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GAPS OF F-YANG-MILLS FIELDS ON SUBMANIFOLDS*

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Abstract. Replacing the integrand of the Yang-Mills functional by $F\left(\frac{\|R^{\nabla}\|^2}{2}\right)$, we define an F-Yang-Mills functional, and hence F-Yang-Mills fields, where F is a non-negative function. The gaps of F-Yang-Mills fields on some submanifolds of the Euclidean spaces and the spheres are investegated in this paper.

1. Introduction

Let $P(M,G)$ be a principal bundle over a compact Riemannian manifold M with structure group G, $E = P \times_\rho V$ a vector bundle assosiated with $P(M,G)$, whose standard fibre is some vector space V, where $\rho : G \to \mathrm{GL}(V)$ is a representation of G. Let $\Omega^p(E) = \Gamma(\wedge^p T^*M \otimes E)$ be the space of E-valued p-forms, ∇ the connection of E. We use $\mathscr{C}_E$ to stand for the set of connections of E. Let $\mathfrak{g}_E = P \times_{\mathrm{Ad}_G} \mathfrak{g}$ be the adjoint vector bundle, where $\mathfrak{g}$ is the Lie algebra of the Lie group G. It is known that, for any $\nabla, \nabla' \in \mathscr{C}_E$, we have $\nabla - \nabla' \in \Omega^1(\mathfrak{g}_E)$. For each $\nabla \in \mathscr{C}_E$, the curvature 2-form $R^\nabla \in \Omega^2(\mathfrak{g}_E)$ is defined by $R^\nabla_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. If G is compact and semisimple, there is a natural invariant metric on $\mathfrak{g}_E$, and this metric induces a one on $\Omega^2(\mathfrak{g}_E)$. With respect to this induced metric, the Yang-Mills functional is defined as follows:

$$\mathscr{S}(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|^2. \tag{1}$$

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If a connection ∇ of E is a critical point of the Yang-Mills functional, we call it a Yang-Mills connection, the associated curvature tensor is called a Yang-Mills field.

An inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ is defined by setting $\langle U, V \rangle = -\frac{1}{2}\,\mathrm{trace}[\rho(U), \rho(V)]$, where $\rho : \mathfrak{g} \to \mathfrak{so}(N)$ is a faithful representation. In the paper [2, 1], Bourguignon and Lawson obtained a well known result on gaps of Yang-Mills fields as follows:

THEOREM 1 ([1]). *Let R^∇ be a Yang-Mills field on S^n $(n \geq 5)$ satisfying that*

$$\|R^\nabla\|^2 \leq \frac{1}{2}\binom{n}{2},$$

then $R^\nabla \equiv 0$.

If the integrand of the Yang-Mills functional is replaced by $\|R^\nabla\|^p$, then we can obtain a p-Yang-Mills functional, the critical points of which are called p-Yang-Mills connections, and the associated curvature tensors are called p-Yang-Mills fields. The article [3] investegated the gaps of p-Yang-Mills fields of Euclidean and sphere submanifolds, and generalized the related results of [1].

Let M^n be a submanifold of N^{n+k}, and $h(\cdot, \cdot)$ the second fundamental form, and let $1 \leq i, j \leq n$; $n+1 \leq \mu \leq n+k$. Choose a local orthonormal frame field $\{e_i \,|\, i = 1, \dots, n+k\}$ on N, such that $\{e_1, \dots, e_n\}$ are tangent to M and $\{e_\mu \,|\, \mu = n+1, \dots, n+k\}$ are normal to M. Set $h(e_i, e_j) = h^\mu_{ij} e_\mu$ and $H^\mu = \sum h^\mu_{ii}$. The article [3] proved the following gap theorem for submanifolds of the Euclidean spaces:

THEOREM 2 ([3]). *Let M^n be a submanifold of $\mathbf{R}^{n+k}$, satisfying the following condition*:

$$\sum[(-H^\mu h^\mu_{jl} + h^\mu_{jm} h^\mu_{ml})\delta_{ki} + h^\mu_{ik} h^\mu_{jl}] \leq (2-n)\delta_{ik}\delta_{jl}.$$

If R^∇ is a p-Yang-Mills field $(p \geq 2)$ with $\|R^\nabla\|^2 \leq \frac{1}{2}\binom{n}{2}$ $(n > 4)$, then we have $R^\nabla \equiv 0$.

If $M^n = S^n \subseteq \mathbf{R}^{n+1}$, then the condition above is satisfied (in fact the equality holds in this case), and the gap theorem is true for p-Yang-Mills fields. Therefore, Theorem 2 generalizes the related result of [1].

