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PSEUDO-PARALLEL *CR* SUBMANIFOLDS OF A COMPLEX SPACE FORM

By

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Abstract. We classify pseudo-parallel proper *CR* submanifolds of a non-flat complex space form with semi-flat normal connection under the condition that the dimension of the holomorphic tangent space is greater than 2.

1. Introduction

There are many results about real hypersurfaces immersed in a complex space form with additional conditions for the second fundamental form A. It is well known that there are no real hypersurface in a complex space form $M^n(c)$, $c \neq 0$, of constant holomorphic sectional curvature $4c$ with parallel second fundamental form. Moreover, Maeda [6] showed that no real hypersurface in $M^n(c)$, $c > 0$, $n \geq 3$, satisfies semi-parallel condition, that is, $R(X, Y) \cdot A = 0$ for any X, Y tangent to the real hypersurface. Niebergall and Ryan [7] also proved the non-existence of semi-parallel real hypersurface in $M^2(c)$, $c \neq 0$.

If the second fundamental form A of a submanifold M satisfies

$$R(X, Y)A = \alpha(X \wedge Y)A$$

for any $X, Y \in TM$, α being a function, then A is said to be *pseudo-parallel*, that generalize the notion of semi-symmetric. In [5], Lobos and Ortega obtained the classification of the pseudo-parallel real hypersurfaces in $M^n(c)$, $c \neq 0$, $n \geq 2$.

A submanifold M of a Kählerian manifold $\tilde{M}$ is called a *CR submanifold* of $\tilde{M}$ if there exists a differentiable distribution $H : x \to H_x \subset T_x(M)$ on M satisfying the conditions that H is holomorphic, i.e., $JH_x = H_x$ for each $x \in M$,

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and the complementary orthogonal distribution $H^{\perp} : x \to H_x^{\perp} \subset T_x(M)$ is anti-invariant, i.e. $JH_x^{\perp} \subset T_x(M)^{\perp}$ for each $x \in M$. Any real hypersurface of a Kählerian manifold is a CR submanifold.

The main purpose of the present paper is to prove the following theorem.

Theorem. *Let M be an n-dimensional proper CR submanifold of a complex space form $M^m(c)$, $c = \pm 1$, with semi-flat normal connection. We suppose that the dimension h of the holomorphic tangent space > 2. If the second fundamental form A satisfies $R(X, Y)A = \alpha(X \wedge Y)A$ for any X and Y tangent to M, α being a function, then α is constant and M is one of the following hypersurfaces of totally geodesic $M^{(n+1)/2}(c)$ in $M^m(c)$;*

i) *If $c = +1$, then $\alpha = \cot^2(r)$, for $0 < r < \pi/2$, and M is an open subset of a geodesic hypersphere of radius r.*
ii) *If $c = -1$, then*
   a) *$1 < \alpha = \coth^2(r)$, for $r > 0$, and M is an open subset of a geodesic hypersphere of radius r;*
   b) *$\alpha = 1$, and M is an open subset of a horosphere;*
   c) *$0 < \alpha = \tanh^2(r) < 1$, for $r > 0$, and M is an open subset of a tube of radius r over a totally geodesic $\mathbf{C}H^{(n-1)/2}$.*

2. Preliminaries

Let $M^m(c)$ denote the complex space form of complex dimension m (real dimension $2m$) with constant holomorphic sectional curvature $4c$. For the sake of simplicity, if $c > 0$, we only use $c = +1$ and call it the complex projective space $\mathbf{C}P^n$, and if $c < 0$, we just consider $c = -1$, so that we call it the complex hyperbolic space $\mathbf{C}H^n$. We denote by J the almost complex structure of $M^m(c)$. The Hermitian metric of $M^m(c)$ is denoted by G.

Let M be a real n-dimensional Riemannian manifold isometrically immersed in $M^m(c)$. We denote by g the Riemannian metric induced on M from G, and by p the codimension of M, that is, $p = 2m - n$.

We denote by $T_x(M)$ and $T_x(M)^{\perp}$ the tangent space and the normal space of M respectively.

