times H_3 \times GL(l), \rho_1 \otimes (\rho_2 \otimes \sigma_2) \otimes \sigma_3 \otimes \Lambda_1, V(m_1) \otimes (V(m_2) \otimes V(r_2)) \otimes V(r_3) \otimes V(l))$ is a PV.

Proposition 3.2. *If $H_2 = \{1\}$, then the triplet $(H \times GL(l), (\rho \otimes \sigma) \otimes \Lambda_1, (V(m) \otimes V(r)) \otimes V(l)) = (H_1 \times H_3 \times GL(l), \rho_1 \otimes \sigma_3 \otimes \Lambda_1, V(m_1) \otimes V(r_3) \otimes V(l))$ is a PV if and only if it is an irreducible PV. The triplet $(H_1 \times H_3 \times GL(l), \rho_1 \otimes \sigma_3 \otimes \Lambda_1, V(m_1) \otimes V(r_3) \otimes V(l))$ with $m_1 l = r_3$ is an irreducible PV if and only if it satisfies one of the following conditions*:

(1) $m_1 l = r_3$ *and* $(H_3, \rho_3, V(r_3)) = (SL(r_3), \Lambda_1, V(r_3))$.

(2) $l = 1$, $m_1 = r_3$ *and* $(H_1, \rho_1, V(m_1)) = (SL(m_1), \Lambda_1, V(m_1))$.

Proof. The first assertion is obvious. Note that the triplet $(H_1 \times H_3 \times GL(l), \rho_1 \otimes \sigma_3 \otimes \Lambda_1, V(m_1) \otimes V(r_3) \otimes V(l))$ $(m_1 l = r_3)$ is reduced. By §7 of [SK], we obtain the second assertion. □

Proposition 3.3. *If $H_2 \neq \{1\}$, then the triplet $(H \times GL(l), (\rho \otimes \sigma) \otimes \Lambda_1, (V(m) \otimes V(r)) \otimes V(l)) = (H_1 \times H_2 \times H_3 \times GL(l), \rho_1 \otimes (\rho_2 \otimes \sigma_2) \otimes \sigma_3 \otimes \Lambda_1, V(m_1) \otimes (V(m_2) \otimes V(r_2)) \otimes V(r_3) \otimes V(l))$ is a PV if and only if it is castling* *equivalent to a trivial PV of the form* $(H_2 \times G \times GL(N), (\rho_2 \otimes \sigma_2) \otimes \tau \otimes \Lambda_1, (V(m_2) \otimes V(r_2)) \otimes V(k) \otimes V(N))$ $(N \geq m_2 r_2 k)$. *If the triplet* $(H \times GL(l), (\rho \otimes \sigma) \otimes \Lambda_1, (V(m) \otimes V(r)) \otimes V(l))$ *with* $H_2 \neq \{1\}$ *is a PV, then* $ml \neq r$.

PROOF. We shall prove the first assertion. We put $T_1 := (H_1 \times H_2 \times H_3 \times GL(l), \rho_1 \otimes (\rho_2 \otimes \sigma_2) \otimes \sigma_3 \otimes \Lambda_1, V(m_1) \otimes (V(m_2) \otimes V(r_2)) \otimes V(r_3) \otimes V(l))$ and $T_2 := (H_1 \times H_2 \times H_2 \times H_3 \times GL(l), \rho_1 \otimes \rho_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \Lambda_1, V(m_1) \otimes V(m_2) \otimes V(r_2) \otimes V(r_3) \otimes V(l))$. Assume that the triplet T_1 is a PV. Then the triplet T_2 is an irreducible PV.

First, by Lemma 2.15 (resp. Lemma 2.19), we see that T_2 is not castling equivalent to $(Sp(n) \times GL(2), \Lambda_1 \otimes 2\Lambda_1, V(2n) \otimes V(3))$ $(n \geq 2)$ (resp. $(SL(n) \times GL(m), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(m))$ $(n \geq 2m \geq 2)$).

Second we shall consider the case where T_2 is castling equivalent to $(G_s \times GL(1), \rho_s \otimes \Lambda_1, V(t) \otimes V(1))$ $(t \geq 3)$ in Lemma 2.13. Since T_2 is non-reduced, by Lemma 2.13, T_2 is of the following form: $(GL(1) \times G_s \times SL(a_1) \times \cdots \times SL(a_r), \Lambda_1 \otimes \rho_s \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(t) \otimes V(a_1) \otimes \cdots \otimes V(a_r))$ for some $r \geq 1$, $a_i < a_{i+1}$ $(1 \leq i \leq r-1)$, $a_1 \geq t-1$ and $a_2 \geq t(t-1)-1$. Since $t(t-1)-1 > t$, we have $G_s = SL(t-1)$ and $a_1 = t-1$. By §7 of [SK], we see that $(G_s \times GL(1), \rho_s \otimes \Lambda_1, V(t) \otimes V(1)) = (SL(2) \times GL(1), 2\Lambda_1 \otimes \Lambda_1, V(3) \otimes V(1))$. Since $T_2 \cong (GL(1) \times SL(2) \times SL(2) \times SL(a_2) \times \cdots \times SL(a_r), \Lambda_1 \otimes 2\Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(3) \otimes V(2) \otimes V(a_2) \otimes \cdots \otimes V(a_r))$, T_1 is castling equivalent to $(SL(2) \times GL(1), (2\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1, (V(3) \otimes V(2)) \otimes V(1))$. Since $\dim(SL(2) \times GL(1)) = 4 < 6 = \dim((V(3) \otimes V(2)) \otimes V(1))$, T_1 is a non PV, which is a contradiction. Hence we see that T_2 is not castling equivalent to $(G_s \times GL(1), \rho_s \otimes \Lambda_1, V(t) \otimes V(1))$ in Lemma 2.13.

Third we shall consider the case where T_2 is castling equivalent to $(G_s \times GL(t), \rho_s \otimes \Lambda_1, V(n) \otimes V(t))$ $(n \geq 2t \geq 4)$ in Lemma 2.14. Similarly, by Lemma 2.14 and [SK, §7], we see that $(G_s \times GL(t), \rho_s \otimes \Lambda_1, V(n) \otimes V(t)) = (SL(4) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(6) \otimes V(2))$ and T_1 is castling equivalent to $(SL(4) \times GL(1), (\Lambda_2 \otimes \Lambda_1^{(*)}) \otimes \Lambda_1, (V(6) \otimes V(4)^{(*)}) \otimes V(1))$. Since $\dim(SL(4) \times GL(1)) = 16 < 24 = \dim((V(6) \otimes V(4)^{(*)}) \otimes V(1))$, T_1 is a non PV, which is a contradiction. Hence we see that T_2 is not castling equivalent to $(G_s \times GL(t), \rho_s \otimes \Lambda_1, V(n) \otimes V(t))$ in Lemma 2.14.

Fourth we shall consider the case where T_2 is castling equivalent to $(SL(2) \times SL(2) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(2) \otimes V(2) \otimes V(2))$ (resp. $(SL(3) \times SL(3) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3) \otimes V(2))$). Similarly, by Lemma 2.16 (resp. Lemma 2.17), we see that T_1 is castling equivalent to $(SL(2) \times GL(2), (\Lambda_1 \otimes \Lambda_1) \otimes \Lambda_1, (V(2) \otimes V(2)) \otimes V(2))$ (resp. $(SL(3) \times GL(2), (\Lambda_1 \otimes \Lambda_1^{(*)}) \otimes$

$\Lambda_1, (V(3) \otimes V(3)^{(*)}) \otimes V(2)))$. Since $\dim(SL(2) \times GL(2)) = 7 < 8 = \dim((V(2) \otimes V(2)) \otimes V(2))$ (resp. $\dim(SL(3) \times GL(2)) = 12 < 18 = \dim((V(3) \otimes V(3)^{(*)}) \otimes V(2)))$, T_1 is a non PV, which is a contradiction. Hence we see that T_2 is not castling equivalent to $(SL(2) \times SL(2) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(2) \otimes V(2) \otimes V(2))$ (resp. $(SL(3) \times SL(3) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3) \otimes V(2)))$.

Fifth we shall consider the case where T_2 is castling equivalent to $(SL(n) \times GL(n), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(n))$ $(n \geq 2)$. Similarly, by Lemma 2.18, we see that T_1 is castling equivalent to $P := (SL(n) \times GL(1), (\Lambda_1 \otimes \Lambda_1^{(*)}) \otimes \Lambda_1, (V(n) \otimes V(n)^{(*)}) \otimes V(1))$. By Lemma 1.11 in [KKTI], P is a non PV, which is a contradiction. Hence we see that T_2 is not castling equivalent to $(SL(n) \times GL(n), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(n))$ $(n \geq 2)$.

Sixth we shall consider the case where T_2 is castling equivalent to $(GL(n) \times G_{ss}, \Lambda_1 \otimes \rho_{ss}, V(n) \otimes V(t))$ $(n \geq t \geq 3)$ in Lemma 2.20. By Lemma 2.20, T_2 is of the following form: $(GL(1) \times K \times SL(a_1) \times \cdots \times SL(a_r), \Lambda_1 \otimes \sigma \otimes \Lambda_1 \otimes \cdots \otimes \Lambda_1, V(1) \otimes V(k) \otimes V(a_1) \otimes \cdots \otimes V(a_r))$ for some $r \geq 1$, $a_i < a_{i+1}$ $(1 \leq i \leq r-1)$, $a_1 \geq n$. Here, for K and σ, there exist a connected semisimple linear algebraic group L and an irreducible rational representation τ of L such that $G_{ss} = K \times L$ and $\rho_{ss} = \sigma \otimes \tau$. Then we see that $H_2 \times H_2$ is a normal subgroup of K. Here, for $H_2 \times H_2$ and $\rho_2 \otimes \sigma_2$, there exist a connected semisimple linear algebraic group $\tilde{K}$ and an irreducible rational representation $\tilde{\sigma}$ of $\tilde{K}$ such that $K = H_2 \times H_2 \times \tilde{K}$ and $\sigma = \rho_2 \otimes \sigma_2 \otimes \tilde{\sigma}$. Then we see that T_1 is castling equivalent to $(H_2 \times \tilde{K} \times L \times GL(n), (\rho_2 \otimes \sigma_2) \otimes \tilde{\sigma} \otimes \tau \otimes \Lambda_1, (V(m_2) \otimes V(r_2)) \otimes V(\deg \tilde{\sigma}) \otimes V(\deg \tau) \otimes V(n))$ $(n \geq t = m_2 r_2 (\deg \tilde{\sigma})(\deg \tau))$, which is a trivial PV. Thus we obtain the first assertion.

We shall prove the second assertion by contradiction. We assume that there exists a PV T_1 such that $H_2 \neq \{1\}$ and $m_1 m_2 l = r_2 r_3$. Since the triplet T_2 is reduced, by the first assertion, T_2 is of the following form: $(H_2 \times H_2 \times G \times GL(N), \rho_2 \otimes \sigma_2 \otimes \tau \otimes \Lambda_1, V(m_2) \otimes V(r_2) \otimes V(k) \otimes V(N))$ $(N \geq m_2 r_2 k)$. Then it satisfies one of the following conditions: (i) $N = l$. (ii) $SL(N)$ is a normal subgroup of H_1. (iii) $SL(N)$ is a normal subgroup of H_3. If it satisfies one of the conditions (i) and (ii), then $ml > r$, which is a contradiction. If it satisfies the condition (iii), then $ml < r$, which is a contradiction. Thus we obtain the second assertion. □

Theorem 3.4. *The triplet $T(H, \rho, \sigma, n, l, 0)^*$ is a PV if and only if it satisfies one of the following conditions*:

(1) $H_2 = \{1\}$, $m_1 l = r_3$ *and* $(H_3, \sigma_3, V(r_3)) = (SL(r_3), \Lambda_1, V(r_3))$. *In this case, the triplet $T(H, \rho, \sigma, n, l, 0)^*$ is a regular PV with* 1 *basic relative invariant.*

(2) $H_2 = \{1\}$, $l = 1$, $m_1 = r_3$ *and* $(H_1, \rho_1, V(m_1)) = (SL(m_1), \Lambda_1, V(m_1))$. *In this case, the triplet* $T(H, \rho, \sigma, n, l, 0)^*$ *is a regular* PV *with* 1 *basic relative invariant.*

(3) $H_2 = \{1\}$, $m_1 l \neq r_3$ *and the triplet* $P := (H_1 \times H_3 \times GL(l), \rho_1 \otimes \sigma_3 \otimes \Lambda_1, V(m_1) \otimes V(r_3) \otimes V(l))$ *is an irreducible* PV. *In this case, the triplet* $T(H, \rho, \sigma, n, l, 0)^*$ *is a non-regular* PV *and the number of basic relative invariants of it is equal to that of* P.

(4) $H_2 \neq \{1\}$ *and the triplet* $(H_1 \times H_2 \times H_3 \times GL(l), \rho_1 \otimes (\rho_2 \otimes \sigma_2) \otimes \sigma_3 \otimes \Lambda_1, V(m_1) \otimes (V(m_2) \otimes V(r_2)) \otimes V(r_3) \otimes V(l))$ *is castling equivalent to a trivial* PV *of the form* $(H_2 \times G \times GL(N), (\rho_2 \otimes \sigma_2) \otimes \tau \otimes \Lambda_1, (V(m_2) \otimes V(r_2)) \otimes V(k) \otimes V(N))$ $(N \geq m_2 r_2 k)$. *In this case, the triplet* $T(H, \rho, \sigma, n, l, 0)^*$ *is a non-regular* PV, *and the number of basic relative invariants of it is* 1 (*resp.* 0) *if* $N = m_2 r_2 k$ (*resp.* $N > m_2 r_2 k$).