For submanifolds of spheres, the following gap theorem is proved in [3]:

THEOREM 3 ([3]). *Let M^n $(n > 4)$ be a submanifold of S^{n+k}, and satisfy the following condition*:

$$(-H^\mu h^\mu_{jl} + h^\mu_{jm} h^\mu_{ml})\delta_{ki} + h^\mu_{ik} h^\mu_{jl} \le b\delta_{ik}\delta_{jl}, \tag{2}$$

where $b \le 0$. If R^∇ is a p-Yang-Mills field on M with $\|R^\nabla\|^2 \le \frac{1}{2}\binom{n}{2}$ and $p \ge 2$, then, we have $R^\nabla \equiv 0$.

REMARK 4. The conditions in Theorems 2 and 3

$$\sum[(-H^\mu h^\mu_{jl} + h^\mu_{jm} h^\mu_{ml})\delta_{ki} + h^\mu_{ik} h^\mu_{jl}] \le a\delta_{ik}\delta_{jl} \tag{3}$$

mean that for any skew-symmetric 2-tensor $A = (A_{ij})$, we have

$$\sum[(-H^\mu h^\mu_{jl} + h^\mu_{jm} h^\mu_{ml})\delta_{ki} + h^\mu_{ik} h^\mu_{jl}]A_{ij}A_{kl} \le a\delta_{ik}\delta_{jl}A_{ij}A_{kl}.$$

In [3], the conditions are

$$\sum[(H^\mu h^\mu_{jl} - h^\mu_{jm} h^\mu_{ml})\delta_{ki} - h^\mu_{ik} h^\mu_{jl}] \le -a\delta_{ik}\delta_{jl} \tag{4}$$

which mean that

$$\sum[(H^\mu h^\mu_{jl} - h^\mu_{jm} h^\mu_{ml})\delta_{ki} - h^\mu_{ik} h^\mu_{jl}]A_{ij}A_{lk} \le -a\delta_{ik}\delta_{jl}A_{ij}A_{lk}.$$

Because A_{ij} is skew-symmetric, i.e. $A_{ij} = -A_{ji}$, The conditions (3) and (4) are the same.

Replacing the integrand of the Yang-Mills functional by $F\left(\frac{\|R^\nabla\|^2}{2}\right)$, where F is a non-negative function, we define an F-Yang-Mills functional, and hence F-Yang-Mills fields, which is a generalization of p-Yang-Mills fields. In this paper, we investegate the gaps of F-Yang-Mills fields on submanifolds of the Euclidean space and the spheres, and our main results are in the following:

THEOREM 5. *Let M^n be a submanifold of $\mathbf{R}^{n+k}$, and satisfy the following condition*:

$$(-H^\mu h^\mu_{jl} + h^\mu_{jm} h^\mu_{ml})\delta_{ki} + h^\mu_{ik} h^\mu_{jl} \le (2-n)\delta_{ik}\delta_{jl}. \tag{5}$$

Suppose that R^∇ is an F-Yang-Mills field on M^n which satisfies $\|R^\nabla\|^2 \le \frac{1}{2}\binom{n}{2}$, where, $F(t) > 0$, $F'(t) > 0$ and $F''(t) \ge 0$ for $t > 0$. Then, we have $\nabla R^\nabla = 0$ for $n \ge 3$ and $R^\nabla = 0$ for $n \ge 5$.

THEOREM 6. *Let M^n be a submanifold of S^{n+k}, and satisfy the following condition*:

$$(-H^{\mu}h^{\mu}_{jl} + h^{\mu}_{jm}h^{\mu}_{ml})\delta_{ki} + h^{\mu}_{ik}h^{\mu}_{jl} \le b\delta_{ik}\delta_{jl}, \tag{6}$$

where, $b \le 0$. If R^{∇} is an F-Yang-Mills field on M with $\|R^{\nabla}\|^2 \le \frac{1}{2}\binom{n}{2}$, where $F(t) > 0$, $F'(t) > 0$ and $F''(t) \ge 0$ for $t > 0$, then, we have $\nabla R^{\nabla} = 0$ for $n \ge 3$ and $R^{\nabla} \equiv 0$ for $n \ge 5$.

These theorems generalize the corresponding theorems of [1, 3].

2. F-Yang-Mills Fields

DEFINITION 7. Let $F : [0, +\infty) \to [0, +\infty)$ be a C^{∞} function. Define $\mathscr{S}_F : \mathscr{C}_E \to R$ as follows: For any $\nabla \in \mathscr{C}_E$, set

$$\mathscr{S}_F(\nabla) = \int_M F\left(\frac{\|R^{\nabla}\|^2}{2}\right), \tag{7}$$

which is called an F-Yang-Mills functional. The critical points of $\mathscr{S}_F$ are called F-Yang-Mills connections, and the associated curvature tensors are called F-Yang-Mills fields. When $F(t) = \frac{1}{p}(2t)^{p/2}$, the F-Yang-Mills fields are the p-Yang-Mills fields.