We denote by $\tilde{\nabla}$ the covariant differentiation in $M^m(c)$, and by ∇ the one in M determined by the induced metric. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad \tilde{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$ of M. We call both A and B the *second fundamental form* of M and are related by $G(B(X,Y),V) = g(A_V X, Y)$. The second fundamental form A and B are symmetric.

For $x \in M$, the *first normal space* $N_1(x)$ is the orthogonal complement in $T_x(M)^{\perp}$ of the set

$$N_0(x) = \{V \in T_x(M)^{\perp} \mid A_V = 0\}.$$

The covariant derivative $(\nabla_X A)_V Y$ of A is defined to be

$$(\nabla_X A)_V Y = \nabla_X(A_V Y) - A_{D_X V} Y - A_V \nabla_X Y.$$

If $(\nabla_X A)_V Y = 0$ for any vector fields X and Y tangent to M, then the second fundamental form of M is said to be *parallel in the direction of the normal vector* V. If the second fundamental form is parallel in any direction, it is said to be *parallel*.

Definition. A submanifold M of a Kählerian manifold $\tilde{M}$ is called a *CR submanifold* of $\tilde{M}$ if there exists a differentiable distribution $H : x \to H_x \subset T_x(M)$ on M satisfying the following conditions:

(i) H is holomorphic, i.e., $JH_x = H_x$ for each $x \in M$, and

(ii) the complementary orthogonal distribution $H^{\perp} : x \to H_x^{\perp} \subset T_x(M)$ is anti-invariant, i.e. $JH_x^{\perp} \subset T_x(M)^{\perp}$ for each $x \in M$.

We call H_x a *holomorphic tangent space*.

In the following, we put $\dim H_x = h$, $\dim H_x^{\perp} = q$. If $q = 0$ (resp. $h = 0$) for any $x \in M$, then the CR submanifold M is a holomorphic submanifold (resp. anti-invariant submanifold or totally real submanifold) of $\tilde{M}$. If a CR submanifold satisfies $h > 0$ and $q > 0$, then it is said to be *proper*.

In the sequel, we assume that M is a CR submanifold of $M^m(c)$. The tangent space $T_x(M)$ of M is decomposed as $T_x(M) = H_x + H_x^{\perp}$ at each point x of M, where $H_x^{\perp}$ denotes the orthogonal complement of H_x in $T_x(M)$. Similarly, we see that $T_x(M)^{\perp} = JH_x^{\perp} + N_x$, where N_x is the orthogonal complement of $JH_x^{\perp}$ in $T_x(M)^{\perp}$.

For any vector field X tangent to M, we put

$$JX = PX + FX,$$

where PX is the tangential part of JX and FX the normal part of JX. Then P is an endomorphism on the tangent bundle $T(M)$ and F is a normal bundle valued 1-form on the tangent bundle $T(M)$. We notice that $P^3 + P = 0$.

For any vector field V normal to M, we put

$$JV = tV + fV,$$

where tV is the tangential part of JV and fV the normal part of JV. Then we see that $FP = 0$, $fF = 0$, $tf = 0$ and $Pt = 0$.

We define the covariant derivatives of P, F, t and f by $(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y$, $(\nabla_X F)Y = D_X(FY) - F\nabla_X Y$, $(\nabla_X t)V = \nabla_X(tV) - tD_X V$ and $(\nabla_X f)V = D_X(fV) - fD_X V$, respectively. We then have

$$(\nabla_X P)Y = A_{FY}X + tB(X, Y), \quad (\nabla_X F)Y = -B(X, PY) + fB(X, Y),$$

$$(\nabla_X t)V = -PA_V X + A_{fV}X, \quad (\nabla_X f)V = -FA_V X - B(X, tV).$$

For any vector fields X and Y in $H_x^\perp = tT(M)^\perp$ we obtain

$$A_{FX}Y = A_{FY}X. \tag{2.1}$$

We denote by R the Riemannian curvature tensor field of M. Then the *equation of Gauss* is given by

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y + g(PY, Z)PX - g(PX, Z)PY - 2g(PX, Y)PZ\} + A_{B(Y,Z)}X - A_{B(X,Z)}Y,$$

for any X, Y and Z tangent to M.

