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HOCHSCHILD COHOMOLOGY RING OF A MAXIMAL ORDER OF THE QUATERNION ALGEBRA

By

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Abstract. We give an efficient bimodule projective resolution of a maximal $\mathbf{Z}$-order Λ of the ordinary quaternion algebra over $\mathbf{Q}$, and therefore we determine the ring structure of the Hochschild cohomology of Λ by calculating the Yoneda products using this resolution.

1. Introduction

Let R be a commutative ring and Λ an R-algebra which is a finitely generated projective R-module. The nth Hochschild cohomology of Λ is defined by $HH^n(\Lambda) := \mathrm{Ext}^n_{\Lambda^e}(\Lambda, \Lambda)$, where $\Lambda^e = \Lambda \otimes_R \Lambda^{\mathrm{op}}$. The Yoneda product gives $HH^*(\Lambda) := \bigoplus_{n \geq 0} HH^n(\Lambda)$ a graded ring structure with $1 \in Z\Lambda \simeq HH^0(\Lambda)$ where $Z\Lambda$ denotes the center of Λ. $HH^*(\Lambda)$ is called the Hochschild cohomology ring of Λ. The Hochschild cohomology ring $HH^*(\Lambda)$ is graded-commutative, that is, for $\alpha \in HH^p(\Lambda)$ and $\beta \in HH^q(\Lambda)$ we have $\alpha\beta = (-1)^{pq}\beta\alpha$ (see [2], [6]).

The Hochschild cohomology has important connections and applications to the representation theory of algebras. For example, under appropriate hypotheses two derived equivalent algebras have isomorphic Hochschild cohomology algebras (see [5, Proposition 2.5]), and the second Hochschild cohomology group is important in deformation theory. The Hochschild cohomology has been studied for various algebras, however it is difficult to compute in general.

We have investigated the Hochschild cohomology of quaternion orders in [3] (see also [6], [7], [4]). Let $A = \mathbf{Q} \oplus \mathbf{Q}i \oplus \mathbf{Q}j \oplus \mathbf{Q}ij$ be the ordinary quaternion algebra over $\mathbf{Q}$ with the relations $i^2 = j^2 = -1$, $ij = -ji$. We set $\Gamma = \mathbf{Z} \oplus \mathbf{Z}i \oplus$

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$\mathbf{Z}j \oplus \mathbf{Z}ij$. Then Γ is a $\mathbf{Z}$-order of A. The multiplicative structure of the Hochschild cohomology ring $HH^*(\Gamma)$ is known in [6] (see [3] for a new proof and a generalization):

$$HH^*(\Gamma) \simeq \mathbf{Z}[X, Y, Z]/(2X, 2Y, 2Z, X^2 + Y^2 + Z^2),$$

where $\deg X = \deg Y = \deg Z = 1$. So it is natural ask to consider the Hochschild cohomology rings of other quaternion orders. On the other hand, Bobovich [1] shows that the Hochschild cohomology of a maximal order of a simple central algebra over the algebraic number field is periodic with period 2. Now we set $a = (1 + i + j + ij)/2$. Then $\Lambda = \mathbf{Z} \oplus \mathbf{Z}i \oplus \mathbf{Z}j \oplus \mathbf{Z}a$ is a maximal $\mathbf{Z}$-order of A. In this article, we give an explicit bimodule projective resolution of a maximal quaternion order Λ, and apply the result to determine the Hochschild cohomology ring of Λ. This resolution is not periodic, but nevertheless we can determine the ring structure of the Hochschild cohomology by using this resolution.

In Section 2, we give a bimodule projective resolution of Λ (Theorem 2.1). In Section 3, we give the module structure of $HH^*(\Lambda)$ using this resolution of Λ (Theorem 3.1). This is a complicated calculation. To compute the Yoneda products on $HH^*(\Lambda)$ we need generators of $HH^*(\Lambda)$ as a module. In Section 4, as a main theorem of this paper, we determine the Hochschild cohomology ring $HH^*(\Lambda)$ (Theorem 4.2).