Let $\nabla^t = \nabla + A^t$ be a variation of $\nabla \in \mathscr{C}_E$, where $A^t \in \Omega^1(\mathfrak{g}_E)$ with $A^0 = 0$. Then the curvature of ∇^t is given by

$$R^{\nabla^t} = R^{\nabla} + d^{\nabla}A^t + \frac{1}{2}[A^t \wedge A^t], \tag{8}$$

where, the operation $[\cdot \wedge \cdot]$ is defined as follows: For $\varphi, \psi \in \Omega(\mathfrak{g}_E)$, $[\varphi \wedge \psi]_{X,Y} = [\varphi_X, \psi_Y] - [\varphi_Y, \psi_X]$. Let d^{∇} be the wedge differentiation. By a straightforward calculation, we have

$$\begin{aligned}
\frac{d}{dt}\mathscr{S}_F(\nabla^t) &= \int_M \frac{d}{dt}F\left(\frac{\|R^{\nabla^t}\|^2}{2}\right) \\
&= \int_M F'\left(\frac{\|R^{\nabla^t}\|^2}{2}\right)\left\langle \frac{d}{dt}R^{\nabla^t}, R^{\nabla^t}\right\rangle \\
&= \int_M F'\left(\frac{\|R^{\nabla^t}\|^2}{2}\right)\left\langle d^{\nabla}\frac{d}{dt}A^t + \left[\frac{d}{dt}A^t \wedge A^t\right], R^{\nabla^t}\right\rangle.
\end{aligned} \tag{9}$$

Let $D = \frac{d}{dt}\nabla^t|_{t=0}$. The above equality becomes as

$$\frac{d}{dt}\mathscr{S}_F(\nabla^t)\Big|_{t=0} = \int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle d^\nabla D, R^\nabla\rangle$$

$$= \int_M \left\langle D, \delta^\nabla F'\left(\frac{\|R^\nabla\|^2}{2}\right) R^\nabla \right\rangle, \tag{10}$$

where δ^∇ is the adjoint operator of d^∇. Hence the Euler-Lagrange equation of $\mathscr{S}_F(.)$ is

$$\delta^\nabla F'\left(\frac{\|R^\nabla\|^2}{2}\right) R^\nabla = 0. \tag{11}$$

3. Lemmas

For $\varphi \in \Omega^2(\mathfrak{g}_E)$, $\omega \in \Omega^2(M) \otimes \mathrm{Hom}(\mathfrak{X}(M), \mathfrak{X}(M))$, where $\mathfrak{X}(M)$ is the set of smooth sections of TM. let

$$(\varphi \circ \omega)_{X,\,Y} = \frac{1}{2}\sum \varphi_{e_j, \omega_{X,\,Y} e_j}. \tag{12}$$

We use R to express the Riemannian curvature tensor of M, Ric for the Ricci operator. On M, we take a local orthonormal frame field $\{e_i\}_{i=1,\dots,n}$, and adopt the Einsteinian convention of summation. The range of the indices i, j, k, l, m is $\{1, \dots, n\}$. Let

$$(\mathrm{Ric} \wedge I)_{X,\,Y} = \mathrm{Ric}(X) \wedge Y + X \wedge \mathrm{Ric}(Y) \tag{13}$$

and

$$\mathfrak{R}^\nabla(\varphi) = \sum \{[R^\nabla_{e_j,\,X}, \varphi_{e_j,\,Y}] - [R^\nabla_{e_j,\,Y}, \varphi_{e_j,\,X}]\}. \tag{14}$$

Here, $\mathrm{Ric} \wedge I \in \Omega^2(M) \otimes \mathrm{Hom}(\mathfrak{X}(M), \mathfrak{X}(M))$, and $X \wedge Y$ is defined as:

$$(X \wedge Y)(Z) = \langle X, Z\rangle Y - \langle Y, Z\rangle X. \tag{15}$$

For any $\varphi \in \Omega^2(\mathfrak{g}_E)$, we have (see [1])

$$\Delta\varphi = \nabla^*\nabla\varphi - \varphi \circ (\mathrm{Ric} \wedge I + 2R) + \mathfrak{R}^\nabla(\varphi). \tag{16}$$

Hence we have

$$\frac{1}{2}\Delta\|\varphi\|^2 = \langle \Delta^\nabla\varphi, \varphi\rangle - \|\nabla\varphi\|^2 - \langle \varphi \circ (\mathrm{Ric} \wedge I + 2R), \varphi\rangle - \langle \mathfrak{R}^\nabla(\varphi), \varphi\rangle. \tag{17}$$