The *equation of Codazzi* of M is given by

$$g((\nabla_X A)_V Y, Z) - g((\nabla_Y A)_V X, Z) = c\{g(Y, PZ)g(X, tV) - g(X, PZ)g(Y, tV) - 2g(X, PY)g(Z, tV)\}.$$

We define the curvature tensor $R^\perp$ of the normal bundle of M by

$$R^\perp(X, Y)V = D_X D_Y V - D_Y D_X V - D_{[X, Y]}V.$$

Then we have the *equation of Ricci*

$$G(R^\perp(X, Y)V, U) + g([A_U, A_V]X, Y) = c\{g(Y, tV)g(X, tU) - g(X, tV)g(Y, tU) - 2g(X, PY)g(V, fU)\}.$$

If $R^{\perp}$ vanishes identically, the normal connection of M is said to be *flat*. If $R^{\perp}(X, Y)V = 2cg(X, PY)fV$, then the normal connection of M is said to be *semi-flat* (see [9]).

We put

$$(R(X, Y)A)_V Z = R(X, Y)A_V Z - A_{R^{\perp}(X, Y)V} Z - A_V R(X, Y)Z.$$

If $(R(X, Y)A)_V = 0$ for any X, Y and Z tangent to M and any V normal to M, then the second fundamental form A is said to be *semi-parallel*. This condition is weaker than $\nabla A = 0$. We call M a *semi-parallel CR submanifold* if its second fundamental form A is semi-parallel. We proved the following theorem [4].

Theorem A. *Let M be an n-dimensional proper CR submanifold of a complex space form $M^m(c)$, $c \neq 0$, with semi-flat normal connection. If the dimension of the holomorphic tangent space is greater than* 2, *then the second fundamental form A is not semi-parallel.*

On the condition that a CR manifold is proper, we note the following.

Remark 2.1. Let M be a complex n-dimensional ($n \geq 2$) holomorphic submanifold of a complex space form $M^m(c)$. If the normal connection of M is semi-flat, then M is either totally geodesic or M is an Einstein Kählerian hypersurface of $M^m(c)$ with scalar curvature n^2c. The latter case occurs only when $c > 0$ (see Ishihara [2]). Then the second fundamental form of M is parallel.

Let M be an n-dimensional anti-invariant submanifold of a complex space form $M^m(c)$. If the normal connection of M is semi-flat, then the normal connection of M is flat by $P = 0$. There exists an anti-invariant submanifold with flat normal connection and parallel second fundamental form. For example, $\pi(S^1(r_1) \times \cdots \times S^1(r_{n+1}))$, $\sum r_i = 1$, where $\pi : S^{2m+1} \to \mathbf{C}P^m$ is the standard fibration, is an anti-invariant submanifold with flat normal connection and parallel second fundamental form of $\mathbf{C}P^m$ (c.f. Yano-Kon [9; p. 237, Theorem 3.17]).

3. Pseudo-Parallel *CR* Submanifolds

In this section, we prove our main theorem. First we prepare some lemmas.

Lemma 3.1. *Let M be an n-dimensional proper CR submanifold of a complex space form $M^m(c)$, $c = \pm 1$, with semi-flat normal connection. If the second* *fundamental form A satisfies $R(X, Y)A = \alpha(X \wedge Y)A$ for any X and Y tangent to M, α being a function, then, for any vector V normal to M, A satisfies*

$$A_{fV}X = 0 \quad \textit{for } X \in T_x(M), \tag{3.1}$$

$$g(A_V X, Y) = 0 \quad \textit{for } X \in H_x,\ Y \in H_x^{\perp}. \tag{3.2}$$

Moreover, if the dimension h of the holomorphic tangent space > 2,

$$g(A_V X, Y) = -\frac{1}{h} \operatorname{tr}(A_V P^2) g(X, Y) \quad \textit{for } X, Y \in H_x, \tag{3.3}$$

$$PA_V = A_V P, \tag{3.4}$$

where tr *denotes the trace of an operator.*

Proof. Since $(R(X, Y)A)_V Z = \alpha((X \wedge Y)A)_V Z$ for any tangent vectors $X, Y, Z \in T_x(M)$, we have

$$\begin{aligned} R(X, Y)A_V Z &= A_{R^{\perp}(X, Y)V} Z + A_V R(X, Y)Z + \alpha((X \wedge Y)A)_V Z \\ &= 2cg(X, PY)A_{fV}Z + A_V R(X, Y)Z + \alpha g(X, A_V Z)Y \\ &\quad - \alpha g(Y, A_V Z)X - \alpha g(X, Z)A_V Y + \alpha g(Y, Z)A_V X. \end{aligned} \tag{3.5}$$