Proof. By Theorem 2.11 and Propositions 3.2 and 3.3, we obtain our assertion. □

4. Generic Isotropy Subgroups

Let H be a connected semisimple linear algebraic group and let ρ be an m-dimensional irreducible rational representation of H with $m \geq 2$. For each m, we define a sequence $\{c_i\}_{i \geq -1}$ by $c_{-1} = -1$, $c_0 = 0$, $c_i = mc_{i-1} - c_{i-2}$ $(i \geq 1)$. Let n and l be positive integers with $n > ml$. We define a triplet $P(H, \rho, n, l, j)$ with $j \geq 0$ (resp. $Q(H, \rho, l, j)$ with $j \geq 0$, $R(H, \rho, n - ml, j)$ with $j \geq 1$) as $(GL(c_{j+1}n - c_j l) \times H \times GL(c_j n - c_{j-1} l), \Lambda_1 \otimes \rho^{[*]} \otimes \Lambda_1, V(c_{j+1}n - c_j l) \otimes V(m)^{[*]} \otimes V(c_j n - c_{j-1} l))$ (resp. $(GL(c_{j+2} l) \times H \times GL(c_{j+1} l), \Lambda_1 \otimes \rho^{[*]} \otimes \Lambda_1, V(c_{j+2} l) \otimes V(m)^{[*]} \otimes V(c_{j+1} l))$, $(GL(c_{j+1}(n - ml)) \times H \times GL(c_j(n - ml)), \Lambda_1 \otimes \rho^{[*]} \otimes \Lambda_1, V(c_{j+1}(n - ml)) \otimes V(m)^{[*]} \otimes V(c_j(n - ml))))$, where $\rho^{[*]} := \begin{cases} \rho & (j \equiv 0 \pmod 2) \\ \rho^* & (j \equiv 1 \pmod 2) \end{cases}$.

The triplet $P(H, \rho, n, l, 0)$ is a non-regular trivial PV and the triplet $P(H, \rho, n, l, j)$ is obtained from the non-regular trivial PV $P(H, \rho, n, l, 0)$ by applying a castling transformation j times. The triplet $Q(H, \rho, l, 0)$ (resp. $R(H, \rho, n - ml, 1)$) is a regular trivial PV and the triplet $Q(H, \rho, l, j)$ (resp. $R(H, \rho, n - ml, j)$) is obtained from the regular trivial PV $Q(H, \rho, l, 0)$ (resp. $R(H, \rho, n - ml, 1)$) by applying a castling transformation j (resp. $j - 1$) times.

The purpose of this section is a calculation of generic isotropy subgroups of three PVs $P(H, \rho, n, l, j)$, $Q(H, \rho, l, j)$ and $R(H, \rho, n - ml, j)$.

First we calculate a generic isotropy subgroup of $P(H, \rho, n, l, j)$. For $M(ml, n-ml) \cong (\mathbf{G}_a)^{ml(n-ml)}$ and $H \times GL(l) \times GL(n-ml)$, we define a homomorphism $f : H \times GL(l) \times GL(n-ml) \to \mathrm{Aut}(M(ml, n-ml))$ of groups by $f((h, A, B))(C) = ({}^tA^{-1} \otimes {}^t\rho(h)^{-1})CB^{-1}$ for $(h, A, B) \in H \times GL(l) \times GL(n-ml)$, $C \in M(ml, n-ml)$. Let $M(ml, n-ml) \rtimes (H \times GL(l) \times GL(n-ml))$ be a semi-direct product of $M(ml, n-ml)$ with $H \times GL(l) \times GL(n-ml)$ relative to f. We shall calculate a generic isotropy subgroup of $P(H, \rho, n, l, 0)$. When the representation space $V(n) \otimes V(m) \otimes V(l)$ is identified with $V_0 := M(n,m) \overbrace{\oplus \cdots \oplus}^{l} M(n,m)$ for $P(H, \rho, n, l, 0)$, the representation $\Lambda_1 \otimes \rho \otimes \Lambda_1$ is given by $V_0 \ni (X_1, \dots, X_l) \mapsto (DX_1{}^t\rho(h), \dots, DX_l{}^t\rho(h)){}^tA \in V_0$ for $(D, h, A) \in GL(n) \times H \times GL(l)$.

Proposition 4.1. *We define* $v_0 = (X_1^{(0)}, \dots, X_l^{(0)}) \in V_0$ *by* $(X_1^{(0)} | \cdots | X_l^{(0)}) = \begin{pmatrix} I_{ml} \\ 0 \end{pmatrix} \in M(n, ml)$. *Put* $D(C, h, A, B) := \begin{pmatrix} {}^tA^{-1} \otimes {}^t\rho(h)^{-1} & CB \\ 0 & B \end{pmatrix}$ *for* $(C, (h, A, B)) \in M(ml, n-ml) \rtimes (H \times GL(l) \times GL(n-ml))$. *Then* $v_0 \in V_0$ *is a generic point of* $P(H, \rho, n, l, 0)$ *and the generic isotropy subgroup at* v_0 *is* $(GL(n) \times H \times GL(l))_{v_0} = \{(D(C, h, A, B), h, A) \mid (C, (h, A, B)) \in M(ml, n-ml) \rtimes (H \times GL(l) \times GL(n-ml))\}$. *Furthermore, there exists the isomorphism* $\Psi : M(ml, n-ml) \rtimes (H \times GL(l) \times GL(n-ml)) \to (GL(n) \times H \times GL(l))_{v_0}$ *of groups such that* $\Psi((C, (h, A, B))) = (D(C, h, A, B), h, A)$ *for* $(C, (h, A, B)) \in M(ml, n-ml) \rtimes (H \times GL(l) \times GL(n-ml))$.

Proof. By direct calculation, we obtain our assertion. □

By Theorem 2.5, a generic isotropy subgroup of $P(H, \rho, n, l, j)$ is isomorphic to $M(ml, n-ml) \rtimes (H \times GL(l) \times GL(n-ml))$. However we need the explicit form of a generic isotropy subgroup as a subgroup of $GL(c_{j+1}n - c_jl) \times H \times GL(c_jn - c_{j-1}l)$. We shall calculate the explicit form of a generic isotropy subgroup of $P(H, \rho, n, l, j)$ with $j \geq 1$. First we shall calculate a generic isotropy subgroup of $P(H, \rho, m+1, 1, j)$ with $j \geq 1$. Next, by the above result, we shall calculate a generic isotropy subgroup of $P(H, \rho, n, l, j)$ with $j \geq 1$.

Note that $c_{j+1}(m+1) - c_j = c_{j+2} + c_{j+1}$ and $c_j(m+1) - c_{j-1} = c_{j+1} + c_j$. When the representation space $V(c_{j+2} + c_{j+1}) \otimes V(m)^{[*]} \otimes V(c_{j+1} + c_j)$ is identified with $V_j := M(c_{j+2} + c_{j+1}, m) \overbrace{\oplus \cdots \oplus}^{c_{j+1}+c_j} M(c_{j+2} + c_{j+1}, m)$ (resp. $W_j := M(c_{j+1} + c_j, m) \overbrace{\oplus \cdots \oplus}^{c_{j+2}+c_{j+1}} M(c_{j+1} + c_j, m)$) for $P(H, \rho, m+1, 1, j)$ with $j \geq 1$, the representation $\Lambda_1 \otimes \rho^{[*]} \otimes \Lambda_1$ is given by $V_j \ni (X_1, \dots, X_{c_{j+1}+c_j}) \mapsto (SX_1{}^t\rho^{[*]}(h), \dots, SX_{c_{j+1}+c_j}{}^t\rho^{[*]}(h)){}^tT \in V_j$ (resp. $W_j \ni (Y_1, \dots, Y_{c_{j+2}+c_{j+1}}) \mapsto (TY_1{}^t\rho^{[*]}(h), \dots,$

$TY_{c_{j+2}+c_{j+1}}{}^t\rho^{[*]}(h))^tS \in W_j)$ for $(S,h,T) \in GL(c_{j+2}+c_{j+1}) \times H \times GL(c_{j+1}+c_j)$.
Here $\rho^{[*]}(h) := \begin{cases} \rho(h) & (j \equiv 0 \pmod 2) \\ {}^t\rho(h)^{-1} & (j \equiv 1 \pmod 2) \end{cases}$.

Proposition 4.2. *There exist generic points* $v_j = (X_1^{(j)}, \dots, X_{c_{j+1}+c_j}^{(j)}) \in V_j$ *and* $w_j = (Y_1^{(j)}, \dots, Y_{c_{j+2}+c_{j+1}}^{(j)}) \in W_j$ *of* $P(H,\rho,m+1,1,j)$ *with* $j \geq 1$ *such that*

$$X_s^{(j)} = \begin{pmatrix} x_s^{(j)} \\ 0 \end{pmatrix} \in M(c_{j+2}+c_{j+1},m), \quad x_s^{(j)} \in M(c_{j+2},m) \text{ for } 1 \leq s \leq c_{j+1},$$

$$X_t^{(j)} = \begin{pmatrix} 0 \\ x_t^{(j)} \end{pmatrix} \in M(c_{j+2}+c_{j+1},m), \quad x_t^{(j)} \in M(c_{j+1},m) \text{ for } c_{j+1}+1 \leq t \leq c_{j+1}+c_j,$$

$$Y_s^{(j)} = \begin{pmatrix} y_s^{(j)} \\ 0 \end{pmatrix} \in M(c_{j+1}+c_j,m), \quad y_s^{(j)} \in M(c_{j+1},m) \text{ for } 1 \leq s \leq c_{j+2},$$

$$Y_t^{(j)} = \begin{pmatrix} 0 \\ y_t^{(j)} \end{pmatrix} \in M(c_{j+1}+c_j,m), \quad y_t^{(j)} \in M(c_j,m) \text{ for } c_{j+2}+1 \leq t \leq c_{j+2}+c_{j+1}.$$

These generic points are defined inductively by the following method:

(1) *Let* $\{y_1^{(1)}, \dots, y_{c_3}^{(1)}\}$ *be a linear basis of* $\langle I_m \rangle^\perp$ *and* $\{y_{c_3+1}^{(1)}, \dots, y_{c_3+c_2}^{(1)}\}$ *a linear basis of* $M(1,m)$. *We define* $w_1 = (Y_1^{(1)}, \dots, Y_{c_3+c_2}^{(1)}) \in W_1$ *by*

$$Y_s^{(1)} = \begin{pmatrix} y_s^{(1)} \\ 0 \end{pmatrix} \in M(c_2+c_1,m), \quad y_s^{(1)} \in M(c_2,m) \text{ for } 1 \leq s \leq c_3,$$

$$Y_t^{(1)} = \begin{pmatrix} 0 \\ y_t^{(1)} \end{pmatrix} \in M(c_2+c_1,m), \quad y_t^{(1)} \in M(c_1,m) \text{ for } c_3+1 \leq t \leq c_3+c_2.$$

Then, for $w_1 \in W_1$, *the point* $v_1 = (X_1^{(1)}, \dots, X_{c_2+c_1}^{(1)}) \in V_1$ *is defined by the isomorphism of vector spaces of Lemma* 2.21.

(2) *Assume that there exists a generic point* $v_i = (X_1^{(i)}, \dots, X_{c_{i+1}+c_i}^{(i)}) \in V_i$ *of* $P(H,\rho,m+1,1,i)$ *with* $i \geq 1$ *such that*

$$X_s^{(i)} = \begin{pmatrix} x_s^{(i)} \\ 0 \end{pmatrix} \in M(c_{i+2}+c_{i+1},m), \quad x_s^{(i)} \in M(c_{i+2},m) \text{ for } 1 \leq s \leq c_{i+1},$$

$$X_t^{(i)} = \begin{pmatrix} 0 \\ x_t^{(i)} \end{pmatrix} \in M(c_{i+2}+c_{i+1},m), \quad x_t^{(i)} \in M(c_{i+1},m)$$

$$\text{for } c_{i+1}+1 \leq t \leq c_{i+1}+c_i.$$

Let $\{y_1^{(i+1)},\ldots,y_{c_{i+3}}^{(i+1)}\}$ *be a linear basis of* $\langle x_1^{(i)},\ldots,x_{c_{i+1}}^{(i)}\rangle^{\perp}$ *and* $\{y_{c_{i+3}+1}^{(i+1)},\ldots,y_{c_{i+3}+c_{i+2}}^{(i+1)}\}$ *a linear basis of* $\langle x_{c_{i+1}+1}^{(i)},\ldots,x_{c_{i+1}+c_i}^{(i)}\rangle^{\perp}$. *We define* $w_{i+1}=(Y_1^{(i+1)},\ldots,Y_{c_{i+3}+c_{i+2}}^{(i+1)})\in W_{i+1}$ *by*

$$Y_s^{(i+1)}=\begin{pmatrix} y_s^{(i+1)} \\ 0 \end{pmatrix}\in M(c_{i+2}+c_{i+1},m),\quad y_s^{(i+1)}\in M(c_{i+2},m)$$

$$\textit{for } 1\le s\le c_{i+3},$$

$$Y_t^{(i+1)}=\begin{pmatrix} 0 \\ y_t^{(i+1)} \end{pmatrix}\in M(c_{i+2}+c_{i+1},m),\quad y_t^{(i+1)}\in M(c_{i+1},m)$$

$$\textit{for } c_{i+3}+1\le t\le c_{i+3}+c_{i+2}.$$

Then, for $w_{i+1}\in W_{i+1}$, *the point* $v_{i+1}=(X_1^{(i+1)},\ldots,X_{c_{i+2}+c_{i+1}}^{(i+1)})\in V_{i+1}$ *is defined by the isomorphism of vector spaces of Lemma* 2.21.

Proof. By Lemma 2.21 and Proposition 4.1, we obtain our assertion. □

Proposition 4.3. *Let* τ_i $(i\ge 1)$ (*resp.* $\tilde{\tau}_i$ $(i\ge 1)$) *be the* c_{i+1}*-dimensional* (*resp. the* c_i*-dimensional*) *rational representations of* H *which are defined inductively by the following method*: $\tau_1=\rho^*$, $\rho^*\otimes\rho=1\oplus\tau_2$, $\tau_{2s}\otimes\rho^*=\tau_{2s-1}\oplus\tau_{2s+1}$ $(s\ge 1)$ *and* $\tau_{2s+1}\otimes\rho=\tau_{2s}\oplus\tau_{2s+2}$ $(s\ge 1)$ (*resp.* $\tilde{\tau}_1=1$, $\tilde{\tau}_2=\rho$, $\tilde{\tau}_{2s}\otimes\rho^*=\tilde{\tau}_{2s-1}\oplus\tilde{\tau}_{2s+1}$ $(s\ge 1)$ *and* $\tilde{\tau}_{2s+1}\otimes\rho=\tilde{\tau}_{2s}\oplus\tilde{\tau}_{2s+2}$ $(s\ge 1)$). *For* $C\in M(m,1)$, *we define* $u_i(C)\in M(c_i,c_{i+1})$ $(i\ge 1)$ *by* $u_1(C)=-{}^tC$ *and* $(u_i(C)y_1^{(i)},\ldots,u_i(C)y_{c_{i+2}}^{(i)})=(y_{c_{i+2}+1}^{(i)},\ldots,y_{c_{i+2}+c_{i+1}}^{(i)})u_{i+1}(C)$ $(i\ge 1)$.