By a straightforward calculation, we get

$$
\begin{aligned}
\Delta F\left(\frac{\|R^\nabla\|^2}{2}\right) &= -\sum \nabla_{e_i}\nabla_{e_i} F\left(\frac{\|R^\nabla\|^2}{2}\right) \\
&= -\sum \nabla_{e_i}\left(F'\left(\frac{\|R^\nabla\|^2}{2}\right)\nabla_{e_i}\frac{\|R^\nabla\|^2}{2}\right) \\
&= -F''\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2\|\nabla\|R^\nabla\|\ \ \|^2 - \frac{1}{2}F'\left(\frac{\|R^\nabla\|^2}{2}\right)\Delta\|R^\nabla\|^2 \\
&= -F''\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2\|\nabla\|R^\nabla\|\ \ \|^2 - F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle \Re^\nabla(R^\nabla), R^\nabla\rangle \\
&\quad + F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle \Delta^\nabla R^\nabla, R^\nabla\rangle - F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|\nabla R^\nabla\|^2 \\
&\quad - F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle R^\nabla \circ (\mathrm{Ric}\wedge I + 2R), R^\nabla\rangle. \qquad (18)
\end{aligned}
$$

LEMMA 8. *For an F-Yang-Mills field R^∇, we have*

$$
\begin{aligned}
&\int_M F''\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2\|\nabla\|R^\nabla\|\ \ \|^2 \\
&\quad + \int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2 + \int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle \Re^\nabla(R^\nabla), R^\nabla\rangle \\
&\quad + \int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle R^\nabla \circ (\mathrm{Ric}\wedge I + 2R), R^\nabla\rangle = 0. \qquad (19)
\end{aligned}
$$

PROOF. Integrating (18) shows that it is sufficient to prove $\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle \Delta^\nabla R^\nabla, R^\nabla\rangle = 0$. By (11) and the Bianchi equality $d^\nabla R^\nabla = 0$, we have

$$
\begin{aligned}
&\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle \Delta^\nabla R^\nabla, R^\nabla\rangle \\
&\quad = \int_M \left\langle d^\nabla \circ \delta^\nabla R^\nabla, F'\left(\frac{\|R^\nabla\|^2}{2}\right)R^\nabla\right\rangle + \int_M \left\langle \delta^\nabla \circ d^\nabla R^\nabla, F'\left(\frac{\|R^\nabla\|^2}{2}\right)R^\nabla\right\rangle
\end{aligned}
$$

$$= \int_M \left\langle \delta^\nabla R^\nabla, \delta^\nabla F'\left(\frac{\|R^\nabla\|^2}{2}\right) R^\nabla \right\rangle + \int_M \left\langle \delta^\nabla \circ d^\nabla R^\nabla, F'\left(\frac{\|R^\nabla\|^2}{2}\right) R^\nabla \right\rangle$$

$$= 0. \tag{20}$$

□

Let $\{X_a\}$ be an orthonormal frame of $\mathfrak{g}_E$, and $\{e_i\}$ on M. Let

$$R^\nabla_{e_i, e_j} = f^a_{ij} X_a, \quad (\nabla_{e_k} R^\nabla)_{e_i, e_j} = f^a_{ijk} X_a. \tag{21}$$

Then we have $f^a_{ij} = -f^a_{ji}$, $f^a_{ijk} = -f^a_{jik}$, $\|R^\nabla\|^2 = \frac{1}{2} f^a_{ij} f^a_{ij}$, $\|\nabla R^\nabla\|^2 = \frac{1}{2} f^a_{ijk} f^a_{ijk}$.

Lemma 9 ([3]). *We have*
(i) *If M^n is a submanifold of $\mathbf{R}^{n+k}$, then*

$$\langle R^\nabla \circ (\mathrm{Ric} \wedge I + 2R), R^\nabla \rangle = [(H^\mu h^\mu_{jl} - h^\mu_{jm} h^\mu_{ml})\delta_{ki} - h^\mu_{ik} h^\mu_{jl}] f^a_{ij} f^a_{kl}; \tag{22}$$

(ii) *If M^n is a submanifold of S^{n+k}, then*

$$\langle R^\nabla \circ (\mathrm{Ric} \wedge I + 2R), R^\nabla \rangle$$
$$= [(H^\mu h^\mu_{jl} - h^\mu_{jm} h^\mu_{ml})\delta_{ki} - h^\mu_{ik} h^\mu_{jl}] f^a_{ij} f^a_{kl} + 2(n-2)\|R^\nabla\|^2. \tag{23}$$

Proof. (i) The Riemannian curvature operator R of M is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