Thus we have

$$\begin{aligned} \operatorname{tr} R(X, Y)A_V A_{fV} &= 2cg(X, PY) \operatorname{tr} A_{fV}^2 + \operatorname{tr} R(X, Y)A_{fV}A_V \\ &\quad + 2\alpha g(X, A_V A_{fV} Y) - 2\alpha g(Y, A_V A_{fV} X). \end{aligned}$$

By the equation of Ricci, we have $A_{fV}A_V = A_V A_{fV}$. Thus we obtain $\operatorname{tr} A_{fV}^2 = 0$, which proves (3.1).

Using the equation of Gauss and (3.5),

$$\begin{aligned} & c(g(Y, A_V Z)X - g(X, A_V Z)Y + g(PY, A_V Z)PX \\ & \quad - g(PX, A_V Z)PY - 2g(PX, Y)PA_V Z) \\ & \quad + A_{B(Y, A_V Z)}X - A_{B(X, A_V Z)}Y \\ & = c(g(Y, Z)A_V X - g(X, Z)A_V Y + g(PY, Z)A_V PX \\ & \quad - g(PX, Z)A_V PY - 2g(PX, Y)A_V PZ) \\ & \quad + A_V A_{B(Y, Z)}X - A_V A_{B(X, Z)}Y + \alpha g(X, A_V Z)Y \\ & \quad - \alpha g(Y, A_V Z)X - \alpha g(X, Z)A_V Y + \alpha g(Y, Z)A_V X. \end{aligned} \tag{3.6}$$

We take an orthonormal basis $\{e_1, \dots, e_h, tv_1 := e_{h+1}, \dots, tv_q := e_n\}$ of $T_x(M)$, where $\{e_1, \dots, e_h\}$ is an orthonormal basis of H_x and $\{v_1, \dots, v_q\}$ is an orthonormal basis of $JH_x^{\perp}$. Then we have

$$
\begin{aligned}
c\sum_i &(g(Pe_i, A_V X)g(e_i, Y) - g(e_i, A_V X)g(Pe_i, Y) + g(P^2 e_i, A_V X)g(Pe_i, Y) \\
&- g(Pe_i, A_V X)(P^2 e_i, Y) - 2g(Pe_i, Pe_i)g(PA_V X, Y)) \\
&+ \sum_i g(A_{B(Pe_i, A_V X)} e_i, Y) - \sum_i g(A_{B(e_i, A_V X)} Pe_i, Y) \\
= c\sum_i &(g(Pe_i, X)g(A_V e_i, Y) - g(e_i, X)g(A_V Pe_i, Y) + g(P^2 e_i, X)g(A_V Pe_i, Y) \\
&- g(Pe_i, X)g(A_V P^2 e_i, Y) - 2g(Pe_i, Pe_i)g(A_V PX, Y)) \\
&+ \sum_i g(A_V A_{B(Pe_i, X)} e_i, Y) - \sum_i g(A_V A_{B(e_i, X)} Pe_i, Y) \\
&- 2\alpha g(A_V X, PY) - 2\alpha g(A_V PX, Y).
\end{aligned}
$$