Put $A_i(C,h,\alpha,\beta):=\begin{pmatrix} \alpha^{-1}\tau_{i+1}(h) & -{}^tu_{i+1}(C)(\beta\tilde{\tau}_{i+1}(h)) \\ 0 & \beta\tilde{\tau}_{i+1}(h)\end{pmatrix}$ $(i\ge 0)$ *for* $(C,(h,\alpha,\beta))\in M(m,1)\rtimes(H\times GL(1)\times GL(1))$.

Then the generic isotropy subgroup at $v_j\in V_j$ *of* $P(H,\rho,m+1,1,j)$ *with* $j\ge 1$ *is given by* $G(H,\rho,m+1,1,j):=\{(A_j(C,h,\alpha,\beta),h,{}^tA_{j-1}(C,h,\alpha,\beta)^{-1})\mid (C,(h,\alpha,\beta))\in M(m,1)\rtimes(H\times GL(1)\times GL(1))\}$. *Furthermore, there exists the isomorphism* $\psi_j:M(m,1)\rtimes(H\times GL(1)\times GL(1))\to G(H,\rho,m+1,1,j)$ *of groups such that* $\psi_j((C,(h,\alpha,\beta)))=(A_j(C,h,\alpha,\beta),h,{}^tA_{j-1}(C,h,\alpha,\beta)^{-1})$ *for* $(C,(h,\alpha,\beta))\in M(m,1)\rtimes(H\times GL(1)\times GL(1))$.

Proof. We shall prove this by induction on the index j.

By Proposition 4.1, we see that $v_0=\begin{pmatrix} I_m \\ 0\end{pmatrix}\in M(m+1,m)$ is the generic point of $P(H,\rho,m+1,1,0)$, the generic isotropy subgroup at v_0 is $(GL(m+1)\times H\times$

$GL(1))_{v_0} = \{(D(C,h,\alpha,\beta),h,\alpha) \mid (C,(h,\alpha,\beta)) \in M(m,1) \rtimes (H \times GL(1) \times GL(1))\}$ and there exists the isomorphism $\Psi : M(m,1) \rtimes (H \times GL(1) \times GL(1)) \to (GL(m+1) \times H \times GL(1))_{v_0}$ of groups such that $\Psi((C,(h,\alpha,\beta))) = (D(C,h,\alpha,\beta), h,\alpha)$ for $(C,(h,\alpha,\beta)) \in M(m,1) \rtimes (H \times GL(1) \times GL(1))$.

We put $G := \{g := (D(C,h,\alpha,\beta),h) \mid (C,(h,\alpha,\beta)) \in M(m,1) \rtimes (H \times GL(1) \times GL(1))\}$. We see that $\langle v_0 \rangle = \left\{ \begin{pmatrix} xI_m \\ 0 \end{pmatrix} \in M(m+1,m) \middle| x \in \mathbf{C} \right\}$ and $\langle v_0 \rangle^{\perp} = \left\{ \begin{pmatrix} Y \\ Z \end{pmatrix} \in M(m+1,m) \middle| \begin{matrix} Y \in M(m),\ \mathrm{Tr}\ Y = 0, \\ Z \in M(1,m) \end{matrix} \right\}$.

Then G acts on $\langle v_0 \rangle$ by $\langle v_0 \rangle \ni \begin{pmatrix} xI_m \\ 0 \end{pmatrix} \mapsto (\Lambda_1 \otimes \rho)(g)\left(\begin{pmatrix} xI_m \\ 0 \end{pmatrix}\right) = \begin{pmatrix} \alpha^{-1}\,{}^t\rho(h)^{-1}(xI_m)\,{}^t\rho(h) \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha^{-1}xI_m \\ 0 \end{pmatrix} \in \langle v_0 \rangle$ for $g \in G$, and G acts on $\langle v_0 \rangle^{\perp}$ by $\langle v_0 \rangle^{\perp} \ni \begin{pmatrix} Y \\ Z \end{pmatrix} \mapsto (\Lambda_1^* \otimes \rho^*)(g)\left(\begin{pmatrix} Y \\ Z \end{pmatrix}\right) = \begin{pmatrix} \alpha\rho(h)Y\rho(h)^{-1} \\ -{}^tC(\alpha\rho(h))Y\rho(h)^{-1} + \beta^{-1}Z\rho(h)^{-1} \end{pmatrix} \in \langle v_0 \rangle^{\perp}$ for $g \in G$.

The representation of H on $\{xI_m \mid x \in \mathbf{C}\}$ which is defined by $xI_m \mapsto {}^t\rho(h)^{-1}(xI_m)\,{}^t\rho(h) = xI_m$ for $h \in H$ is a unit representation 1. Since H is semisimple, there exists the c_3-dimensional rational representation τ_2 of H such that $\rho^* \otimes \rho = 1 \oplus \tau_2$. Then we see that the representation of H on $\{Y \in M(m) \mid \mathrm{Tr}\ Y = 0\}$ which is defined by $Y \mapsto \rho(h)Y\rho(h)^{-1}$ for $h \in H$ is the representation τ_2^*.

We shall calculate the representation matrix $\begin{pmatrix} U & 0 \\ S & T \end{pmatrix}$ where $U \in M(c_3)$, $S \in M(c_2,c_3)$ and $T \in M(c_2)$ of the linear transformation $(\Lambda_1^* \otimes \rho^*)(g) : \langle v_0 \rangle^{\perp} \to \langle v_0 \rangle^{\perp}$ with respect to a basis $\{Y_1^{(1)}, \ldots, Y_{c_3+c_2}^{(1)}\}$.

We see that $U = \alpha\tau_2^*(h)$ and $T = \beta^{-1}\rho^*(h)$. Since $(-{}^tC(\alpha\rho(h))y_1^{(1)}\rho(h)^{-1}, \ldots, -{}^tC(\alpha\rho(h))y_{c_3}^{(1)}\rho(h)^{-1}) = (y_{c_3+1}^{(1)}, \ldots, y_{c_3+c_2}^{(1)})S$ and $(\alpha\rho(h)y_1^{(1)}\rho(h)^{-1}, \ldots, \alpha\rho(h)y_{c_3}^{(1)}\rho(h)^{-1}) = (y_1^{(1)}, \ldots, y_{c_3}^{(1)})\alpha\tau_2^*(h)$, we have $(-{}^tCy_1^{(1)}, \ldots, -{}^tCy_{c_3}^{(1)}) = (y_{c_3+1}^{(1)}, \ldots, y_{c_3+c_2}^{(1)})S(\alpha\tau_2^*(h))^{-1}$. We put $u_2(C) := S(\alpha\tau_2^*(h))^{-1}$. Then we have $S = u_2(C)(\alpha\tau_2^*(h))$.

Therefore the generic isotropy subgroup at $w_1 = (Y_1^{(1)}, \ldots, Y_{c_3+c_2}^{(1)}) \in W_1$ of $(GL(c_2+c_1) \times H \times GL(c_3+c_2), \Lambda_1^* \otimes \rho^* \otimes \Lambda_1, V(c_2+c_1)^* \otimes V(m)^* \otimes V(c_3+c_2))$ is $(GL(c_2+c_1) \times H \times GL(c_3+c_2))_{w_1} = \{(D(C,h,\alpha,\beta),h,A_1(C,h,\alpha,\beta)) \mid (C,(h,\alpha,\beta)) \in M(m,1) \rtimes (H \times GL(1) \times GL(1))\}$ and there exists the isomorphism $\delta_1 : (GL(m+1) \times H \times GL(1))_{v_0} \to (GL(c_2+c_1) \times H \times GL(c_3+c_2))_{w_1}$ of groups such that $\delta_1((D(C,h,\alpha,\beta),h,\alpha)) = (D(C,h,\alpha,\beta),h,A_1(C,h,\alpha,\beta))$ for $(C,(h,\alpha,\beta)) \in M(m,1) \rtimes (H \times GL(1) \times GL(1))$. Thus we see that our assertion holds for $j = 1$.

Assume that the generic isotropy subgroup at $v_k \in V_k$ of $P(H, \rho, m+1, 1, k)$ with $k \geq 1$ is given by $G(H, \rho, m+1, 1, k) = \{(A_k(C, h, \alpha, \beta), h, {}^tA_{k-1}(C, h, \alpha, \beta)^{-1}) \mid (C, (h, \alpha, \beta)) \in M(m, 1) \rtimes (H \times GL(1) \times GL(1))\}$ and there exists the isomorphism $\psi_k : M(m, 1) \rtimes (H \times GL(1) \times GL(1)) \to G(H, \rho, m+1, 1, k)$ of groups such that $\psi_k((C, (h, \alpha, \beta))) = (A_k(C, h, \alpha, \beta), h, {}^tA_{k-1}(C, h, \alpha, \beta)^{-1})$ for $(C, (h, \alpha, \beta)) \in M(m, 1) \rtimes (H \times GL(1) \times GL(1))$.

We put $G := \{g := (A_k(C, h, \alpha, \beta), h) \mid (C, (h, \alpha, \beta)) \in M(m, 1) \rtimes (H \times GL(1) \times GL(1))\}$. We see that $\langle X_1^{(k)}, \ldots, X_{c_{k+1}+c_k}^{(k)} \rangle = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \in M(c_{k+2} + c_{k+1}, m) \;\middle|\; \begin{matrix} X \in \langle x_1^{(k)}, \ldots, x_{c_{k+1}}^{(k)} \rangle, \\ Y \in \langle x_{c_{k+1}+1}^{(k)}, \ldots, x_{c_{k+1}+c_k}^{(k)} \rangle \end{matrix} \right\}$ and $\langle X_1^{(k)}, \ldots, X_{c_{k+1}+c_k}^{(k)} \rangle^{\perp} = \langle Y_1^{(k+1)}, \ldots, Y_{c_{k+3}+c_{k+2}}^{(k+1)} \rangle = \left\{ \begin{pmatrix} Z \\ W \end{pmatrix} \in M(c_{k+2} + c_{k+1}, m) \;\middle|\; \begin{matrix} Z \in \langle y_1^{(k+1)}, \ldots, y_{c_{k+3}}^{(k+1)} \rangle, \\ W \in \langle y_{c_{k+3}+1}^{(k+1)}, \ldots, y_{c_{k+3}+c_{k+2}}^{(k+1)} \rangle \end{matrix} \right\}$.

Then G acts on $\langle X_1^{(k)}, \ldots, X_{c_{k+1}+c_k}^{(k)} \rangle$ by $\langle X_1^{(k)}, \ldots, X_{c_{k+1}+c_k}^{(k)} \rangle \ni \begin{pmatrix} X \\ Y \end{pmatrix} \mapsto (\Lambda_1 \otimes \rho^{[*]})(g)\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right) = \begin{pmatrix} \alpha^{-1}\tau_{k+1}(h) X\, {}^t\rho^{[*]}(h) - {}^tu_{k+1}(C)(\beta\tilde{\tau}_{k+1}(h))\, Y\, {}^t\rho^{[*]}(h) \\ \beta\tilde{\tau}_{k+1}(h)\, Y\, {}^t\rho^{[*]}(h) \end{pmatrix} \in \langle X_1^{(k)}, \ldots, X_{c_{k+1}+c_k}^{(k)} \rangle$ for $g \in G$, and G acts on $\langle X_1^{(k)}, \ldots, X_{c_{k+1}+c_k}^{(k)} \rangle^{\perp}$ by

$$\langle X_1^{(k)}, \ldots, X_{c_{k+1}+c_k}^{(k)} \rangle^{\perp} \ni \begin{pmatrix} Z \\ W \end{pmatrix} \mapsto (\Lambda_1^* \otimes \rho^{[*]*})(g)\left(\begin{pmatrix} Z \\ W \end{pmatrix}\right)$$
$$= \begin{pmatrix} \alpha\, {}^t\tau_{k+1}(h)^{-1} Z \rho^{[*]}(h)^{-1} \\ u_{k+1}(C)(\alpha\, {}^t\tau_{k+1}(h)^{-1}) Z \rho^{[*]}(h)^{-1} + \beta^{-1}\, {}^t\tilde{\tau}_{k+1}(h)^{-1} W \rho^{[*]}(h)^{-1} \end{pmatrix}$$
$$\in \langle X_1^{(k)}, \ldots, X_{c_{k+1}+c_k}^{(k)} \rangle^{\perp}$$

for $g \in G$. Here $\rho^{[*]}(h) := \begin{cases} \rho(h) & (k \equiv 0 \pmod 2) \\ {}^t\rho(h)^{-1} & (k \equiv 1 \pmod 2) \end{cases}$.

Since $(\tau_{k+1}(h) x_1^{(k)}\, {}^t\rho^{[*]}(h), \ldots, \tau_{k+1}(h) x_{c_{k+1}}^{(k)}\, {}^t\rho^{[*]}(h)) = (x_1^{(k)}, \ldots, x_{c_{k+1}}^{(k)}) \tau_k(h)$ for $h \in H$ (resp. $(\tilde{\tau}_{k+1}(h) x_{c_{k+1}+1}^{(k)}\, {}^t\rho^{[*]}(h), \ldots, \tilde{\tau}_{k+1}(h) x_{c_{k+1}+c_k}^{(k)}\, {}^t\rho^{[*]}(h)) = (x_{c_{k+1}+1}^{(k)}, \ldots, x_{c_{k+1}+c_k}^{(k)}) \tilde{\tau}_k(h)$ for $h \in H$), the representation of H on $\langle x_1^{(k)}, \ldots, x_{c_{k+1}}^{(k)} \rangle$ (resp. $\langle x_{c_{k+1}+1}^{(k)}, \ldots, x_{c_{k+1}+c_k}^{(k)} \rangle$) which is defined by $X \mapsto \tau_{k+1}(h) X\, {}^t\rho^{[*]}(h)$ for $h \in H$ (resp. $Y \mapsto \tilde{\tau}_{k+1}(h)\, Y\, {}^t\rho^{[*]}(h)$ for $h \in H$) is τ_k (resp. $\tilde{\tau}_k$). Since H is semisimple, there exists the c_{k+3}-dimensional rational representation τ_{k+2} (resp. the c_{k+2}-dimensional rational representation $\tilde{\tau}_{k+2}$) of H such that $\tau_{k+1} \otimes \rho^{[*]} = \tau_k \oplus \tau_{k+2}$ (resp. $\tilde{\tau}_{k+1} \otimes \rho^{[*]} = \tilde{\tau}_k \oplus \tilde{\tau}_{k+2}$). Then we see that the representation of H on $\langle x_1^{(k)}, \ldots, x_{c_{k+1}}^{(k)} \rangle^{\perp} = \langle y_1^{(k+1)}, \ldots, y_{c_{k+3}}^{(k+1)} \rangle$ (resp. $\langle x_{c_{k+1}+1}^{(k)}, \ldots, x_{c_{k+1}+c_k}^{(k)} \rangle^{\perp} = \langle y_{c_{k+3}+1}^{(k+1)}, \ldots, y_{c_{k+3}+c_{k+2}}^{(k+1)} \rangle$) which is defined by $Z \mapsto {}^t\tau_{k+1}(h)^{-1} Z \rho^{[*]}(h)^{-1}$ for $h \in H$ (resp. $W \mapsto {}^t\tilde{\tau}_{k+1}(h)^{-1} W \rho^{[*]}(h)^{-1}$ for $h \in H$) is the representation τ_{k+2}^* (resp. $\tilde{\tau}_{k+2}^*$).