Let $R_{ijkl} = g(R(e_k, e_l)e_j, e_i)$ and $r_{jl} = \sum R_{ijil}$ are the Riemannian curvature tensor and the Ricci curvature tensor of M^n respectively, h^μ_{ij} the second fundamental tensor, and $H^\mu = \sum_{i=1}^n h^\mu_{ii}$. Because the Riemannian curvature of $\mathbf{R}^{n+k}$ vanishes, by Gaussian equation we get

$$R_{ijkl} = h^\mu_{lj} h^\mu_{ik} - h^\mu_{il} h^\mu_{jk}, \quad r_{jl} = H^\mu h^\mu_{jl} - h^\mu_{ij} h^\mu_{il}. \tag{24}$$

Since

$$(\mathrm{Ric} \wedge I)_{e_k, e_l} = \mathrm{Ric}(e_k) \wedge e_l + e_k \wedge \mathrm{Ric}(e_l) = r_{ki} e_i \wedge e_l + r_{li} e_k \wedge e_i, \tag{25}$$

we have

$$\langle R^\nabla \circ (\mathrm{Ric} \wedge I), R^\nabla \rangle$$
$$= \frac{1}{2} \sum \langle (R^\nabla \circ (\mathrm{Ric} \wedge I))_{e_k, e_l}, R^\nabla_{e_k, e_l} \rangle$$

$$= \frac{1}{4}\sum \langle (R^{\nabla}_{e_j,(\mathrm{Ric}\wedge I)_{e_k,e_l}e_j}), R^{\nabla}_{e_k,e_l}\rangle$$
$$= \frac{1}{4}\sum \langle (R^{\nabla}_{e_j,(r_{ki}e_i\wedge e_l + r_{li}e_k\wedge e_i)e_j}), R^{\nabla}_{e_k,e_l}\rangle$$
$$= \frac{1}{4}\sum r_{ki}\langle (R^{\nabla}_{e_j,(e_i\wedge e_l)e_j}), R^{\nabla}_{e_k,e_l}\rangle + \frac{1}{4}\sum r_{li}\langle (R^{\nabla}_{e_j,(e_k\wedge e_i)e_j}), R^{\nabla}_{e_k,e_l}\rangle$$
$$= \frac{1}{2} r_{ki}\langle R^{\nabla}_{e_i,e_l}, R^{\nabla}_{e_k,e_l}\rangle + \frac{1}{2} r_{li}\langle R^{\nabla}_{e_k,e_i}, R^{\nabla}_{e_k,e_l}\rangle$$
$$= r_{li}\langle R^{\nabla}_{e_k,e_i}, R^{\nabla}_{e_k,e_l}\rangle = -r_{li}\langle R^{\nabla}_{e_i,e_k}, R^{\nabla}_{e_k,e_l}\rangle. \tag{26}$$

Similarly,

$$\langle R^{\nabla}\circ 2R, R^{\nabla}\rangle = \frac{1}{2}\sum \langle (R^{\nabla}\circ 2R)_{e_k,e_l}, R^{\nabla}_{e_k,e_l}\rangle$$
$$= \frac{1}{2}\sum \langle R^{\nabla}_{e_j,R(e_k,e_l)e_j}, R^{\nabla}_{e_k,e_l}\rangle$$
$$= \frac{1}{2}\sum R_{ijkl}\langle R^{\nabla}_{e_j,e_i}, R^{\nabla}_{e_k,e_l}\rangle. \tag{27}$$

So we have

$$\langle R^{\nabla}\circ(\mathrm{Ric}\wedge I + 2R), R^{\nabla}\rangle$$
$$= \langle R^{\nabla}\circ(\mathrm{Ric}\wedge I), R^{\nabla}\rangle + \langle R^{\nabla}\circ 2R, R^{\nabla}\rangle$$
$$= -r_{lj}\langle R^{\nabla}_{e_j,e_k}, R^{\nabla}_{e_k,e_l}\rangle + \frac{1}{2}R_{ijkl}\langle R^{\nabla}_{e_j,e_i}, R^{\nabla}_{e_k,e_l}\rangle$$
$$= \frac{1}{2}[-2r_{lj}\langle R^{\nabla}_{e_j,e_k}, R^{\nabla}_{e_k,e_l}\rangle + R_{ijkl}\langle R^{\nabla}_{e_j,e_i}, R^{\nabla}_{e_k,e_l}\rangle]. \tag{28}$$