By the straightforward computation,

$$
\begin{aligned}
(3.7)\qquad &(hc + 2c + \alpha)g(A_V X, PY) + (hc + 2c + \alpha)g(A_V PX, Y) \\
&- \sum_a g(A_a PA_a A_V X, Y) + \sum_a g(A_V A_a PA_a X, Y) = 0,
\end{aligned}
$$

where A_a is the second fundamental form in the direction of v_a. Similarly, putting $Y = e_i$, $Z = Pe_i$ into (3.6) and taking inner product with Y and summation,

$$
\begin{aligned}
(3.8)\qquad c\bigg(&\left(1 + \frac{\alpha}{c}\right)g(PA_V X, Y) - \mathrm{tr}(P^2 A_V)g(PX, Y) + g(P^2 A_V PX, Y) \\
&- 2g(PA_V P^2 X, Y) - \left(h + 2 + \frac{\alpha}{c}\right)g(A_V PX, Y)\bigg) \\
&+ \sum_a \mathrm{tr}(A_a A_V P)g(A_a X, Y) + \sum_a g(A_a PA_V A_a X, Y) \\
&- \sum_a g(A_V A_a PA_a X, Y) = 0.
\end{aligned}
$$

Since the normal connection of M is semi-flat, the equation of Ricci gives

$$A_a A_b X = A_b A_a X$$

for any $X \in H_x$. So we have $\mathrm{tr}(A_a A_V P) = 0$. Moreover, we obtain

$$g(A_a P A_V A_a X, Y) = -g(X, A_a A_V P A_a Y) = g(X, A_V A_a P A_a Y)$$
$$= g(A_a P A_a A_V X, Y)$$

for any $X, Y \in T_x(M)$. Thus, using (3.7) and (3.8), we have

$$-(h+1)cg(PA_V X, Y) - c\,\mathrm{tr}(P^2 A_V)g(PX, Y) + cg(P^2 A_V PX, Y) - 2cg(PA_V P^2 X, Y) = 0 \tag{3.9}$$

for any $X, Y \in T_x(M)$. When $X \in H_x^{\perp}$ and $Y \in H_x$, from (3.9),

$$g(PA_V X, Y) = -g(A_V X, PY) = 0. \tag{3.10}$$

So we have $g(A_V X, Y) = 0$ for $X \in H_x^{\perp}$ and $Y \in H_x$.

Next we consider the case that $X, Y \in H_x$. Since $PX, PY \in H_x$, using (3.9),

$$-(h-1)cg(PA_V X, Y) - cg(A_V PX, Y) - c\,\mathrm{tr}(P^2 A_V)g(PX, Y) = 0,$$
$$-(h-1)cg(A_V PX, Y) + cg(A_V X, PY) - c\,\mathrm{tr}(P^2 A_V)g(PX, Y) = 0.$$

From these equations and the assumption that $h > 2$, we get

$$g(PA_V X, Y) - g(A_V PX, Y) = 0.$$

From this and (3.10), we have $PA_V = A_V P$ for any V normal to M.

Thus we obtain

$$g(A_V X, Y) = -\frac{1}{h}\,\mathrm{tr}(A_V P^2)g(X, Y)$$

for $X, Y \in H_x$. □

By the similar method of Lemma 2.2 of [8], we have

Lemma 3.2. *Let M be a CR submanifold of $M^m(c)$ with semi-flat normal connection. If $A_{fV} = 0$ and $PA_V = A_V P$ for any vector field V normal to M, then*

$$g(A_U X, A_V Y) = cg(X, Y)g(tU, tV) - cg(FX, U)g(FY, V)$$
$$- \sum_i g(A_U tV, e_i)g(A_{Fe_i} X, Y).$$

Using these lemmas, we prove

Lemma 3.3. *Let M be an n-dimensional proper CR submanifold of a complex space form $M^m(c)$, $c = \pm 1$, with semi-flat normal connection. We suppose that the dimension h of the holomorphic tangent space > 2. If the second fundamental form A satisfies $R(X,Y)A = \alpha(X \wedge Y)A$ for any X and Y tangent to M, α being a function, then* $\dim H_x^{\perp} = 1$.