We shall calculate the representation matrix $\begin{pmatrix} U & 0 \\ S & T \end{pmatrix}$ where $U \in M(c_{k+3})$, $S \in M(c_{k+2}, c_{k+3})$ and $T \in M(c_{k+2})$ of the linear transformation $(\Lambda_1^* \otimes \rho^{[*]*})(g) : \langle X_1^{(k)}, \dots, X_{c_{k+1}+c_k}^{(k)} \rangle^\perp \to \langle X_1^{(k)}, \dots, X_{c_{k+1}+c_k}^{(k)} \rangle^\perp$ with respect to a basis $\{ Y_1^{(k+1)}, \dots, Y_{c_{k+3}+c_{k+2}}^{(k+1)} \}$.

We see that $U = \alpha \tau_{k+2}^*(h)$ and $T = \beta^{-1} \tilde{\tau}_{k+2}^*(h)$.

Since $(u_{k+1}(C)(\alpha {}^t\tau_{k+1}(h)^{-1}) y_1^{(k+1)} \rho^{[*]}(h)^{-1}, \dots, u_{k+1}(C)(\alpha {}^t\tau_{k+1}(h)^{-1}) y_{c_{k+3}}^{(k+1)} \cdot \rho^{[*]}(h)^{-1}) = (y_{c_{k+3}+1}^{(k+1)}, \dots, y_{c_{k+3}+c_{k+2}}^{(k+1)}) S$ and $(\alpha {}^t\tau_{k+1}(h)^{-1} y_1^{(k+1)} \rho^{[*]}(h)^{-1}, \dots, \alpha {}^t\tau_{k+1}(h)^{-1} y_{c_{k+3}}^{(k+1)} \rho^{[*]}(h)^{-1}) = (y_1^{(k+1)}, \dots, y_{c_{k+3}}^{(k+1)}) \alpha \tau_{k+2}^*(h)$, we have $(u_{k+1}(C) y_1^{(k+1)}, \dots, u_{k+1}(C) y_{c_{k+3}}^{(k+1)}) = (y_{c_{k+3}+1}^{(k+1)}, \dots, y_{c_{k+3}+c_{k+2}}^{(k+1)}) S(\alpha \tau_{k+2}^*(h))^{-1}$. We put $u_{k+2}(C) := S(\alpha \tau_{k+2}^*(h))^{-1}$. Then we have $S = u_{k+2}(C)(\alpha \tau_{k+2}^*(h))$.

Therefore the generic isotropy subgroup at $w_{k+1} = (Y_1^{(k+1)}, \dots, Y_{c_{k+3}+c_{k+2}}^{(k+1)}) \in W_{k+1}$ of $(GL(c_{k+2} + c_{k+1}) \times H \times GL(c_{k+3} + c_{k+2}), \Lambda_1^* \otimes \rho^{[*]*} \otimes \Lambda_1, V(c_{k+2} + c_{k+1})^* \otimes V(m)^{[*]*} \otimes V(c_{k+3} + c_{k+2}))$ is $(GL(c_{k+2} + c_{k+1}) \times H \times GL(c_{k+3} + c_{k+2}))_{w_{k+1}} = \{(A_k(C, h, \alpha, \beta), h, A_{k+1}(C, h, \alpha, \beta)) \mid (C, (h, \alpha, \beta)) \in M(m, 1) \rtimes (H \times GL(1) \times GL(1))\}$ and there exists the isomorphism $\delta_{k+1} : G(H, \rho, m+1, 1, k) \to (GL(c_{k+2} + c_{k+1}) \times H \times GL(c_{k+3} + c_{k+2}))_{w_{k+1}}$ of groups such that $\delta_{k+1}((A_k(C, h, \alpha, \beta), h, {}^tA_{k-1}(C, h, \alpha, \beta)^{-1})) = (A_k(C, h, \alpha, \beta), h, A_{k+1}(C, h, \alpha, \beta))$ for $(C, (h, \alpha, \beta)) \in M(m, 1) \rtimes (H \times GL(1) \times GL(1))$. Thus we see that our assertion holds for $j = k+1$. Hence our assertion holds by induction on j. □

When the representation space $V(c_{j+1}n - c_j l) \otimes V(m)^{[*]} \otimes V(c_j n - c_{j-1} l)$ is identified with $V_j := M(c_{j+1}n - c_j l, m) \overbrace{\oplus \cdots \oplus}^{c_j n - c_{j-1} l} M(c_{j+1}n - c_j l, m)$ (resp. $W_j := M(c_j n - c_{j-1} l, m) \overbrace{\oplus \cdots \oplus}^{c_{j+1} n - c_j l} M(c_j n - c_{j-1} l, m)$) for $P(H, \rho, n, l, j)$ with $j \geq 1$, the representation $\Lambda_1 \otimes \rho^{[*]} \otimes \Lambda_1$ is given by $V_j \ni (X_1, \dots, X_{c_j n - c_{j-1} l}) \mapsto (SX_1 {}^t\rho^{[*]}(h), \dots, SX_{c_j n - c_{j-1} l} {}^t\rho^{[*]}(h)) {}^tT \in V_j$ (resp. $W_j \ni (Y_1, \dots, Y_{c_{j+1} n - c_j l}) \mapsto (TY_1 {}^t\rho^{[*]}(h), \dots, TY_{c_{j+1} n - c_j l} {}^t\rho^{[*]}(h)) {}^tS \in W_j$) for $(S, h, T) \in GL(c_{j+1}n - c_j l) \times H \times GL(c_j n - c_{j-1} l)$. Here $\rho^{[*]}(h) := \begin{cases} \rho(h) & (j \equiv 0 \pmod 2) \\ {}^t\rho(h)^{-1} & (j \equiv 1 \pmod 2) \end{cases}$. Note that $c_{j+1}n - c_j l = c_{j+2} l + c_{j+1}(n - ml)$ and $c_j n - c_{j-1} l = c_{j+1} l + c_j (n - ml)$.

PROPOSITION 4.4. *We define* $\tilde{x}_{c_{j+1}\alpha+\beta}^{(j)} \in M(c_{j+2} l, m)$ $(j \geq 1,\ 0 \leq \alpha \leq l-1,\ 1 \leq \beta \leq c_{j+1})$ *as* $(0_{(c_{j+2}\alpha, m)}, x_\beta^{(j)}, 0_{(c_{j+2}(l-\alpha-1), m)})'$, *where* $x_\beta^{(j)}$ $(1 \leq \beta \leq c_{j+1})$ *are the matrices which are defined in Proposition* 4.2. *We define* $\tilde{x}_{c_{j+1}l + c_j\gamma + \delta}^{(j)} \in M(c_{j+1}(n - ml), m)$ $(j \geq 1,\ 0 \leq \gamma \leq n - ml - 1,\ 1 \leq \delta \leq c_j)$ *as* $(0_{(c_{j+1}\gamma, m)}, x_{c_{j+1}+\delta}^{(j)}, 0_{(c_{j+1}(n-ml-\gamma-1), m)})'$, *where* $x_{c_{j+1}+\delta}^{(j)}$ $(1 \leq \delta \leq c_j)$ *are the matrices which are defined in*

Proposition 4.2. *We define* $\tilde{y}^{(j)}_{c_{j+2}\alpha+\beta} \in M(c_{j+1}l, m)$ $(j \geq 1,\ 0 \leq \alpha \leq l-1,\ 1 \leq \beta \leq c_{j+2})$ *as* $(0_{(c_{j+1}\alpha, m)}, y^{(j)}_\beta, 0_{(c_{j+1}(l-\alpha-1), m)})'$, *where* $y^{(j)}_\beta$ $(1 \leq \beta \leq c_{j+2})$ *are the matrices which are defined in Proposition* 4.2. *We define* $\tilde{y}^{(j)}_{c_{j+2}l+c_{j+1}\gamma+\delta} \in M(c_j(n-ml), m)$ $(j \geq 1,\ \ 0 \leq \gamma \leq n-ml-1,\ \ 1 \leq \delta \leq c_{j+1})$ *as* $(0_{(c_j\gamma, m)}, y^{(j)}_{c_{j+2}+\delta}, 0_{(c_j(n-ml-\gamma-1), m)})'$, *where* $y^{(j)}_{c_{j+2}+\delta}$ $(1 \leq \delta \leq c_{j+1})$ *are the matrices which are defined in Proposition* 4.2. *We define* $v_j = (X^{(j)}_1, \dots, X^{(j)}_{c_j n - c_{j-1} l}) \in V_j$ $(j \geq 1)$ *by* $X^{(j)}_s = \begin{pmatrix} \tilde{x}^{(j)}_s \\ 0 \end{pmatrix} \in M(c_{j+1}n - c_j l, m)$ *for* $1 \leq s \leq c_{j+1}l$, $X^{(j)}_t = \begin{pmatrix} 0 \\ \tilde{x}^{(j)}_t \end{pmatrix} \in M(c_{j+1}n - c_j l, m)$ *for* $c_{j+1}l + 1 \leq t \leq c_j n - c_{j-1}l$. *We defined* $w_j = (Y^{(j)}_1, \dots, Y^{(j)}_{c_{j+1}n - c_j l}) \in W_j$ $(j \geq 1)$ *by* $Y^{(j)}_s = \begin{pmatrix} \tilde{y}^{(j)}_s \\ 0 \end{pmatrix} \in M(c_j n - c_{j-1}l, m)$ *for* $1 \leq s \leq c_{j+2}l$, $Y^{(j)}_t = \begin{pmatrix} 0 \\ \tilde{y}^{(j)}_t \end{pmatrix} \in M(c_j n - c_{j-1}l, m)$ *for* $c_{j+2}l + 1 \leq t \leq c_{j+1}n - c_j l$. *Then* v_j *corresponds to* w_j *by the isomorphism of vector spaces of Lemma* 2.21.

Proof. By Lemma 2.21 and Proposition 4.2, we obtain our assertion. □

Proposition 4.5. *For* $C = (C_{pq})_{1 \leq p \leq l, 1 \leq q \leq n-ml} \in M(ml, n-ml)$ *where* $C_{pq} \in M(m, 1)$ $(1 \leq p \leq l,\ \ 1 \leq q \leq n-ml)$, *we define* $U_i(C)$ $(i \geq 1)$ *as* $(u_i(C_{pq}))_{1 \leq q \leq n-ml, 1 \leq p \leq l} \in M(c_i(n-ml), c_{i+1}l)$.

Put $A_i(C, h, A, B) := \begin{pmatrix} {}^tA^{-1} \otimes \tau_{i+1}(h) & -{}^tU_{i+1}(C)(B \otimes \tilde{\tau}_{i+1}(h)) \\ 0 & B \otimes \tilde{\tau}_{i+1}(h) \end{pmatrix}$ $(i \geq 0)$ *for* $(C, (h, A, B)) \in M(ml, n-ml) \rtimes (H \times GL(l) \times GL(n-ml))$.

Then $v_j \in V_j$ *is a generic point of* $P(H, \rho, n, l, j)$ *with* $j \geq 1$ *and the generic isotropy subgroup at* $v_j \in V_j$ *of* $P(H, \rho, n, l, j)$ *with* $j \geq 1$ *is given by* $G(H, \rho, n, l, j) := \{(A_j(C, h, A, B), h, {}^tA_{j-1}(C, h, A, B)^{-1}) \mid (C, (h, A, B)) \in M(ml, n-ml) \rtimes (H \times GL(l) \times GL(n-ml))\}$. *Furthermore, there exists the isomorphism* $\Psi_j : M(ml, n-ml) \rtimes (H \times GL(l) \times GL(n-ml)) \to G(H, \rho, n, l, j)$ *of groups such that* $\Psi_j((C, (h, A, B))) = (A_j(C, h, A, B), h, {}^tA_{j-1}(C, h, A, B)^{-1})$ *for* $(C, (h, A, B)) \in M(ml, n-ml) \rtimes (H \times GL(l) \times GL(n-ml))$.

Proof. Similarly to the proof of Proposition 4.3, we obtain our assertion. □

Second we calculate the explicit form of a generic isotropy subgroup of $Q(H, \rho, l, j)$ as a subgroup of $GL(c_{j+2}l) \times H \times GL(c_{j+1}l)$. When the representation space $V(c_{j+2}l) \otimes V(m)^{[*]} \otimes V(c_{j+1}l)$ is identified with $V_j := M(c_{j+2}l, m)$

$\overbrace{\oplus\cdots\oplus}^{c_{j+1}l} M(c_{j+2}l,m)$ (resp. $W_j := M(c_{j+1}l,m)\overbrace{\oplus\cdots\oplus}^{c_{j+2}l} M(c_{j+1}l,m)$) for $Q(H,\rho,l,j)$, the representation $\Lambda_1\otimes\rho^{[*]}\otimes\Lambda_1$ is given by $V_j \ni (X_1,\dots,X_{c_{j+1}l}) \mapsto (SX_1{}^t\rho^{[*]}(h),\dots,SX_{c_{j+1}l}{}^t\rho^{[*]}(h)){}^tT \in V_j$ (resp. $W_j \ni (Y_1,\dots,Y_{c_{j+2}l}) \mapsto (TY_1{}^t\rho^{[*]}(h),\dots,TY_{c_{j+2}l}{}^t\rho^{[*]}(h)){}^tS \in W_j$) for $(S,h,T)\in GL(c_{j+2}l)\times H\times GL(c_{j+1}l)$. Here $\rho^{[*]}(h) := \begin{cases}\rho(h) & (j\equiv 0 \pmod 2)\\ {}^t\rho(h)^{-1} & (j\equiv 1 \pmod 2)\end{cases}$.

PROPOSITION 4.6. *We define* $v_0 = (X_1^{(0)},\dots,X_l^{(0)})\in V_0$ *by* $(X_1^{(0)}|\cdots|X_l^{(0)}) = I_{ml}$. *Then* $v_0\in V_0$ *is a generic point of* $Q(H,\rho,l,0)$ *and the generic isotropy subgroup at* v_0 *is* $\{({}^tA^{-1}\otimes{}^t\rho(h)^{-1},h,A)\,|\,h\in H, A\in GL(l)\}\cong H\times GL(l)$.