2. Bimodule Projective Resolution

Let $A = \mathbf{Q} \oplus \mathbf{Q}i \oplus \mathbf{Q}j \oplus \mathbf{Q}ij$ be the ordinary quaternion algebra over $\mathbf{Q}$ with the relations $i^2 = j^2 = -1$, $ij = -ji$. We put $a = (1 + i + j + ij)/2$. Then $\Lambda = \mathbf{Z} \oplus \mathbf{Z}i \oplus \mathbf{Z}j \oplus \mathbf{Z}a$ is a maximal order of A. Note that the following equations hold:

$$ia = a - 1 - j = i - 1 - ai, \quad ja + aj = j - 1, \quad a^2 = a - 1, \quad a^3 = -1.$$

It is easy to see that 1, i, a, ia are linearly independent over $\mathbf{Z}$, so we take $\{1, i, a, ia\}$ as a $\mathbf{Z}$-basis of Λ.

In this section we give an efficient bimodule projective resolution of Λ. For each $q \geq 0$, let Y_q be the direct sum of $q + 1$ copies of $\Lambda \otimes \Lambda$. As elements of Y_q, we set

$$c_q^s = \begin{cases} (0, \dots, 0, 1 \overset{s}{\check{\otimes}} 1, 0, \dots, 0) & (\text{if } 1 \leq s \leq q + 1), \\ 0 & (\text{otherwise}). \end{cases}$$

Then we have $Y_q = \bigoplus_{k=1}^{q+1} \Lambda c_q^k \Lambda$. Define left Λ^{e}-homomorphisms $\pi : Y_0 \to \Lambda$; $c_0^1 \mapsto 1$ and $\delta_q : Y_q \to Y_{q-1}$ $(q > 0)$ given by

$$\delta_q(c_q^s) = \begin{cases} ic_{q-1}^s + c_{q-1}^s i + ac_{q-1}^{s-1} + c_{q-1}^{s-1}a - c_{q-1}^{s-1} & \text{for } q \text{ even,} \\ ic_{q-1}^s - c_{q-1}^s i + ac_{q-1}^{s-1} - c_{q-1}^{s-1}a & \text{for } q \text{ odd.} \end{cases}$$

Theorem 2.1. *The above* (Y, π, δ) *is a* Λ^{e}*-projective resolution of* Λ.

Proof. First, we must check that $\pi \cdot \delta_1 = 0$ and $\delta_q \cdot \delta_{q+1} = 0$ for $q \geq 1$. Clearly $\pi \cdot \delta_1 = 0$ holds. If $q(\geq 2)$ is even, we have the following:

$$\begin{aligned} \delta_{q-1} \cdot \delta_q(c_q^s) &= \delta_{q-1}(ic_{q-1}^s + c_{q-1}^s i + ac_{q-1}^{s-1} + c_{q-1}^{s-1}a - c_{q-1}^{s-1}) \\ &= i(ic_{q-2}^s - c_{q-2}^s i) + (ic_{q-2}^s - c_{q-2}^s i)i + i(ac_{q-2}^{s-1} - c_{q-2}^{s-1}a) \\ &\quad + (ac_{q-2}^{s-1} - c_{q-2}^{s-1}a)i + a(ic_{q-2}^{s-1} - c_{q-2}^{s-1}i) + (ic_{q-2}^{s-1} - c_{q-2}^{s-1}i)a \\ &\quad - (ic_{q-2}^{s-1} - c_{q-2}^{s-1}i) + a(ac_{q-2}^{s-2} - c_{q-2}^{s-2}a) + (ac_{q-2}^{s-2} - c_{q-2}^{s-2}a)a \\ &\quad - (ac_{q-2}^{s-2} - c_{q-2}^{s-2}a) \\ &= (ia + ai - i)c_{q-2}^{s-1} - c_{q-2}^{s-1}(ai + ia - i) + (a^2 - a)c_{q-2}^{s-2} - c_{q-2}^{s-2}(a^2 - a) \\ &= 0. \end{aligned}$$

The case q odd is similar.