Substituting (24) into the above yields

$$\langle R^{\nabla}\circ(\mathrm{Ric}\wedge I + 2R), R^{\nabla}\rangle$$
$$= \frac{1}{2}[-2(H^{\mu}h^{\mu}_{jl} - h^{\mu}_{ji}h^{\mu}_{il})\langle R^{\nabla}_{e_j,e_k}, R^{\nabla}_{e_k,e_l}\rangle + (h^{\mu}_{ik}h^{\mu}_{jl} - h^{\mu}_{il}h^{\mu}_{jk})\langle R^{\nabla}_{e_j,e_i}, R^{\nabla}_{e_k,e_l}\rangle]$$
$$= -(H^{\mu}h^{\mu}_{jl} - h^{\mu}_{ji}h^{\mu}_{il})f^{a}_{jk}f^{a}_{kl} + \frac{1}{2}(h^{\mu}_{ik}h^{\mu}_{jl} - h^{\mu}_{il}h^{\mu}_{jk})f^{a}_{ji}f^{a}_{kl}$$
$$= [-(H^{\mu}h^{\mu}_{jl} - h^{\mu}_{jm}h^{\mu}_{ml})\delta_{ki} + h^{\mu}_{ik}h^{\mu}_{jl}]f^{a}_{ji}f^{a}_{kl}$$
$$= [(H^{\mu}h^{\mu}_{jl} - h^{\mu}_{jm}h^{\mu}_{ml})\delta_{ki} - h^{\mu}_{ik}h^{\mu}_{jl}]f^{a}_{ij}f^{a}_{kl}. \tag{29}$$

(ii) If M^n is a submanifold of S^{n+k}, the Riemannian and the Ricci tensors can be respectively written as

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h^\mu_{ik}h^\mu_{jl} - h^\mu_{il}h^\mu_{jk}) \tag{30}$$

and

$$r_{jl} = (n-1)\delta_{jl} + H^\mu h^\mu_{jl} - h^\mu_{il}h^\mu_{ji}. \tag{31}$$

By (28), (30) and (31), we have

$$\begin{aligned}
&\langle R^\nabla \circ (\mathrm{Ric} \wedge I + 2R), R^\nabla \rangle \\
&= \frac{1}{2}[-2r_{lj}\langle R^\nabla_{e_j,e_k}, R^\nabla_{e_k,e_l}\rangle + R_{ijkl}\langle R^\nabla_{e_j,e_i}, R^\nabla_{e_k,e_l}\rangle] \\
&= -((n-1)\delta_{jl} + H^\mu h^\mu_{jl} - h^\mu_{ji}h^\mu_{il})f^a_{jk}f^a_{kl} + \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + h^\mu_{ik}h^\mu_{jl} - h^\mu_{il}h^\mu_{jk})f^a_{ji}f^a_{kl} \\
&= -(H^\mu h^\mu_{jl} - h^\mu_{ji}h^\mu_{il})f^a_{jk}f^a_{kl} + \frac{1}{2}(h^\mu_{ik}h^\mu_{jl} - h^\mu_{il}h^\mu_{jk})f^a_{ji}f^a_{kl} \\
&\quad - (n-1)\delta_{jl}f^a_{jk}f^a_{kl} + \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})f^a_{ji}f^a_{kl} \\
&= -(H^\mu h^\mu_{jl} - h^\mu_{ji}h^\mu_{il})f^a_{jk}f^a_{kl} + \frac{1}{2}(h^\mu_{ik}h^\mu_{jl} - h^\mu_{il}h^\mu_{jk})f^a_{ji}f^a_{kl} + 2(n-2)\|R^\nabla\|^2 \\
&= [-(H^\mu h^\mu_{jl} - h^\mu_{jm}h^\mu_{ml})\delta_{ki} + h^\mu_{ik}h^\mu_{jl}]f^a_{ji}f^a_{kl} + 2(n-2)\|R^\nabla\|^2 \\
&= [(H^\mu h^\mu_{jl} - h^\mu_{jm}h^\mu_{ml})\delta_{ki} - h^\mu_{ik}h^\mu_{jl}]f^a_{ij}f^a_{kl} + 2(n-2)\|R^\nabla\|^2.
\end{aligned} \tag{32}$$

□

Taking $L = R^\nabla$ in Lemma 5.6 of [1], we have

Lemma 10 ([1]). *If* $\|R^\nabla\|^2 \leqq \frac{1}{2}\binom{n}{2}$, *and* $n \geqq 3$, *then*

$$|\langle [R^\nabla_{e_k,e_i}, R^\nabla_{e_i,e_j}], R^\nabla_{e_j,e_k}\rangle| \leqq 2(n-2)\|R^\nabla\|^2. \tag{33}$$

Furthermore, when $n \geqq 5$ *and* $R^\nabla \neq 0$, *the above inequality is strict.*

4. Gaps of F-Yang-Mills Fields

Theorem 11. *Let* M^n *be a submanifold of* $\mathbf{R}^{n+k}$ *and satisfy the following condition*:

$$(-H^\mu h^\mu_{jl} + h^\mu_{jm}h^\mu_{ml})\delta_{ki} + h^\mu_{ik}h^\mu_{jl} \leq (2-n)\delta_{ik}\delta_{jl}. \tag{34}$$

Suppose that R^∇ *is an* F*-Yang-Mills field on* M^n *which satisfies that* $\|R^\nabla\|^2 \leq \frac{1}{2}\binom{n}{2}$, *where,* $F'(t) > 0$ *and* $F''(t) \geq 0$ *for* $t > 0$. *Then we have* $\nabla R^\nabla = 0$ *for* $n \geq 3$, *or* $R^\nabla = 0$ *for* $n \geq 5$.