PROOF. By direct calculation, we obtain our assertion. □

PROPOSITION 4.7. *We put* $v_j := (\tilde{x}_1^{(j)},\dots,\tilde{x}_{c_{j+1}l}^{(j)})\in V_j$ $(j\ge 1)$ *and* $w_j := (\tilde{y}_1^{(j)},\dots,\tilde{y}_{c_{j+2}l}^{(j)})\in W_j$ $(j\ge 1)$, *where* $\tilde{x}_s^{(j)}\in M(c_{j+2}l,m)$ $(1\le s\le c_{j+1}l)$ *and* $\tilde{y}_t^{(j)}\in M(c_{j+1}l,m)$ $(1\le t\le c_{j+2}l)$ *are the matrices which are defined in Proposition* 4.4. *Then* v_j *corresponds to* w_j *by the isomorphism of vector spaces of Lemma* 2.21, $v_j\in V_j$ *is a generic point of* $Q(H,\rho,l,j)$ *with* $j\ge 1$ *and the generic isotropy subgroup at* $v_j\in V_j$ *of* $Q(H,\rho,l,j)$ *with* $j\ge 1$ *is* $\{({}^tA^{-1}\otimes\tau_{j+1}(h),h,A\otimes{}^t\tau_j(h)^{-1})\,|\,h\in H, A\in GL(l)\}\cong H\times GL(l)$.

PROOF. Similarly to the proof of Proposition 4.3, we obtain our assertion. □

Third we calculate the explicit form of a generic isotropy subgroup of $R(H,\rho,n-ml,j)$ as a subgroup of $GL(c_{j+1}(n-ml))\times H\times GL(c_j(n-ml))$. When the representation space $V(c_{j+1}(n-ml))\otimes V(m)^{[*]}\otimes V(c_j(n-ml))$ is identified with $V_j := M(c_{j+1}(n-ml),m)\overbrace{\oplus\cdots\oplus}^{c_j(n-ml)} M(c_{j+1}(n-ml),m)$ (resp. $W_j := M(c_j(n-ml),m)\overbrace{\oplus\cdots\oplus}^{c_{j+1}(n-ml)} M(c_j(n-ml),m)$) for $R(H,\rho,n-ml,j)$, the representation $\Lambda_1\otimes\rho^{[*]}\otimes\Lambda_1$ is given by $V_j\ni(X_1,\dots,X_{c_j(n-ml)})\mapsto(SX_1{}^t\rho^{[*]}(h),\dots,SX_{c_j(n-ml)}{}^t\rho^{[*]}(h)){}^tT\in V_j$ (resp. $W_j\ni(Y_1,\dots,Y_{c_{j+1}(n-ml)})\mapsto(TY_1{}^t\rho^{[*]}(h),\dots,TY_{c_{j+1}(n-ml)}{}^t\rho^{[*]}(h)){}^tS\in W_j$) for $(S,h,T)\in GL(c_{j+1}(n-ml))\times H\times GL(c_j(n-ml))$. Here $\rho^{[*]}(h) := \begin{cases}\rho(h) & (j\equiv 0 \pmod 2)\\ {}^t\rho(h)^{-1} & (j\equiv 1 \pmod 2)\end{cases}$.

PROPOSITION 4.8. *We put* $v_j := (\tilde{x}_{c_{j+1}l+1}^{(j)},\dots,\tilde{x}_{c_jn-c_{j-1}l}^{(j)})\in V_j$ $(j\ge 1)$ *and* $w_j := (\tilde{y}_{c_{j+2}l+1}^{(j)},\dots,\tilde{y}_{c_{j+1}n-c_jl}^{(j)})\in W_j$ $(j\ge 1)$, *where* $\tilde{x}_s^{(j)}\in M(c_{j+1}(n-ml),m)$ $(c_{j+1}l+1\le$

$s \leq c_j n - c_{j-1} l)$ and $\tilde{y}_t^{(j)} \in M(c_j(n-ml), m)$ $(c_{j+2} l + 1 \leq t \leq c_{j+1} n - c_j l)$ are the matrices which are defined in Proposition 4.4. Then v_j corresponds to w_j by the isomorphism of vector spaces of Lemma 2.21, $v_j \in V_j$ is a generic point of $R(H, \rho, n-ml, j)$ and the generic isotropy subgroup at $v_j \in V_j$ of $R(H, \rho, n-ml, j)$ is $\{(B \otimes \tilde{\tau}_{j+1}(h), h, {}^tB^{-1} \otimes {}^t\tilde{\tau}_j(h)^{-1}) \mid h \in H, B \in GL(n-ml)\} \cong H \times GL(n-ml)$.

Proof. Similarly to the proof of Proposition 4.3, we obtain our assertion. □

5. Some Results about a Classification of all Reductive PVs of the Form $T(H, \rho, \sigma, n, l, j)^{(*)}$ with $j \geq 1$

For the triplet $T(H, \rho, \sigma, n, l, j)^*$ with $j \geq 1$ (resp. $T(H, \rho, \sigma, n, l, j)$ with $j \geq 1$), we define a triplet $A(H, \rho, \sigma, l, j)$ (resp. $B(H, \rho, \sigma, n-ml, j)$) as $(H \times GL(c_{j+1} l) \times GL(c_{j+2} l), \rho \otimes \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 \otimes \Lambda_1^*, V(m) \otimes V(c_{j+1} l) \otimes V(c_{j+2} l) + V(r) \otimes V(c_{j+2} l)^*)$ (resp. $(H \times GL(c_j(n-ml)) \times GL(c_{j+1}(n-ml)), \rho \otimes \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 \otimes \Lambda_1, V(m) \otimes V(c_j(n-ml)) \otimes V(c_{j+1}(n-ml)) + V(r) \otimes V(c_{j+1}(n-ml))))$.

Necessary and sufficient conditions for the prehomogeneity of two triplets $A(H, \rho, \sigma, l, j)$, $B(H, \rho, \sigma, n-ml, j)$ are given in [Ka1]. Especially for the triplet $B(H, \rho, \sigma, n-ml, 1)$, we give a necessary and sufficient condition for its prehomogeneity which is different from that of [Ka1].

Theorem 5.1 (Kasai). *The triplet $A(H, \rho, \sigma, l, j)$ with $(m_1, m_2) = (2, 1)$ (resp. $B(H, \rho, \sigma, n-ml, j)$ with $j \geq 2$ and $(m_1, m_2) = (2, 1)$), namely, $(SL(2) \times GL((j+1)l) \times GL((j+2)l) \times H_3, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1^* \otimes \sigma_3, V(2) \otimes V((j+1)l) \otimes V((j+2)l) + V((j+2)l)^* \otimes V(r_3))$ (resp. $(SL(2) \times GL(j(n-ml)) \times GL((j+1)(n-ml)) \times H_3, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \sigma_3, V(2) \otimes V(j(n-ml)) \otimes V((j+1)(n-ml)) + V((j+1)(n-ml)) \otimes V(r_3))$) is a PV if and only if $T_1 := (GL(l) \times SL(2) \times H_3, \Lambda_1 \otimes (j+1)\Lambda_1 \otimes \sigma_3, V(l) \otimes V(j+2) \otimes V(r_3))$ (resp. $T_2 := (GL(n-ml) \times SL(2) \times H_3, \Lambda_1 \otimes j\Lambda_1 \otimes \sigma_3, V(n-ml) \otimes V(j+1) \otimes V(r_3))$) is an irreducible PV. Assume that T_1 (resp. T_2) is an irreducible PV. We denote by N_1 (resp. N_2) the number of basic relative invariants of T_1 (resp. T_2). Then $A(H, \rho, \sigma, l, j)$ with $(m_1, m_2) = (2, 1)$ (resp. $B(H, \rho, \sigma, n-ml, j)$ with $j \geq 2$ and $(m_1, m_2) = (2, 1)$) is regular if and only if T_1 (resp. T_2) is regular, and the number of basic relative invariants of $A(H, \rho, \sigma, l, j)$ with $(m_1, m_2) = (2, 1)$ (resp. $B(H, \rho, \sigma, n-ml, j)$ with $j \geq 2$ and $(m_1, m_2) = (2, 1)$) is $N_1 + 1$ (resp. $N_2 + 1$).*

Proof. For the proof, see [Ka1, Theorem 3.22] and Correction 1.2. □

THEOREM 5.2 (Kasai). *The triplet* $A(H,\rho,\sigma,l,j)$ *with* $(m_1,m_2) \neq (2,1)$ (*resp.* $B(H,\rho,\sigma,n-ml,j)$ *with* $j \geq 2$ *and* $(m_1,m_2) \neq (2,1)$) *is a PV if and only if the triplet* $(H_1 \times GL(c_{j+1}) \times GL(c_{j+2}) \times H_2 \times SL(l) \times H_3, \rho_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \rho_2 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1^* \otimes \sigma_2 \otimes \Lambda_1 \otimes \sigma_3, V(m_1) \otimes V(c_{j+1}) \otimes V(c_{j+2}) \otimes V(m_2) + V(c_{j+2})^* \otimes V(r_2) \otimes V(l) \otimes V(r_3))$ (*resp.* $(H_1 \times GL(c_j) \times GL(c_{j+1}) \times H_2 \times SL(n-ml) \times H_3, \rho_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \rho_2 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1 \otimes \sigma_2 \otimes \Lambda_1 \otimes \sigma_3, V(m_1) \otimes V(c_j) \otimes V(c_{j+1}) \otimes V(m_2) + V(c_{j+1}) \otimes V(r_2) \otimes V(n-ml) \otimes V(r_3))$) *is castling equivalent to a PV of the form* $(H_1 \times GL(c_{j+1}) \times GL(c_{j+2}) \times H_2 \times SL(N) \times K, \rho_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \rho_2 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1^{(*)} \otimes \sigma_2^{(*)} \otimes \Lambda_1 \otimes \tau, V(m_1) \otimes V(c_{j+1}) \otimes V(c_{j+2}) \otimes V(m_2) + V(c_{j+2})^{(*)} \otimes V(r_2)^{(*)} \otimes V(N) \otimes V(k))$ *with* $N \geq c_{j+2}r_2k$ (*resp. a PV of the form* $(H_1 \times GL(c_j) \times GL(c_{j+1}) \times H_2 \times SL(N') \times K', \rho_1 \otimes \Lambda_1 \otimes \Lambda_1 \otimes \rho_2 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \Lambda_1^{(*)} \otimes \sigma_2^{(*)} \otimes \Lambda_1 \otimes \tau', V(m_1) \otimes V(c_j) \otimes V(c_{j+1}) \otimes V(m_2) + V(c_{j+1})^{(*)} \otimes V(r_2)^{(*)} \otimes V(N') \otimes V(k'))$ *with* $N' \geq c_{j+1}r_2k'$). *Assume that* $A(H,\rho,\sigma,l,j)$ *with* $(m_1,m_2) \neq (2,1)$ (*resp.* $B(H,\rho,\sigma,n-ml,j)$ *with* $j \geq 2$ *and* $(m_1,m_2) \neq (2,1)$) *is a PV. If* $N = c_{j+2}r_2k$ (*resp.* $N' = c_{j+1}r_2k'$), *then* $A(H,\rho,\sigma,l,j)$ *with* $(m_1,m_2) \neq (2,1)$ (*resp.* $B(H,\rho,\sigma,n-ml,j)$ *with* $j \geq 2$ *and* $(m_1,m_2) \neq (2,1)$) *is a regular PV with* 2 *basic relative invariants, and if* $N > c_{j+2}r_2k$ (*resp.* $N' > c_{j+1}r_2k'$), *then* $A(H,\rho,\sigma,l,j)$ *with* $(m_1,m_2) \neq (2,1)$ (*resp.* $B(H,\rho,\sigma,n-ml,j)$ *with* $j \geq 2$ *and* $(m_1,m_2) \neq (2,1)$) *is a non-regular PV with* 1 *basic relative invariant.*

PROOF. For the proof, see Theorem 3.23 in [Ka1]. □

THEOREM 5.3. *The triplet* $B(H,\rho,\sigma,n-ml,1)$ *is a PV if and only if it satisfies one of the following conditions*:

(1) $H_2 = \{1\}$ *and the triplet* $(H_1 \times H_3 \times GL(n-ml), \rho_1^* \otimes \sigma_3 \otimes \Lambda_1, V(m_1)^* \otimes V(r_3) \otimes V(n-ml))$ *is a regular irreducible PV. In this case,* $B(H,\rho,\sigma,n-ml,1)$ *is a regular PV with* 2 *basic relative invariants.*

(2) $H_2 \neq \{1\}$ *and the triplet* $(H_1 \times H_2 \times H_3 \times GL(n-ml), \rho_1^* \otimes (\rho_2^* \otimes \sigma_2) \otimes \sigma_3 \otimes \Lambda_1, V(m_1)^* \otimes (V(m_2)^* \otimes V(r_2)) \otimes V(r_3) \otimes V(n-ml))$ *is castling equivalent to a regular trivial PV of the form* $(H_2 \times G \times GL(N), (\rho_2^* \otimes \sigma_2) \otimes \tau \otimes \Lambda_1, (V(m_2)^* \otimes V(r_2)) \otimes V(k) \otimes V(N))$ *with* $N = m_2r_2k$. *In this case,* $B(H,\rho,\sigma,n-ml,1)$ *is a regular PV with* 2 *basic relative invariants.*

(3) $H_2 = \{1\}$ *and the triplet* $T := (H_1 \times H_3 \times GL(n-ml), \rho_1^* \otimes \sigma_3 \otimes \Lambda_1, V(m_1)^* \otimes V(r_3) \otimes V(n-ml))$ *is a non-regular irreducible PV. In this case,* $B(H,\rho,\sigma,n-ml,1)$ *is a non-regular PV and the number of basic relative invariants of it is* $N+1$, *where* N *stands for the number of basic relative invariants of* T.