Next we state a contracting homotopy. We define right Λ-homomorphisms $T_{-1} : \Lambda \to Y_0$ and $T_q : Y_q \to Y_{q+1}$ $(q \geq 0)$ as follows:

$$T_{-1}(\lambda) = c_0^1 \lambda \quad (\text{for } \lambda \in \Lambda);$$

$$T_q(i^m c_q^s) = \begin{cases} mc_{q+1}^1 & (s = 1), \\ 0 & (s \geq 2), \end{cases}$$

$$T_q(i^m ac_q^s) = \begin{cases} mc_{q+1}^1 - mc_{q+1}^1 a + i^m c_{q+1}^2 & (s = 1 \text{ and } q \text{ odd}), \\ mc_{q+1}^1 a + i^m c_{q+1}^2 & (s = 1 \text{ and } q \text{ even}), \\ i^m c_{q+1}^{s+1} & (s \geq 2), \end{cases}$$

where $m = 0, 1$. These homomorphisms are right Λ-homomorphism. So it permits us to cut down the number of cases. Thus we must check that

$$(\delta_{q+1} T_q + T_{q-1} \delta_q)(i^m a^n c_q^s) = i^m a^n c_q^s$$

holds for $m = 0, 1$; $n = 0, 1$; $1 \leq s \leq q+1$, where we set $\delta_0 = \pi$. In the case $q = 0$, we have the following:

$$\delta_1 T_0(i^m a^n c_0^1) + T_{-1}\pi(i^m a^n c_0^1) = (i^m a^n c_0^1 - c_0^1 i^m a^n) + c_0^1 i^m a^n = i^m a^n c_0^1.$$

Note that the computation of $\delta_1 T_0(i^m a^n c_0^1)$ is divided into four cases.

In the case q odd, we have the following:

Case $s = 1$:

$$\delta_{q+1} T_q(i^m c_q^1) = \delta_{q+1}(m c_{q+1}^1) = m(ic_q^1 + c_q^1 i) \quad (m = 0, 1),$$

$$T_{q-1}\delta_q(i^m c_q^1) = T_{q-1}(i^m(ic_{q-1}^1 - c_{q-1}^1 i)) = \begin{cases} c_q^1 & (m = 0), \\ -c_q^1 i & (m = 1), \end{cases}$$

$$\begin{aligned} \delta_{q+1} T_q(i^m a c_q^1) &= \delta_{q+1}(m c_{q+1}^1 - m c_{q+1}^1 a + i^m c_{q+1}^2) \\ &= \begin{cases} ac_q^1 + c_q^1 a - c_q^1 + ic_q^2 + c_q^2 i & (m = 0), \\ iac_q^1 + c_q^1 i - c_q^1 ia + ic_q^2 i - c_q^2 & (m = 1), \end{cases} \end{aligned}$$

$$\begin{aligned} T_{q-1}\delta_q(i^m a c_q^1) &= T_{q-1}(i^m a(ic_{q-1}^1 - c_{q-1}^1 i)) \\ &= T_{q-1}(i^m(i - 1 - ia)c_{q-1}^1 - c_{q-1}^1 i)) \\ &= \begin{cases} c_q^1 - c_q^1 a - ic_q^2 - c_q^2 i & (m = 0), \\ -c_q^1 - c_q^1 ai + c_q^2 - ic_q^2 i & (m = 1) \end{cases} \\ &= \begin{cases} c_q^1 - c_q^1 a - ic_q^2 - c_q^2 i & (m = 0), \\ -c_q^1 i + c_q^1 ia + c_q^2 - ic_q^2 i & (m = 1). \end{cases} \end{aligned}$$