Proof. According to (19) we have

$$\int_M F''\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2\|\nabla\|R^\nabla\|\ \|^2 + \int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|\nabla R^\nabla\|^2$$
$$= -\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle R^\nabla \circ (\mathrm{Ric} \wedge I + 2R), R^\nabla\rangle$$
$$- \int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle \Re^\nabla(R^\nabla), R^\nabla\rangle \equiv (I) + (II). \tag{35}$$

By (22) and the condition (34), we get

$$(I) = -\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)[(H^\mu h^\mu_{jl} - h^\mu_{jm}h^\mu_{ml})\delta_{ki} - h^\mu_{ik}h^\mu_{jl}]f^a_{ij}f^a_{kl}$$
$$= \int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)[(-H^\mu h^\mu_{jl} + h^\mu_{jm}h^\mu_{ml})\delta_{ki} + h^\mu_{ik}h^\mu_{jl}]f^a_{ij}f^a_{kl}$$
$$\leq \int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)(2-n)\delta_{ik}\delta_{jl}f^a_{ij}f^a_{kl}$$
$$= 2(2-n)\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2. \tag{36}$$

Taking $\varphi = R^\nabla$ in the definition of $\Re^\nabla(\varphi)$, (see (14)), we have

$$\Re^\nabla(R^\nabla)_{e_j,e_k} = [R^\nabla_{e_i,e_j}, R^\nabla_{e_i,e_k}] - [R^\nabla_{e_i,e_k}, R^\nabla_{e_i,e_j}] = 2[R^\nabla_{e_k,e_i}, R^\nabla_{e_i,e_j}]. \tag{37}$$

For $n \geq 3$, from (33) we have

$$(II) = -\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle \Re^\nabla(R^\nabla), R^\nabla\rangle$$
$$= -\frac{1}{2}\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle \Re^\nabla(R^\nabla)_{e_j,e_k}, R^\nabla_{e_j,e_k}\rangle$$

$$= -\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle [R^\nabla_{e_k,e_i}, R^\nabla_{e_i,e_j}], R^\nabla_{e_j,e_k}\rangle$$

$$\leq 2(n-2)\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2, \tag{38}$$

where, the inequality (38) is strict by Lemma 10 if $n \geq 5$ and $R^\nabla \neq 0$. Therefore, we have

$$\int_M F''\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2\|\nabla\|R^\nabla\|\ \ \|^2 + \int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|\nabla R^\nabla\|^2$$

$$\leq 2(2-n+n-2)\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2 = 0. \tag{39}$$

Hence we have $\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|\nabla R^\nabla\|^2 \leq 0$. If $\nabla R^\nabla \neq 0$ at some point, then $\nabla R^\nabla \neq 0$ on some neighborhood U. Because $\int_U F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|\nabla R^\nabla\|^2 \leq 0$, we have $F'\left(\frac{\|R^\nabla\|^2}{2}\right) = 0$, and hence $R^\nabla = 0$ on U, which is a contradiction to $\nabla R^\nabla \neq 0$. Therefore we have $\nabla R^\nabla \equiv 0$ everywhere when $n \geq 3$. When $n \geq 5$ and $R^\nabla \neq 0$, the inequality (39) is strict which is imposible. □

Corollary 12. *Let M^n be a hypersurface of $\mathbf{R}^{n+1}$, the principal curvatures λ_i of which satisfy the following ordinary inequalities*:

$$-H\lambda_j + \lambda_j\lambda_l + \lambda_i\lambda_j \leq 2-n, \quad i,j,l = 1,2,\dots,n. \tag{40}$$

Suppose that R^∇ is an F-Yang-Mills field on M^n with $\|R^\nabla\|^2 \leqq \frac{1}{2}\binom{n}{2}$, where, $F(t) > 0$, $F'(t) > 0$ and $F''(t) \geq 0$ for $t > 0$. Then, $\nabla R^\nabla = 0$ for $n \geq 3$ or $R^\nabla = 0$ for $n \geq 5$.

Especially, if $M^n = S^n$, the equality holds in the condition (40). Hence Corollary 12 is valid for S^n.