(4) *$H_2 \neq \{1\}$ and the triplet $(H_1 \times H_2 \times H_3 \times GL(n-ml), \rho_1^* \otimes (\rho_2^* \otimes \sigma_2) \otimes \sigma_3 \otimes \Lambda_1, V(m_1)^* \otimes (V(m_2)^* \otimes V(r_2)) \otimes V(r_3) \otimes V(n-ml))$ is castling equivalent to a non-regular trivial PV of the form $(H_2 \times G \times GL(N), (\rho_2^* \otimes \sigma_2) \otimes \tau \otimes \Lambda_1, (V(m_2)^* \otimes V(r_2)) \otimes V(k) \otimes V(N))$ with $N > m_2 r_2 k$. In this case, $B(H, \rho, \sigma, n-ml, 1)$ is a non-regular PV with 1 basic relative invariant.*

Proof. Since a generic isotropy subgroup of $(H \times GL(n-ml) \times GL(m(n-ml)), \rho \otimes \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(n-ml) \otimes V(m(n-ml)))$ is $\{(h, A, \rho^*(h) \otimes {}^tA^{-1}) \mid h \in H, A \in GL(n-ml)\}$, by Proposition 2.4, we see that $B(H, \rho, \sigma, n-ml, 1)$ is a PV if and only if $(H \times GL(n-ml), (\rho^* \otimes \sigma) \otimes \Lambda_1, (V(m)^* \otimes V(r)) \otimes V(n-ml))$ is a PV. By Propositions 3.2 and 3.3, we obtain our assertion. □

Theorem 5.4. (1) *If the triplet $T(H, \rho, \sigma, n, l, j)^*$ with $j \geq 1$ is a PV, then the triplet $A(H, \rho, \sigma, l, j)$ is a PV. Furthermore, if the triplet $A(H, \rho, \sigma, l, j)$ is a PV, then the triplet $T(H, \rho, \sigma, n, l, j)^*$ with $j \geq 1$ and $n \geq ml + c_{j+1}r$ is a PV.*

(2) *If the triplet $T(H, \rho, \sigma, n, l, j)$ with $j \geq 1$ is a PV, then the triplet $B(H, \rho, \sigma, n-ml, j)$ is a PV. Furthermore, if the triplet $B(H, \rho, \sigma, n-ml, j)$ is a PV, then the triplet $T(H, \rho, \sigma, n, l, j)$ with $j \geq 1$ and $l \geq c_{j+2}r$ is a PV.*

(3) *The triplet $T(H, \rho, \sigma, n, l, j)^*$ with $j \geq 1$ and $l \geq c_{j+2}r$ is a non-regular PV and the number of basic relative invariants of it is 1 (resp. 0) if $l = c_{j+2}r$ (resp. $l > c_{j+2}r$).*

(4) *The triplet $T(H, \rho, \sigma, n, l, j)$ with $j \geq 1$ and $n \geq ml + c_{j+1}r$ is a non-regular PV and the number of basic relative invariants of it is 1 (resp. 0) if $n = ml + c_{j+1}r$ (resp. $n > ml + c_{j+1}r$).*

Proof. (1) Since H is semisimple, $T(H, \rho, \sigma, n, l, j)^*$ with $j \geq 1$ is isomorphic to $T(H, \rho^{[*]}, \sigma^{[*]}, n, l, j)^*$ where $(\rho^{[*]}, \sigma^{[*]}) := \begin{cases} (\rho, \sigma) & (j \equiv 0 \pmod 2) \\ (\rho^*, \sigma^*) & (j \equiv 1 \pmod 2) \end{cases}$.

By Proposition 4.5, the $(GL(c_{j+1}n - c_j l) \times H)$-part of a generic isotropy subgroup of $(GL(c_{j+1}n - c_j l) \times H \times GL(c_j n - c_{j-1} l), \Lambda_1 \otimes \rho^{[*]} \otimes \Lambda_1, V(c_{j+1}n - c_j l) \otimes V(m)^{[*]} \otimes V(c_j n - c_{j-1} l))$ is given by $G_j := \{g_j := (A_j(C, h, A, B), h) \mid (C, (h, A, B)) \in M(ml, n-ml) \rtimes (H \times GL(l) \times GL(n-ml))\}$.

By Proposition 2.4, $T(H, \rho^{[*]}, \sigma^{[*]}, n, l, j)^*$ is a PV if and only if $(G_j, (\Lambda_1^* \otimes \sigma^{[*]})|_{G_j}, M(c_{j+1}n - c_j l, r))$ is a PV. The action $(\Lambda_1^* \otimes \sigma^{[*]})|_{G_j}$ is given by

$$M(c_{j+1}n - c_j l, r) \ni \begin{pmatrix} X \\ Y \end{pmatrix} \mapsto (\Lambda_1^* \otimes \sigma^{[*]})(g_j)\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right)$$

$$= \begin{pmatrix} (A \otimes {}^t\tau_{j+1}(h)^{-1}) X\, {}^t\sigma^{[*]}(h) \\ U_{j+1}(C)(A \otimes {}^t\tau_{j+1}(h)^{-1}) X\, {}^t\sigma^{[*]}(h) + ({}^tB^{-1} \otimes {}^t\tilde{\tau}_{j+1}(h)^{-1}) Y\, {}^t\sigma^{[*]}(h) \end{pmatrix}$$

$$\in M(c_{j+1}n - c_j l, r) \quad \text{for } X \in M(c_{j+2}l, r),\ Y \in M(c_{j+1}(n - ml), r).$$

Here we put $K_j := \{({}^tA^{-1} \otimes \tau_{j+1}(h), h) \mid h \in H, A \in GL(l)\}$, $L_j := \{(B \otimes \tilde{\tau}_{j+1}(h), h) \mid h \in H, B \in GL(n - ml)\}$ and define a map $\varphi : M(c_{j+1}n - c_j l, r) \to M(c_{j+2}l, r)$ by $\varphi\left(\begin{pmatrix} X \\ Y \end{pmatrix}\right) = X$ for $X \in M(c_{j+2}l, r)$, $Y \in M(c_{j+1}(n - ml), r)$.

If $(G_j, (\Lambda_1^* \otimes \sigma^{[*]})|_{G_j}, M(c_{j+1}n - c_j l, r))$ is a PV, then by applying Lemma 2.3 to $\varphi : M(c_{j+1}n - c_j l, r) \to M(c_{j+2}l, r)$, we see that $(K_j, (\Lambda_1^* \otimes \sigma^{[*]})|_{K_j}, M(c_{j+2}l, r))$ is a PV. By Proposition 4.7, we see that $A(H, \rho, \sigma, l, j)$ is a PV if and only if $(K_j, (\Lambda_1^* \otimes \sigma^{[*]})|_{K_j}, M(c_{j+2}l, r))$ is a PV.

Note that if $n \geq ml + c_{j+1}r$, then $(L_j, (\Lambda_1^* \otimes \sigma^{[*]})|_{L_j}, M(c_{j+1}(n - ml), r))$ is a trivial PV. If $(K_j, (\Lambda_1^* \otimes \sigma^{[*]})|_{K_j}, M(c_{j+2}l, r))$ is a PV, then by applying Lemma 2.3 to $\varphi : M(c_{j+1}n - c_j l, r) \to M(c_{j+2}l, r)$, we see that $(G_j, (\Lambda_1^* \otimes \sigma^{[*]})|_{G_j}, M(c_{j+1}n - c_j l, r))$ with $n \geq ml + c_{j+1}r$ is a PV. Thus we obtain our assertion.

(2) By Proposition 4.8 and an argument similar to that in (1), we obtain our assertion.

(3) If $j \geq 1$ and $l \geq c_{j+2}r$, then by Lemma 4.10 in [KKTI], a triplet $(GL(c_j n - c_{j-1}l) \times GL(c_{j+1}n - c_j l), \Lambda_1 \otimes \Lambda_1 \overbrace{+ \cdots +}^{m} \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1^* \overbrace{+ \cdots +}^{r} 1 \otimes \Lambda_1^*, V(c_j n - c_{j-1}l) \otimes V(c_{j+1}n - c_j l) \overbrace{+ \cdots +}^{m} V(c_j n - c_{j-1}l) \otimes V(c_{j+1}n - c_j l) + V(c_{j+1}n - c_j l)^* \overbrace{+ \cdots +}^{r} V(c_{j+1}n - c_j l)^*)$ is a non-regular PV and the number of basic relative invariants of it is 1 (resp. 0) if $l = c_{j+2}r$ (resp. $l > c_{j+2}r$). By Proposition 4.5, we obtain our assertion.

(4) Assume that $j \geq 1$ and $n \geq ml + c_{j+1}r$. First we shall consider the case where $j = 1$. We obtain $P := (H \times GL(n) \times GL(r + l), \rho \otimes \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 \otimes \Lambda_1, V(m) \otimes V(n) \otimes V(r + l) + V(r) \otimes V(r + l))$ by applying a castling transformation to $T(H, \rho, \sigma, n, l, 1)$. Since $n \geq ml + mr$ and $r + l > r$, P is a non-regular PV and the number of basic relative invariants of P is 1 (resp. 0) if $n = ml + mr$ (resp. $n > ml + mr$). Thus we see that our assertion holds for $j = 1$.

Second we shall consider the case where $j = 2$. We obtain $P_1 := (H \times GL(nm - l) \times GL(n + r), \rho \otimes \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 \otimes \Lambda_1, V(m) \otimes V(nm - l) \otimes V(n + r) + V(r) \otimes V(n + r))$ by applying a castling transformation to $T(H, \rho, \sigma, n, l, 2)$.

Furthermore, we obtain $P_2 := (H \times GL(mr+l) \times GL(n+r), \rho^* \otimes \Lambda_1 \otimes \Lambda_1^* + \sigma \otimes 1 \otimes \Lambda_1, V(m)^* \otimes V(mr+l) \otimes V(n+r)^* + V(r) \otimes V(n+r))$ by applying a castling transformation to P_1. Note that $n+r \geq m(mr+l)$. By Proposition 2.4 and Theorem 2.11, we see that P_2 is a non-regular PV and the number of basic relative invariants of P_2 is 1 (resp. 0) if $n+r = m(mr+l)$ (resp. $n+r > m(mr+l)$). Thus we see that our assertion holds for $j=2$.

Third we shall consider the case where $j \geq 3$. We obtain $P_3 := (H \times GL(c_j n - c_{j-1} l) \times GL(c_{j-1} n - c_{j-2} l + r), \rho \otimes \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 \otimes \Lambda_1, V(m) \otimes V(c_j n - c_{j-1} l) \otimes V(c_{j-1} n - c_{j-2} l + r) + V(r) \otimes V(c_{j-1} n - c_{j-2} l + r))$ by applying a castling transformation to $T(H, \rho, \sigma, n, l, j)$. Furthermore, we obtain $P_4 := (H \times GL(c_{j-2} n - c_{j-3} l + mr) \times GL(c_{j-1} n - c_{j-2} l + r), \rho^* \otimes \Lambda_1 \otimes \Lambda_1^* + \sigma \otimes 1 \otimes \Lambda_1, V(m)^* \otimes V(c_{j-2} n - c_{j-3} l + mr) \otimes V(c_{j-1} n - c_{j-2} l + r)^* + V(r) \otimes V(c_{j-1} n - c_{j-2} l + r))$ by applying a castling transformation to P_3. Since $n \geq ml + c_{j+1} r$, we have $n + c_{j-1} r \geq m(l + c_j r)$. We put $\tilde{n} := n + c_{j-1} r$ and $\tilde{l} := l + c_j r$. Then we see that $\tilde{n} \geq m\tilde{l}$. Since $\begin{pmatrix} c_{i+1} & c_{i+3} \\ c_{i+2} & c_{i+4} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & m \end{pmatrix} \begin{pmatrix} c_i & c_{i+2} \\ c_{i+1} & c_{i+3} \end{pmatrix}$ $(i \geq -1)$ and $\det \begin{pmatrix} c_{-1} & c_1 \\ c_0 & c_2 \end{pmatrix} = -m$, we have $c_i c_{i+3} - c_{i+1} c_{i+2} = -m$ $(i \geq -1)$. Since $\begin{pmatrix} c_{i+1} & c_{i+2} \\ c_{i+2} & c_{i+3} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & m \end{pmatrix} \cdot \begin{pmatrix} c_i & c_{i+1} \\ c_{i+1} & c_{i+2} \end{pmatrix}$ $(i \geq -1)$ and $\det \begin{pmatrix} c_{-1} & c_0 \\ c_0 & c_1 \end{pmatrix} = -1$, we have $c_i c_{i+2} - (c_{i+1})^2 = -1$ $(i \geq -1)$. Then we see that $c_{j-2} n - c_{j-3} l + mr = c_{j-2} \tilde{n} - c_{j-3} \tilde{l}$ and $c_{j-1} n - c_{j-2} l + r = c_{j-1} \tilde{n} - c_{j-2} \tilde{l}$. P_4 is isomorphic to $P_5 := (H \times GL(c_{j-2} \tilde{n} - c_{j-3} \tilde{l}) \times GL(c_{j-1} \tilde{n} - c_{j-2} \tilde{l}), \rho^* \otimes \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 \otimes \Lambda_1^*, V(m)^* \otimes V(c_{j-2} \tilde{n} - c_{j-3} \tilde{l}) \otimes V(c_{j-1} \tilde{n} - c_{j-2} \tilde{l}) + V(r) \otimes V(c_{j-1} \tilde{n} - c_{j-2} \tilde{l})^*)$. First we shall consider the case where $n = ml + c_{j+1} r$. Since $\tilde{n} = m\tilde{l}$, we have $c_{j-2} \tilde{n} - c_{j-3} \tilde{l} = c_{j-1} \tilde{l}$ and $c_{j-1} \tilde{n} - c_{j-2} \tilde{l} = c_j \tilde{l}$. Note that $\tilde{l} > c_j r$. By Theorems 5.1 and 5.2, P_5 is a non-regular PV with 1 basic relative invariant. Next we shall consider the case where $n > ml + c_{j+1} r$. Since $\tilde{n} > m\tilde{l}$ and $c_{j-1} \tilde{n} - c_{j-2} \tilde{l} > \tilde{l} > c_j r > r$, by (3) of Theorem 5.4, P_5 is a non-regular PV and the number of basic relative invariants of P_5 is 0. Thus we see that our assertion holds for $j \geq 3$. □

6. A Classification of all 3-Simple PVs of the Form $T(H, \rho, \sigma, n, l, j)^{(*)}$ with $j \geq 1$

In this section, we shall classify all reductive PVs of the form $T(H, \rho, \sigma, n, l, j)^{(*)}$ with $j \geq 1$ where H is a simple linear algebraic group.

THEOREM 6.1. *Assume that H is a simple linear algebraic group and $l < c_{j+2} r$. Then the triplet $T(H, \rho, \sigma, n, l, j)^*$ with $j \geq 1$ is a PV if and only if it is one of the following PVs. Here we denote by N the number of basic relative invariants.*

(I) *Regular PVs*

(1) $(SL(2) \times GL(n) \times GL(2n-1), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(2) \otimes V(n) \otimes V(2n-1) + V(2n-1)^*)$ *with* $n \geq 3$. $N = 1$.