Case $s \geq 2$:

$$\begin{aligned} \delta_{q+1} T_q(i^m c_q^s) &= 0, \\ T_{q-1}\delta_q(i^m c_q^s) &= T_{q-1}(i^m(ic_{q-1}^s - c_{q-1}^s i + ac_{q-1}^{s-1} - c_{q-1}^{s-1} a)) = i^m c_q^s, \\ \delta_{q+1} T_q(i^m a c_q^s) &= \delta_{q+1}(i^m c_{q+1}^{s+1}) \\ &= i^m(ic_q^{s+1} + c_q^{s+1} i + ac_q^s + c_q^s a - c_q^s), \\ T_{q-1}\delta_q(i^m a c_q^s) &= T_{q-1}(i^m((i - 1 - ia)c_{q-1}^s - ac_{q-1}^s i + (a - 1)c_{q-1}^{s-1} - ac_{q-1}^{s-1} a)) \\ &= -i^m(ic_q^{s+1} + c_q^{s+1} i + c_q^s a - c_q^s). \end{aligned}$$

Note that the computations of $T_{q-1}\delta_q$ are divided into some subcases. The case $q(\geq 2)$ even is handled by a similar way. □

It seems to be difficult (or impossible) to construct a periodic bimodule projective resolution of Λ. The resolution in Theorem 2.1 is not periodic, but nevertheless we can determine the ring structure of the Hochschild cohomology by using this resolution.

3. Module Structure

In this section, we determine the module structure of $HH^n(\Lambda)$. To calculate products on $HH^*(\Lambda)$ by the Yoneda product, we need an explicit module generator of $HH^n(\Lambda)$. Let Λ^{q+1} be a direct sum of $q+1$ copies of Λ. As elements of Λ^{q+1}, we set

$$\iota_q^s = \begin{cases} (0,\dots,0,\overset{s}{\check{1}},0,\dots,0) & (\text{if } 1 \leq s \leq q+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Thus we have $\Lambda^{q+1} = \bigoplus_{k=1}^{q+1} \Lambda\iota_q^k$.

Applying the functor $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$ to the resolution (Y,π,δ), we have the following complex, where we identify $\mathrm{Hom}_{\Lambda^e}(Y_q,\Lambda)$ with Λ^{q+1} using an isomorphism $\mathrm{Hom}_{\Lambda^e}(Y_q,\Lambda) \to \Lambda^{q+1}$; $f \mapsto \sum_{k=1}^{q+1} f(c_q^k)\iota_q^k$:

$$(\mathrm{Hom}_{\Lambda^e}(Y,\Lambda),\delta^{\#}):\quad 0 \longrightarrow \Lambda \xrightarrow{\delta_1^{\#}} \Lambda^2 \xrightarrow{\delta_2^{\#}} \Lambda^3 \xrightarrow{\delta_3^{\#}} \Lambda^4 \xrightarrow{\delta_4^{\#}} \Lambda^5 \longrightarrow \cdots,$$

$$\delta_{q+1}^{\#}(\lambda\iota_q^s) = \begin{cases} i\lambda\iota_{q+1}^s - \lambda i\iota_{q+1}^s + a\lambda\iota_{q+1}^{s+1} - \lambda a\iota_{q+1}^{s+1} & \text{for } q \text{ odd}, \\ i\lambda\iota_{q+1}^s + \lambda i\iota_{q+1}^s + a\lambda\iota_{q+1}^{s+1} + \lambda a\iota_{q+1}^{s+1} - \lambda\iota_{q+1}^{s+1} & \text{for } q \text{ even}, \end{cases}$$

for $\lambda \in \Lambda$. In the above, note that

$$\lambda\iota_q^s = \begin{cases} (0,\dots,0,\overset{s}{\check{\lambda}},0,\dots,0) & (\text{if } 1 \leq s \leq q+1), \\ 0 & (\text{otherwise}), \end{cases}$$

and so on. In the following, if z is a cocycle, we also denote its cohomology class by z for brevity.