Proof. Let

$$h^{n+1}_{ij} \equiv h_{ij} = \lambda_i\delta_{ij}, \quad H \equiv H^{n+1} = \sum_i \lambda_i. \tag{41}$$

Then we have (i, j, k, l not summation)

$$(-H^\mu h^\mu_{jl} + h^\mu_{jm}h^\mu_{ml})\delta_{ki} + h^\mu_{ik}h^\mu_{jl} = (-H\lambda_j + \lambda_j\lambda_l + \lambda_i\lambda_j)\delta_{ki}\delta_{jl}.$$

By Theorem 11, when

$$(-H\lambda_j + \lambda_j\lambda_l + \lambda_i\lambda_j)\delta_{ki}\delta_{jl} \le (2-n)\delta_{ki}\delta_{jl}, \tag{42}$$

the conclusions of Corollary 12 hold. Condition (42) means that for any skew-symmetric tensor A_{ij}, we have

$$(-H\lambda_j + \lambda_j\lambda_l + \lambda_i\lambda_j)\delta_{ki}\delta_{jl}A_{ij}A_{kl} \le (2-n)\delta_{ki}\delta_{jl}A_{ij}A_{kl}, \tag{43}$$

which is equivalent to (40) as an ordinary inequality. □

REMARK 13. In Corollary 3.3 of [3], the condition

$$H\lambda_j - \lambda_j\lambda_l - \lambda_i\lambda_j \le n-2$$

means that for any slew-symmetric tensor A_{ij}, the following inequality holds:

$$(H\lambda_j - \lambda_j\lambda_l - \lambda_i\lambda_j)\delta_{ki}\delta_{jl}A_{ji}A_{kl} \le (n-2)\delta_{ki}\delta_{jl}A_{ji}A_{kl}$$

which is equivalent to (40) as an ordinary inequality.

THEOREM 14. *Let M^n be a submanifold of S^{n+k}, and satisfy the following condition*:

$$(-H^\mu h^\mu_{jl} + h^\mu_{jm}h^\mu_{ml})\delta_{ki} + h^\mu_{ik}h^\mu_{jl} \le b\delta_{ik}\delta_{jl}, \tag{44}$$

where $b \le 0$. If R^∇ is an F-Yang-Mills field on M with $\|R^\nabla\|^2 \le \frac{1}{2}\binom{n}{2}$, where $F(t) > 0$, $F'(t) > 0$ and $F''(t) \ge 0$ for $t > 0$, then, we have $\nabla R^\nabla = 0$ for $n \ge 3$ and $R^\nabla \equiv 0$ for $n \ge 5$.

PROOF. By Lemma 9 (ii) and condition (44), we get

(I) of RHS of (35)

$$= -\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\langle R^\nabla \circ (\mathrm{Ric}\wedge I + 2R), R^\nabla\rangle$$

$$= \int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)[(-H^\mu h^\mu_{jl} + h^\mu_{jm}h^\mu_{ml})\delta_{ki} + h^\mu_{ik}h^\mu_{jl}]f^a_{ij}f^a_{kl}$$

$$+ 2(2-n)\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2$$

$$\leq \int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right) b\delta_{ik}\delta_{jl}f_{ij}^a f_{kl}^a + 2(2-n)\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2$$

$$= \int_M 2(b+2-n)F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2. \tag{45}$$

According to (35), (38), (45) and Lemma 10, for $n \geq 3$ we have

$$\int_M F''\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2\|\nabla\|R^\nabla\|\ \ \|^2 + \int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|\nabla R^\nabla\|^2$$

$$\leq \int_M 2(b+2-n)F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2 + 2(n-2)\int_M F'\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2$$

$$= 2\int_M bF'\left(\frac{\|R^\nabla\|^2}{2}\right)\|R^\nabla\|^2 \leq 0. \tag{46}$$

For the rest proof see that of Theorem 11. □

For a hypersurface of a sphere, we have a result similar to Corollary 12, i.e.

Corollary 15. *Suppose that M^n is a hypersurface of S^{n+1}, the principal curvatures of which satisfies the following inequalities*:

$$-H\lambda_j + \lambda_j\lambda_l + \lambda_i\lambda_j \leq 0, \quad i,j = 1,2,\ldots,n. \tag{47}$$

If R^∇ is an F-Yang-Mills field on M^n with $\|R^\nabla\|^2 \leqq \frac{1}{2}\binom{n}{2}$, where, $F(t) > 0$, $F'(t) > 0$ and $F''(t) \geq 0$ for $t > 0$, then, we have $\nabla R^\nabla = 0$ for $n \geq 3$ or $R^\nabla = 0$ for $n \geq 5$.

The proof of this corollary is similar to that of Corollary 12, and we omit the details.

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