(2) $(SL(2) \times GL(5) \times GL(7), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(2) \otimes V(5) \otimes V(7) + V(7)^*)$. $N = 2$.

(II) *Non-regular PVs*

(1) $(SL(2) \times GL(n) \times GL(2n-2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(2) \otimes V(n) \otimes V(2n-2) + V(2n-2)^*)$ *with* $n \geq 5$. $N = 1$.

(2) $(SL(2) \times GL(2n-1) \times GL(3n-2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(2) \otimes V(2n-1) \otimes V(3n-2) + V(3n-2)^*)$ *with* $n \geq 4$. $N = 1$.

(3) $(SL(2) \times GL(2n-3) \times GL(3n-6), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(2) \otimes V(2n-3) \otimes V(3n-6) + V(3n-6)^*)$ *with* $n \geq 7$. $N = 1$.

Proof. We assume that H is a simple linear algebraic group, $l < c_{j+2}r$ and the triplet $T(H, \rho, \sigma, n, l, j)^*$ with $j \geq 1$ is a PV. By Correction 1.2 and Theorems 5.1, 5.2 and 5.4, we see that $(m_1, m_2) = (2, 1)$; and (j, l) is one of $(1, 1)$, $(1, 2)$, $(2, 1)$ and $(2, 3)$. Then the triplet $T(H, \rho, \sigma, n, l, j)^*$ with $j \geq 1$ is of the following form: $T := (SL(2) \times GL(jn - (j-1)l) \times GL((j+1)n - jl), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(2) \otimes V(jn - (j-1)l) \otimes V((j+1)n - jl) + V((j+1)n - jl)^*)$ with $n > 2l$ where (j, l) is one of $(1, 1)$, $(1, 2)$, $(2, 1)$ and $(2, 3)$. By (1) of Theorem 5.4, we see that the triplet T with $n \geq 2l + j + 1$ is a PV. By Propositions 2.4, 4.3 and 4.5, we see that the triplet T where (n, j, l) is one of $(3, 1, 1)$, $(5, 1, 2)$, $(3, 2, 1)$, $(4, 2, 1)$, $(7, 2, 3)$ and $(8, 2, 3)$ is a PV. Thus we obtain our assertion. □

Theorem 6.2. *Assume that $H = H_1$ is a simple linear algebraic group. Then, for the triplet $T(H, \rho, \sigma, n, l, j)$ with $j \geq 1$, the castling transform at $GL(c_{j+1}n - c_j l)$ has smaller dimension.*

Proof. Note that $m(c_j n - c_{j-1} l) + 1 - (c_{j+1} n - c_j l) = c_{j-1} n - c_{j-2} l + 1 > 0$. By Lemma 2.22, we see that $c_{j+1} n - c_j l - (c_{j-1} n - c_{j-2} l + 1) > n - 1 > 0$. Thus we obtain our assertion. □

Theorem 6.2 means that in the classification of reductive PVs with two irreducible components, it is not necessary to consider the triplet $T(H, \rho, \sigma, n, l, j)$ with $j \geq 1$ where $H = H_1$ is a simple linear algebraic group.

Theorem 6.3. *Assume that $H = H_2$ is a simple linear algebraic group. Then the triplet $T(H, \rho, \sigma, n, l, j)$ with $j \geq 1$ is a PV if and only if $n \geq ml + c_{j+1} r$.*

Proof. By Theorems 5.2, 5.3 and 5.4, we obtain our assertion. □

Theorem 6.4. *Assume that $H = H_2$ is a simple linear algebraic group and the triplet $T(H, \rho, \sigma, n, l, j)$ with $j \geq 1$ is a PV. Then, for $T(H, \rho, \sigma, n, l, j)$ with $j \geq 1$, the castling transform at $GL(c_{j+1}n - c_j l)$ has smaller dimension.*

Proof. Note that $m(c_j n - c_{j-1} l) + r - (c_{j+1} n - c_j l) = c_{j-1} n - c_{j-2} l + r > 0$. By Lemma 2.22 and Theorem 6.3, we see that $c_{j+1} n - c_j l - (c_{j-1} n - c_{j-2} l + r) > n - r > 0$. Thus we obtain our assertion. □

7. List

Any 3-simple PV with two irreducible components is castling equivalent to either a simple PV with two irreducible components or a 2-simple PV with two irreducible components or a reduced 3-simple PV with two irreducible components. Since all simple PVs and 2-simple PVs are completely classified (See [K2], [KKIY] and [KKTI]), we can complete the classification of 3-simple PVs with two irreducible components by giving the complete list of reduced 3-simple PVs with two irreducible components.

Definition 7.1. *For two triplets $T_i := (G_i, \rho_i, V_i)$ $(i = 1, 2)$, we define the direct sum $T_1 \oplus T_2$ of T_1 and T_2 as $(G_1 \times G_2, \rho_1 \otimes 1 + 1 \otimes \rho_2, V_1 \oplus V_2)$. A triplet is said to be indecomposable if it is not the direct sum of some pair of triplets (cf. Definition 1.5 in [Ka1]).*

Definition 7.2. *Assume that a triplet (G, ρ, V) is a PV. Let $\sigma : G \to GL(W)$ be a rational representation. For a positive integer n satisfying $n \geq \dim W$, the triplet $(G \times GL(n), \rho \otimes 1 + \sigma \otimes \Lambda_1, V + W \otimes V(n))$ is always a PV. We call such a triplet a PV of trivial type (cf. Definition 1.6 in [Ka1]).*

Let G_{ss} be a connected semisimple linear algebraic group and let ρ_i $(i = 1, 2)$ be its irreducible rational representation on the finite dimensional vector space V_i. We simply write $(GL(1)^2 \times G_{ss}, \rho_1 + \rho_2, V_1 + V_2)$ instead of a triplet $(GL(1)^2 \times G_{ss}, \Lambda_1 \otimes 1 \otimes \rho_1 + 1 \otimes \Lambda_1 \otimes \rho_2, V_1 \oplus V_2)$.

In this section, we shall give the complete list of indecomposable reduced 3-simple PVs with two irreducible components which are neither trivial PVs nor PVs of trivial type. Here we denote by N the number of basic relative invariants.

(I) Regular *PVs*

(1) $(GL(1)^2 \times SL(4) \times SL(2) \times SL(2), \Lambda_2 \otimes \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1, V(6) \otimes V(2) + V(4) \otimes V(2))$. $N = 2$.

(2) $(GL(1)^2 \times SL(4) \times SL(3) \times SL(2), \Lambda_2 \otimes \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1, V(6) \otimes V(3) + V(4) \otimes V(2))$. $N = 2$.

(3) $(GL(1)^2 \times SL(5) \times SL(2) \times SL(2), \Lambda_2 \otimes \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1, V(10) \otimes V(2) + V(5) \otimes V(2))$. $N = 1$.

(4) $(GL(1)^2 \times SL(5) \times SL(9) \times SL(2), \Lambda_2 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes \Lambda_1, V(10) \otimes V(9) + V(9)^* \otimes V(2))$. $N = 2$.

(5) $(GL(1)^2 \times SL(6) \times SL(2) \times SL(3), \Lambda_2 \otimes \Lambda_1 \otimes 1 + \Lambda_1^{(*)} \otimes 1 \otimes \Lambda_1, V(15) \otimes V(2) + V(6)^{(*)} \otimes V(3))$. $N = 2$.

(6) $(GL(1)^2 \times SL(7) \times SL(2) \times SL(2), \Lambda_2 \otimes \Lambda_1 \otimes 1 + \Lambda_1^* \otimes 1 \otimes \Lambda_1, V(21) \otimes V(2) + V(7)^* \otimes V(2))$. $N = 2$.

(7) $(GL(1)^2 \times SL(2) \times SL(3) \times SL(3), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^{(*)}, V(2) \otimes V(3) \otimes V(3) + V(3)^{(*)})$. $N = 2$.

(8) $(GL(1)^2 \times Sp(n) \times SL(2m) \times SL(2l), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m) + V(2m) \otimes V(2l))$ with $n > m \geq 2l \geq 2$ and $n - 2m + l \geq 0$. $N = 2$.

(9) $(GL(1)^2 \times Sp(n) \times SL(2m) \times SL(2l), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes \Lambda_1, V(2n) \otimes V(2m) + V(2m)^* \otimes V(2l))$ with $n > m \geq 2l \geq 2$. $N = 2$.

(10) $(GL(1)^2 \times SO(n) \times SL(m) \times SL(l), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(m) + V(m) \otimes V(l))$ with $n > m \geq 2l \geq 4$ and $n - 2m + l \geq 0$. $N = 2$.

(11) $(GL(1)^2 \times SO(n) \times SL(m) \times SL(l), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes \Lambda_1, V(n) \otimes V(m) + V(m)^* \otimes V(l))$ with $n > m \geq 2l \geq 4$. $N = 2$.

(12) $(GL(1)^2 \times \mathrm{Spin}(7) \times SL(7) \times SL(2)$, *the spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes \Lambda_1, V(8) \otimes V(7) + V(7)^* \otimes V(2))$. $N = 2$.

(13) $(GL(1)^2 \times \mathrm{Spin}(10) \times SL(15) \times SL(2)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes \Lambda_1, V(16) \otimes V(15) + V(15)^* \otimes V(2))$. $N = 2$.

(14) $(GL(1)^2 \times H_1 \times SL(2) \times H_2, \rho_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \tau \otimes \rho_2, V_1 \otimes V(2) + V \otimes V_2)$ where H_i $(i = 1, 2)$ are simple linear algebraic groups, $\rho_i : H_i \to GL(V_i)$ $(i = 1, 2)$ and $\tau : SL(2) \to GL(V)$ are nontrivial irreducible rational representations, (H_1, ρ_1, V_1) is one of $(Sp(n), \Lambda_1, V(2n))$ $(n \geq 2)$ and $(\mathrm{Spin}(10)$, *a half-spin rep.*, $V(16))$; and $(GL(1) \times H_2 \times SL(2), \Lambda_1 \otimes \rho_2 \otimes \tau, V_2 \otimes V)$ is a regular non-trivial 2-simple irreducible *PV* with $(H_2, \rho_2, \tau) \neq (SL(2), \Lambda_1, 2\Lambda_1)$, $(SL(3), \Lambda_1, 3\Lambda_1)$. $N = 2$.

(15) $(GL(1)^2 \times H_1 \times H_2 \times SL(n), \rho_1 \otimes \rho_2 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V_1 \otimes V_2 \otimes V(n) + V(n)^*)$ where H_i $(i = 1, 2)$ are simple linear algebraic groups, $\rho_i : H_i \to GL(V_i)$ $(i = 1, 2)$ are nontrivial irreducible rational representations, $n = \deg(\rho_1 \otimes \rho_2)$ and $(GL(1) \times H_1 \times H_2, \Lambda_1 \otimes \rho_1 \otimes \rho_2, V_1 \otimes V_2)$ is a regular 2-simple irreducible PV. $N = 2$.

(16) $(GL(1)^2 \times H_1 \times SL(n) \times H_2, \rho_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes \rho_2, V_1 \otimes V(n) + V(n)^* \otimes V_2)$ where H_i $(i = 1, 2)$ are simple linear algebraic groups, $\rho_i : H_i \to GL(V_i)$ $(i = 1, 2)$ are nontrivial irreducible rational representations with $n = \deg \rho_1 > \deg \rho_2$ and $(GL(1) \times H_1 \times H_2, \Lambda_1 \otimes \rho_1 \otimes \rho_2, V_1 \otimes V_2)$ is a regular non-trivial reduced 2-simple irreducible PV. $N = 2$.

(17) $(GL(1)^2 \times SL(2) \times SL((j+1)L) \times SL((j+2)L), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(2) \otimes V((j+1)L) \otimes V((j+2)L) + V((j+2)L)^*)$ where j and L are positive integers and $(SL(2) \times GL(L), (j+1)\Lambda_1 \otimes \Lambda_1, V(j+2) \otimes V(L))$ is a regular irreducible PV. $N = 2$.

(18) $(GL(1)^2 \times H \times SL(c_{j+1}L) \times SL(c_{j+2}L), \rho \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(m) \otimes V(c_{j+1}L) \otimes V(c_{j+2}L) + V(c_{j+2}L)^*)$ where H is a simple linear algebraic group, ρ is its m-dimensional irreducible rational representation with $m \geq 3$, j is a positive integer and $L = c_{j+2}$. Here, for each m, a sequence $\{c_i\}_{i \geq -1}$ is defined by $c_{-1} = -1$, $c_0 = 0$ and $c_i = mc_{i-1} - c_{i-2}$ $(i \geq 1)$. $N = 2$.

(19) $(GL(1)^2 \times H \times SL(c_{j+1}L) \times SL(c_{j+2}L), \rho \otimes \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 \otimes \Lambda_1^*, V(m) \otimes V(c_{j+1}L) \otimes V(c_{j+2}L) + V(r) \otimes V(c_{j+2}L)^*)$ where H is a simple linear algebraic group, ρ (resp. σ) is its m-dimensional (resp. r-dimensional) irreducible rational representation with $m \geq 2$ (resp. $r \geq 2$), j is a positive integer and $L = c_{j+2}r$. Here, for each m, a sequence $\{c_i\}_{i \geq -1}$ is defined by $c_{-1} = -1$, $c_0 = 0$ and $c_i = mc_{i-1} - c_{i-2}$ $(i \geq 1)$. $N = 2$.

(20) $(GL(1)^2 \times SL(m_1) \times SL(m_1 m_2) \times G, \Lambda_1 \otimes \Lambda_1^* \otimes \rho + 1 \otimes \Lambda_1 \otimes \tau, V(m_1) \otimes V(m_1 m_2)^* \otimes V(m_2) + V(m_1 m_2) \otimes V(m))$ with $m_1 = m_2 m$ where G is a simple linear algebraic group, $\rho : G \to GL(V(m_2))$ is a nontrivial irreducible rational representation and $\tau : G \to GL(V(m))$ is a nontrivial irreducible rational representation. $N = 2$.

(21) $(GL(1)^2 \times SL(m) \times SL(n) \times G, \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes \tau, V(m) \otimes V(n) + V(n)^* \otimes V(m))$ with $n \geq 2m \geq 4$ where G is a simple linear algebraic group and $\tau : G \to GL(V(m))$ is a nontrivial irreducible rational representation. $N = 1$.