Theorem 3.1. *The module structure of* $HH^n(\Lambda)$ *is as follows*:

$$HH^n(\Lambda) = \begin{cases} \mathbf{Z} & (n=0), \\ 0 & (n \text{ odd}), \\ (\mathbf{Z}/2\mathbf{Z})\iota_n^1 & (n(\neq 0) \text{ even}). \end{cases}$$

Proof. Let $\lambda_k = b_k + c_k i + d_k a + e_k ia$ $(b_k, c_k, d_k, e_k \in \mathbf{Z})$ be any element of Λ. Then we have

$$\begin{aligned}
i\lambda_k &= b_k i - c_k + d_k ia - e_k a,\\
\lambda_k i &= b_k i - c_k + d_k(i - 1 - ia) + e_k(a - 1 - i)\\
&= -(c_k + d_k + e_k) + (b_k + d_k - e_k)i + e_k a - d_k ia,\\
a\lambda_k &= b_k a + c_k(i - 1 - ia) + d_k(a - 1) + e_k(i - a)\\
&= -(c_k + d_k) + (c_k + e_k)i + (b_k + d_k - e_k)a - c_k ia,\\
\lambda_k a &= b_k a + c_k ia + d_k(a - 1) + e_k i(a - 1)\\
&= -d_k - e_k i + (b_k + d_k)a + (c_k + e_k)ia.
\end{aligned}$$

(i) The case $n = 0$: We see that $HH^0(\Lambda) = \operatorname{Ker}\delta_1^\# = \mathbf{Z}$. In fact, for $\lambda_1 = b_1 + c_1 i + d_1 a + e_1 ia$ $(b_1, c_1, d_1, e_1 \in \mathbf{Z})$, we have

$$\begin{aligned}
\lambda_1 \in \operatorname{Ker}\delta_1^\# &\Leftrightarrow \begin{cases} i\lambda_1 - \lambda_1 i = 0,\\ a\lambda_1 - \lambda_1 a = 0\end{cases}\\
&\Leftrightarrow \begin{cases} (d_1 + e_1) + (e_1 - d_1)i - 2e_1 a + 2d_1 ia = 0,\\ -c_1 + (c_1 + 2e_1)i - e_1 a - (2c_1 + e_1)ia = 0\end{cases}\\
&\Leftrightarrow c_1 = d_1 = e_1 = 0.
\end{aligned}$$

(ii) The case n odd: We show $HH^n(\Lambda) = 0$. Since the cohomology module is periodic with period 2, it suffices to show $HH^1(\Lambda) = 0$. For $\lambda_k = b_k + c_k i + d_k a + e_k ia$ $(b_k, c_k, d_k, e_k \in \mathbf{Z})$, we have

$$\begin{aligned}
(\lambda_1, \lambda_2) \in \operatorname{Ker}\delta_2^\# &\Leftrightarrow \begin{cases} i\lambda_1 + \lambda_1 i = 0,\\ a\lambda_1 + \lambda_1 a - \lambda_1 + i\lambda_2 + \lambda_2 i = 0,\\ a\lambda_2 + \lambda_2 a - \lambda_2 = 0\end{cases}\\
&\Leftrightarrow \begin{cases} -(2c_1 + d_1 + e_1) + (2b_1 + d_1 - e_1)i = 0,\\ -(b_1 + c_1 + 2d_1 + 2c_2 + d_2 + e_2)\\ \quad +(2b_2 + d_2 - e_2)i + (2b_1 + d_1 - e_1)a = 0,\\ -(b_2 + c_2 + 2d_2) + (2b_2 + d_2 - e_2)a = 0\end{cases}\\
&\Leftrightarrow \begin{cases} 2c_1 + d_1 + e_1 = 0,\\ 2b_1 + d_1 - e_1 = 0,\\ b_1 + c_1 + 2d_1 + 2c_2 + d_2 + e_2 = 0,\\ b_2 + c_2 + 2d_2 = 0,\\ 2b_2 + d_2 - e_2 = 0\end{cases}
\end{aligned}$$

$$\Leftrightarrow \begin{cases} b_1 = -c_1 - 2d_2, \\ d_1 = 2d_2, \\ e_1 = -2c_1 - 2d_2, \\ c_2 = -b_2 - 2d_2, \\ e_2 = 2b_2 + d_2. \end{cases}$$

If this is the case, then we have

$$(\lambda_1, \lambda_2) = c_1(i - 1 - 2ia, 0) - d_2(2 - 2a + 2ia, 2i - a - ia) + b_2(0, 1 - i + 2ia).$$

Therefore we have that

$$\operatorname{Ker} \delta_2^{\#} = \mathbf{Z}(i - 1 - 2ia, 0) \oplus \mathbf{Z}(2 - 2a + 2ia, 2i - a - ia) \oplus \mathbf{Z}(0, i - 1 - 2ia).$$

Since for $\lambda_1 = b_1 + c_1 i + d_1 a + e_1 ia$ $(b_1, c_1, d_1, e_1 \in \mathbf{Z})$,

$$\begin{aligned} \delta_1^{\#}(\lambda_1) &= (i\lambda_1 - \lambda_1 i, a\lambda_1 - \lambda_1 a) \\ &= d_1(1 - i + 2ia, 0) + e_1(1 + i - 2a, 2i - a - ia) + c_1(0, i - 1 - 2ia) \\ &= (e_1 - d_1)(i - 1 - 2ia, 0) + e_1(2 - 2a + 2ia, 2i - a - ia) \\ &\quad + c_1(0, i - 1 - 2ia), \end{aligned}$$

it follows that $\operatorname{Ker} \delta_2^{\#} = \operatorname{Im} \delta_1^{\#}$ holds.

(iii) The case $n (\geq 2)$ even: We show $HH^n(\Lambda) = \operatorname{Ker} \delta_{n+1}^{\#} / \operatorname{Im} \delta_n^{\#} = (\mathbf{Z}/2\mathbf{Z})\iota_n^1$. First we calculate $\operatorname{Ker} \delta_{n+1}^{\#}$. Let $\lambda_k = b_k + c_k i + d_k a + e_k ia$ $(b_k, c_k, d_k, e_k \in \mathbf{Z})$. Since

$$(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) \in \operatorname{Ker} \delta_{n+1}^{\#}$$

$$\Leftrightarrow \begin{cases} i\lambda_1 - \lambda_1 i = 0, \\ a\lambda_k - \lambda_k a + i\lambda_{k+1} - \lambda_{k+1} i = 0 \quad (k = 1, 2, \ldots, n), \\ a\lambda_{n+1} - \lambda_{n+1} a = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (d_1 + e_1) + (e_1 - d_1)i - 2e_1 a + 2d_1 ia = 0, \\ (-c_k + d_{k+1} + e_{k+1}) + (c_k + 2e_k - d_{k+1} + e_{k+1})i \\ -(e_k + 2e_{k+1})a - (2c_k + e_k - 2d_{k+1})ia = 0 \quad (k = 1, 2, \ldots, n), \\ -c_{n+1} + (c_{n+1} + 2e_{n+1})i - e_{n+1} a - (2c_{n+1} + e_{n+1})ia = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} d_1 = 0, \\ e_k = 0 \qquad (k = 1, 2, \ldots, n+1), \\ c_k = d_{k+1} \quad (k = 1, 2, \ldots, n), \\ c_{n+1} = 0 \end{cases}$$

$$\Leftrightarrow (\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) = \sum_{k=1}^{n+1} b_k \iota_n^k + \sum_{k=1}^{n} c_k (i\iota_n^k + a\iota_n^{k+1}),$$

we have

$$\operatorname{Ker}\delta_{n+1}^{\#}=\bigoplus_{k=1}^{n+1}\mathbf{Z}\iota_n^k\oplus\bigoplus_{k=1}^{n}\mathbf{Z}(i\iota_n^k+a\iota_n^{k+1})$$

$$=\bigoplus_{k=1}^{n+1}\mathbf{Z}\iota_n^k\oplus\bigoplus_{k=1}^{n}\mathbf{Z}((i+1)\iota_n^k+a\iota_n^{k+1}). \tag{3.1}$$