(22) $(GL(1)^2 \times SL(2) \times SL(n) \times SL(2n-1), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(2) \otimes V(n) \otimes V(2n-1) + V(2n-1)^*)$ with $n \geq 3$. $N = 1$.

(23) $(GL(1)^2 \times SL(2) \times SL(5) \times SL(7), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(2) \otimes V(5) \otimes V(7) + V(7)^*)$. $N = 2$.

(II) Non-regular PVs

(1) $(GL(1)^2 \times SL(2n+1) \times SL(2) \times SL(2), \Lambda_2 \otimes 1 \otimes 1 + \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(n(2n+1)) + V(2n+1) \otimes V(2) \otimes V(2))$ with $n \geq 2$. $N = 2$.

(2) $(GL(1)^2 \times SL(5) \times SL(2) \times SL(2), \Lambda_2 \otimes \Lambda_1 \otimes 1 + \Lambda_1^* \otimes 1 \otimes \Lambda_1, V(10) \otimes V(2) + V(5)^* \otimes V(2))$. $N = 1$.

(3) $(GL(1)^2 \times SL(2n+1) \times SL(2) \times H, \Lambda_2 \otimes \Lambda_1 \otimes 1 + 1 \otimes \tau \otimes \rho, V(n(2n+1)) \otimes V(2) + V \otimes W)$ with $n \geq 2$ where H is a simple linear algebraic group, $\rho : H \to GL(W)$ and $\tau : SL(2) \to GL(V)$ are nontrivial irreducible rational representations and $(GL(1) \times H \times SL(2), \Lambda_1 \otimes \rho \otimes \tau, W \otimes V)$ is a non-trivial 2-simple irreducible PV with $(H, \rho, \tau) \neq (SL(2), \Lambda_1, 2\Lambda_1)$, $(SL(3), \Lambda_1, 3\Lambda_1)$. N is equal to the number of basic relative invariants of $(GL(1) \times H \times SL(2), \Lambda_1 \otimes \rho \otimes \tau, W \otimes V)$.

(4) $(GL(1)^2 \times Sp(n) \times SL(m) \times SL(2), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes 2\Lambda_1, V(2n) \otimes V(m) + V(m) \otimes V(3))$ with $n+1 \geq m \geq 3$. $N = 2$.

(5) $(GL(1)^2 \times Sp(n) \times SL(m) \times SL(2), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes 2\Lambda_1, V(2n) \otimes V(m) + V(m)^* \otimes V(3))$ with $2n > m \geq 3$. $N = 1$ (resp. $N = 2$) if $m \geq 5$ is odd (resp. $m = 3$ or m is even).

(6) $(GL(1)^2 \times Sp(n) \times SL(2m) \times SL(2l+1), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m) + V(2m) \otimes V(2l+1))$ with $n > m \geq 2l+1 \geq 3$ and $2n - 4m + 2l + 1 \geq 0$. $N = 1$.

(7) $(GL(1)^2 \times Sp(n) \times SL(2m) \times SL(2l+1), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes \Lambda_1, V(2n) \otimes V(2m) + V(2m)^* \otimes V(2l+1))$ with $n > m \geq 2l+1 \geq 3$. $N = 1$.

(8) $(GL(1)^2 \times Sp(n) \times SL(2m+1) \times SL(l), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m+1) + V(2m+1) \otimes V(l))$ with $n > m \geq l \geq 2$ and $2n - 4m + l - 2 \geq 0$. $N = 1$ (resp. $N = 0$) if l is odd (resp. l is even).

(9) $(GL(1)^2 \times Sp(n) \times SL(2m+1) \times SL(l), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes \Lambda_1, V(2n) \otimes V(2m+1) + V(2m+1)^* \otimes V(l))$ with $n > m \geq l \geq 2$. $N = 1$ (resp. $N = 0$) if l is even (resp. l is odd).

(10) $(GL(1)^2 \times Sp(n) \times SL(2m+1) \times Sp(l), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m+1) + V(2m+1) \otimes V(2l))$ with $n > m \geq 2$ and $2l > 2m+1$. $N = 0$.

(11) $(GL(1)^2 \times Sp(n) \times SL(2m+1) \times Sp(l), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m+1) + V(2m+1) \otimes V(2l))$ where $n > m \geq 2$, $2m+1 > 2l \geq 4$ and $n - 2m + l - 1 \geq 0$. $N = 0$.

(12) $(GL(1)^2 \times Sp(n) \times SL(3) \times SL(5), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^{(*)} \otimes \Lambda_2, V(2n) \otimes V(3) + V(3)^{(*)} \otimes V(10))$ with $n \geq 2$. $N = 2$.

(13) $(GL(1)^2 \times Sp(n) \times SL(3) \times Sp(m), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(3) + V(3) \otimes V(2m))$ with $n \geq 2$ and $m \geq 2$. $N = 0$.

(14) $(GL(1)^2 \times Sp(n) \times SL(3) \times Sp(m), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes \Lambda_1, V(2n) \otimes V(3) + V(3)^* \otimes V(2m))$ with $n \geq 2$ and $m \geq 2$. $N = 1$.

(15) $(GL(1)^2 \times Sp(n) \times SL(3) \times SO(m), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^{(*)} \otimes \Lambda_1, V(2n) \otimes V(3) + V(3)^{(*)} \otimes V(m))$ with $n \geq 2$ and $m \geq 5$. $N = 2$.

(16) $(GL(1)^2 \times Sp(n) \times SL(3) \times \mathrm{Spin}(7), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^{(*)} \otimes$ *the spin rep.*, $V(2n) \otimes V(3) + V(3)^{(*)} \otimes V(8))$ with $n \geq 2$. $N = 2$.

(17) $(GL(1)^2 \times Sp(n) \times SL(3) \times \mathrm{Spin}(10), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^{(*)} \otimes$ *a half-spin rep.*, $V(2n) \otimes V(3) + V(3)^{(*)} \otimes V(16))$ with $n \geq 2$. $N = 2$.

(18) $(GL(1)^2 \times Sp(n) \times SL(2m) \times SL(2l+1), \Lambda_1 \otimes \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1, V(2n) \otimes V(2m) + V(2n) \otimes V(2l+1))$ with $n \geq 2m \geq 2$ and $n \geq 2l+1 \geq 3$. $N = 1$.

(19) $(GL(1)^2 \times H_1 \times H_2 \times SL(n), \rho_1 \otimes \rho_2 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V_1 \otimes V_2 \otimes V(n) + V(n)^*)$ where H_i $(i = 1, 2)$ are simple linear algebraic groups, $\rho_i : H_i \to GL(V_i)$ $(i = 1, 2)$ are nontrivial irreducible rational representations, $n = \deg(\rho_1 \otimes \rho_2)$ and $(GL(1) \times H_1 \times H_2, \Lambda_1 \otimes \rho_1 \otimes \rho_2, V_1 \otimes V_2)$ is a non-regular 2-simple irreducible PV. $N - 1$ is equal to the number of basic relative invariants of $(GL(1) \times H_1 \times H_2, \Lambda_1 \otimes \rho_1 \otimes \rho_2, V_1 \otimes V_2)$.

(20) $(GL(1)^2 \times H_1 \times SL(n) \times H_2, \rho_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes \rho_2, V_1 \otimes V(n) + V(n)^* \otimes V_2)$ where H_i $(i = 1, 2)$ are simple linear algebraic groups, $\rho_i : H_i \to GL(V_i)$ $(i = 1, 2)$ are nontrivial irreducible rational representations with $n = \deg \rho_1 > \deg \rho_2$ and $(GL(1) \times H_1 \times H_2, \Lambda_1 \otimes \rho_1 \otimes \rho_2, V_1 \otimes V_2)$ is a non-regular non-trivial reduced 2-simple irreducible PV. $N - 1$ is equal to the number of basic relative invariants of $(GL(1) \times H_1 \times H_2, \Lambda_1 \otimes \rho_1 \otimes \rho_2, V_1 \otimes V_2)$.

(21) $(GL(1)^2 \times H \times SL(c_{j+1}L) \times SL(c_{j+2}L), \rho \otimes \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 \otimes \Lambda_1^*, V(m) \otimes V(c_{j+1}L) \otimes V(c_{j+2}L) + V(r) \otimes V(c_{j+2}L)^*)$ where H is a simple linear algebraic group, ρ is its m-dimensional irreducible rational representation with $m \geq 2$, σ is its r-dimensional irreducible rational representation and j and L are positive integers with $L >$

$c_{j+2}r$. Here, for each m, a sequence $\{c_i\}_{i\geq -1}$ is defined by $c_{-1} = -1$, $c_0 = 0$ and $c_i = mc_{i-1} - c_{i-2}$ $(i \geq 1)$. $N = 1$.

(22) $(GL(1)^2 \times H \times SL(2) \times Sp(m), \rho \otimes \Lambda_1 \otimes 1 + 1 \otimes 2\Lambda_1 \otimes \Lambda_1, V \otimes V(2) + V(3) \otimes V(2m))$ with $m \geq 2$ where H is a simple linear algebraic groups, $\rho : H \to GL(V)$ is a nontrivial irreducible rational representation and (H, ρ, V) is one of $(Sp(n), \Lambda_1, V(2n))$ $(n \geq 2)$ and (Spin(10), *a half-spin rep.*, $V(16)$). $N = 2$.

(23) $(GL(1)^2 \times SL(m_1) \times SL(m_1m_2) \times G, \Lambda_1 \otimes \Lambda_1^* \otimes \rho + 1 \otimes \Lambda_1 \otimes \tau, V(m_1) \otimes V(m_1m_2)^* \otimes V(m_2) + V(m_1m_2) \otimes V(m))$ with $m_1 > m_2m$ where G is a simple linear algebraic group, $\rho : G \to GL(V(m_2))$ is a nontrivial irreducible rational representation and $\tau : G \to GL(V(m))$ is a nontrivial irreducible rational representation. $N = 1$.

(24) $(GL(1)^2 \times G_1 \times SL(n) \times G_2, \tau_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1^* \otimes \tau_2, V(k_1) \otimes V(n) + V(n)^* \otimes V(k_2))$ with $n > k_1 > k_2 \geq 2$ where G_i $(i = 1, 2)$ are simple linear algebraic groups, $\rho_i : G_i \to GL(V(k_i))$ $(i = 1, 2)$ are nontrivial irreducible rational representations and $(GL(1) \times G_1 \times G_2, \Lambda_1 \otimes \tau_1 \otimes \tau_2, V(k_1) \otimes V(k_2))$ is a 2-simple irreducible PV. Here, if $(G_1, \tau_1, V(k_1)) = (SL(k_1), \Lambda_1, V(k_1))$ (resp. $(G_2, \tau_2, V(k_2)) = (SL(k_2), \Lambda_1, V(k_2))$), then $n \geq 2k_1$ (resp. $n \geq 2k_2$). N is equal to the number of basic relative invariants of $(GL(1) \times G_1 \times G_2, \Lambda_1 \otimes \tau_1 \otimes \tau_2, V(k_1) \otimes V(k_2))$.

(25) $(GL(1)^2 \times G_1 \times G_2 \times SL(n), \tau_1 \otimes \tau_2 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(k_1) \otimes V(k_2) \otimes V(n) + V(n)^*)$ with $n > k_1k_2$ where G_i $(i = 1, 2)$ are simple linear algebraic groups, $\rho_i : G_i \to GL(V(k_i))$ $(i = 1, 2)$ are nontrivial irreducible rational representations and $(GL(1) \times G_1 \times G_2, \Lambda_1 \otimes \tau_1 \otimes \tau_2, V(k_1) \otimes V(k_2))$ is a 2-simple irreducible PV. N is equal to the number of basic relative invariants of $(GL(1) \times G_1 \times G_2, \Lambda_1 \otimes \tau_1 \otimes \tau_2, V(k_1) \otimes V(k_2))$.

(26) $(GL(1)^2 \times H \times SL(l) \times SL(n), \rho \otimes \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 \otimes \Lambda_1^*, V(m) \otimes V(l) \otimes V(n) + V(r) \otimes V(n)^*)$ where H is a simple linear algebraic group, ρ (resp. σ) is its m-dimensional (resp. r-dimensional) irreducible rational representation with $m \geq 2$ (resp. $r \geq 2$) and l and n are positive integers with $n > ml$ and $l \geq mr$. $N = 1$ (resp. $N = 0$) if $l = mr$ (resp. $l > mr$).

(27) $(GL(1)^2 \times H \times SL(c_jn - c_{j-1}l) \times SL(c_{j+1}n - c_jl), \rho \otimes \Lambda_1 \otimes \Lambda_1 + \sigma \otimes 1 \otimes \Lambda_1^*, V(m) \otimes V(c_jn - c_{j-1}l) \otimes V(c_{j+1}n - c_jl) + V(r) \otimes V(c_{j+1}n - c_jl)^*)$ where H is a simple linear algebraic group, ρ is its m-dimensional irreducible representation with $m \geq 2$, σ is its

r-dimensional irreducible representation and l, n and j are positive integers with $n > ml$ and $l \geq c_{j+2}r$. Here, for each m, a sequence $\{c_i\}_{i \geq -1}$ is defined by $c_{-1} = -1$, $c_0 = 0$ and $c_i = mc_{i-1} - c_{i-2}$ $(i \geq 1)$. $N = 1$ (resp. $N = 0$) if $l = c_{j+2}r$ (resp. $l > c_{j+2}r$).

(28) $(GL(1)^2 \times SL(2) \times SL(n) \times SL(2n-2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(2) \otimes V(n) \otimes V(2n-2) + V(2n-2)^*)$ with $n \geq 5$. $N = 1$.

(29) $(GL(1)^2 \times SL(2) \times SL(2n-1) \times SL(3n-2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(2) \otimes V(2n-1) \otimes V(3n-2) + V(3n-2)^*)$ with $n \geq 4$. $N = 1$.

(30) $(GL(1)^2 \times SL(2) \times SL(2n-3) \times SL(3n-6), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^*, V(2) \otimes V(2n-3) \otimes V(3n-6) + V(3n-6)^*)$ with $n \geq 7$. $N = 1$.

Remark 7.3. For example, $(GL(1)^2 \times SL(5) \times SL(9) \times SL(2), \Lambda_2 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1, V(10) \otimes V(9) + V(9) \otimes V(2))$ (See (I)-(13) in §4 of [Ka1]) is not reduced.

Remark 7.4. For (I)-(1)~(19) and (II)-(1)~(21), see [Ka1, §4] and Correction 1.4. For (I)-(20), (II)-(22), (23), see Correction 1.3. For (I)-(21)~(23) and (II)-(24)~(30), see Theorems 3.4, 5.4 and 6.1.

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