Next we prove

$$\operatorname{Im}\delta_n^{\#}=\mathbf{Z}(2\iota_n^1+\iota_n^2)\oplus\bigoplus_{k=2}^{n+1}\mathbf{Z}\iota_n^k\oplus\bigoplus_{k=1}^{n}\mathbf{Z}((i+1)\iota_n^k+a\iota_n^{k+1}). \tag{3.2}$$

For $\lambda_k=b_k+c_ki+d_ka+e_kia$ $(b_k,c_k,d_k,e_k\in\mathbf{Z})$, we have

$$\begin{aligned}\delta_n^{\#}(\lambda_1,\lambda_2,\dots,\lambda_n)&=\sum_{k=1}^{n}b_k(2i\iota_n^k+(2a-1)\iota_n^{k+1})-\sum_{k=1}^{n}c_k(2\iota_n^k+\iota_n^{k+1})\\&\quad+\sum_{k=1}^{n}d_k((i-1)\iota_n^k+(a-2)\iota_n^{k+1})-\sum_{k=1}^{n}e_k((i+1)\iota_n^k+a\iota_n^{k+1})\\&=\sum_{k=1}^{n}(2b_k+d_k-e_k)((i+1)\iota_n^k+a\iota_n^{k+1})\\&\quad-\sum_{k=1}^{n}(b_k+c_k+d_k)(2\iota_n^k+\iota_n^{k+1})-\sum_{k=1}^{n}d_k\iota_n^{k+1}\\&=\sum_{k=1}^{n}(2b_k+d_k-e_k)((i+1)\iota_n^k+a\iota_n^{k+1})\\&\quad-(b_1+c_1+d_1)(2\iota_n^1+\iota_n^2)-(d_1+2(b_2+c_2+d_2))\iota_n^2\\&\quad-\sum_{k=2}^{n-1}(b_k+c_k+2(d_k+b_{k+1}+c_{k+1}+d_{k+1}))\iota_n^{k+1}\\&\quad-(b_n+c_n+2d_n)\iota_n^{n+1}.\end{aligned}$$

Then $\delta_n^{\#}(\lambda_1,\lambda_2,\dots,\lambda_n)$ is an element of the right-hand side of (3.2). Conversely, by choosing e_k $(1\le k\le n)$, b_1,d_1,b_k $(2\le k\le n)$ properly, $\delta_n^{\#}(\lambda_1,\lambda_2,\dots,\lambda_n)$ is to be any element of the right-hand side of (3.2). Therefore we have

$$\operatorname{Im}\delta_n^{\#}=2\mathbf{Z}\iota_n^1\oplus\bigoplus_{k=2}^{n+1}\mathbf{Z}\iota_n^k\oplus\bigoplus_{k=1}^{n}\mathbf{Z}((i+1)\iota_n^k+a\iota_n^{k+1}). \tag{3.3}$$

Hence by (3.1) and (3.3) we have the results. □

4. Hochschild Cohomology Ring of Λ

In this section, we determine the ring structure of the Hochschild cohomology ring $HH^*(\Lambda)$.

Recall the Yoneda product in $HH^*(\Lambda)$. Let $\alpha \in HH^n(\Lambda)$ and $\beta \in HH^m(\Lambda)$, where α and β are represented by cocycles $f_\alpha : Y_n \to \Lambda$ and $f_\beta : Y_m \to \Lambda$, respectively. There exists the commutative diagram of Λ^e-modules:

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{\delta_{n+m+1}} & Y_{n+m} & \xrightarrow{\delta_{n+m}} & \cdots & \xrightarrow{\delta_{m+2}} & Y_{m+1} & \xrightarrow{\delta_{m+1}} & Y_m & \xrightarrow{f_\beta} & \Lambda & & \\
 & & \downarrow{\scriptstyle \mu_n} & & & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_0} & & \| & & \\
\cdots & \xrightarrow[\delta_{n+1}]{} & Y_n & \xrightarrow[\delta_n]{} & \cdots & \xrightarrow[\delta_2]{} & Y_1 & \xrightarrow[\delta_1]{} & Y_0 & \xrightarrow[\pi]{} & \Lambda & \longrightarrow & 0,
\end{array}$$

where μ_l $(0 \leq l \leq n)$ are liftings of f_β. We define the product $\alpha \cdot \beta \in HH^{n+m}(\Lambda)$ by the cohomology class of $f_\alpha \mu_n$. This product is independent of the choice of representatives f_α and f_β, and liftings μ_l $(0 \leq l \leq n)$.

Let $\alpha = \iota_2^1 (\in HH^2(\Lambda))$. Then α is represented by the Λ^e-homomorphism $f_\alpha : Y_2 \to \Lambda$ given by $f_\alpha(c_2^1) = 1$, $f_\alpha(c_2^2) = f_\alpha(c_2^3) = 0$. Then the following lemma holds.

Lemma 4.1. *A lifting* $\mu_n : Y_{n+2} \to Y_n$ *of* f_α *is given by* $\mu_n(c_{n+2}^k) = c_n^k$ *for* $n \geq 0$.

Proof. Clearly $\pi \cdot \mu_0 = f_\alpha$ holds. If $n \geq 1$ is odd, we have

$$\begin{aligned}
\mu_{n-1}\delta_{n+2}(c_{n+2}^k) &= \mu_{n-1}(ic_{n+1}^k - c_{n+1}^k i + ac_{n+1}^{k-1} - c_{n+1}^{k-1}a) \\
&= ic_{n-1}^k - c_{n-1}^k i + ac_{n-1}^{k-1} - c_{n-1}^{k-1}a \\
&= \delta_n(c_n^k) = \delta_n \mu_n(c_{n+2}^k).
\end{aligned}$$

The case $n \geq 2$ even is similar. □

Let $\beta = \iota_{2n}^1 \in HH^{2n}(\Lambda)$ for $n \geq 1$. Then β is represented by the Λ^e-homomorphism $f_\beta : Y_{2n} \to \Lambda$; $c_{2n}^k \mapsto \iota_0^k$. Since $f_\beta \cdot \mu_{2n}(c_{2n+2}^k) = f_\beta(c_{2n}^k) = \iota_0^k$, it follows that $\alpha\beta = \iota_{2n+2}^1 \in HH^{2n+2}(\Lambda)$ holds. Therefore we have $\alpha^n = \iota_{2n}^1 \in HH^{2n}(\Lambda)$ for $n \geq 1$, and the following theorem holds.

Theorem 4.2. *The Hochschild cohomology ring* $HH^*(\Lambda)$ *is isomorphic to* $\mathbf{Z}[\alpha]/(2\alpha)$, *where* $\deg \alpha = 2$.

Remark. Let $\Gamma = \mathbf{Z} \oplus \mathbf{Z}i \oplus \mathbf{Z}j \oplus \mathbf{Z}ij$ be the ordinary quaternion $\mathbf{Z}$-algebra. The multiplicative structure of the Hochschild cohomology ring $HH^*(\Gamma)$ is known in [6], [3]:

$$HH^*(\Gamma) \simeq \mathbf{Z}[X, Y, Z]/(2X, 2Y, 2Z, X^2 + Y^2 + Z^2),$$

where $\deg X = \deg Y = \deg Z = 1$. $HH^*(\Gamma)$ is not a periodic cohomology. It is known that Γ is not a maximal order of the ordinary quaternion $\mathbf{Q}$ algebra A. In general, there does not exist a ring homomorphism between two Hochschild cohomology rings except special cases. However, by Theorem 4.2, we get an injective ring homomorphism $HH^*(\Lambda) \to HH^*(\Gamma)$ given by $\alpha \mapsto X^2$.

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