Proof. We suppose $\dim H_x^{\perp} \geq 2$. We can take an orthonormal basis $\{v_1, \dots, v_q, v_{q+1}, \dots, v_p\}$ of $T_x(M)^{\perp}$, where $v_1, \dots, v_q \in FH_x^{\perp}$ and $v_{q+1}, \dots, v_p \in N_x$. Since A_{v_1} is symmetric, taking a suitable orthonormal basis $\{e_1, \dots, e_h, e_{h+1}, \dots, e_{h+q}\}$ of $T_x(M)$, where $e_1, \dots, e_h \in H_x$ and $e_{h+1}, \dots, e_{h+q} \in H_x^{\perp}$, $A_{v_1} = A_1$ can be represented by a matrix form

$$A_1 = \left(\begin{array}{ccc|ccc} a_1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & a_1 & & & \\ \hline & & & b_1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & b_q \end{array}\right), \tag{3.11}$$

where $a_1 = -(1/h)\operatorname{tr}(A_1 P^2)$. In the following, we use integers $s, t, \dots$ for $A_1 e_s = a_1 e_s$ and $x, y, \dots$ for $A_1 e_x = b_x e_x$, respectively.

Putting $X = e_x$, $Y = e_y$ and $Z = e_y$ in (3.5) and taking an inner product with e_x, by the straightforward computation,

$$(b_y - b_x)(g(R(e_x, e_y)e_y, e_x) + \alpha) = 0. \tag{3.12}$$

Using (2.1), (3.2) and the equation of Gauss, for any $x \neq y$,

$$\begin{aligned} &g(R(e_x, e_y)e_y, e_x) \\ &\quad = c + g(A_{B(e_y,e_y)}e_x, e_x) - g(A_{B(e_x,e_y)}e_y, e_x) \\ &\quad = c + \sum_a g(A_a e_x, e_x) g(A_a e_y, e_y) - \sum_a g(A_a e_y, e_x) g(A_a e_x, e_y) \\ &\quad = c + \sum_a g(A_{Fe_x} t v_a, e_x) g(A_{Fe_y} t v_a, e_y) - \sum_a g(A_{Fe_y} t v_a, e_x) g(A_{Fe_x} t v_a, e_y) \\ &\quad = c + g(A_{Fe_y} e_y, A_{Fe_x} e_x) - g(A_{Fe_y} e_x, A_{Fe_x} e_y). \end{aligned}$$

From Lemma 3.2 and (2.1), we have

$$\begin{aligned} g(A_{Fe_y}e_x, A_{Fe_x}e_y) &= g(A_{Fe_x}e_y, A_{Fe_x}e_y) \\ &= c - \sum_i g(A_{Fe_x}tFe_x, e_i)g(A_{Fe_i}e_y, e_y) \\ &= c + g(A_{Fe_x}e_x, A_{Fe_y}e_y). \end{aligned}$$

From these equations, we see that $g(R(e_x, e_y)e_y, e_x) = 0$. By (3.12) and Theorem 2.1, we have $b_x = b_y$ for any $x \neq y$, that is, $A_1X = b_1X$ for any $X \in H_x^{\perp}$.

By the similar computation, we see that $A_xX = b_xX$ $(x = 2, \dots, q)$ for $X \in H_x^{\perp}$, where $b_2, \dots, b_q$ are functions. Thus we have

$$A_x tv_y = b_x tv_y.$$

On the other hand, since $A_VtU = A_UtV$ for any $U, V \in FH_x^{\perp}$, we have

$$A_x tv_y = A_y tv_x = b_y tv_x.$$

Since tv_x and tv_y are linearly independent, we have $b_1 = \cdots = b_q = 0$. So we have $[A_U, A_V]X = 0$ for any U and V normal to M and $X \in H_x^{\perp}$. Thus, by the equation of Ricci, we have

$$0 = c\{g(Y, tV)g(X, tU) - g(X, tV)g(Y, tU)\}$$

for any $X, Y \in H_x^{\perp}$. Since $\dim H_x^{\perp} \geq 2$, we can take U and V orthogonal to each other. Putting $X = tU$ and $Y = tV$, we have $c = 0$. This is a contradiction. Consequently, we obtain $\dim H_x^{\perp} = 1$. □

Lemma 3.4. *Let M be an n-dimensional proper CR submanifold of a complex space form $M^m(c)$, $c = \pm 1$, with semi-flat normal connection. We suppose that the dimension h of the holomorphic tangent space > 2. If the second fundamental form A satisfies $R(X, Y)A = \alpha(X \wedge Y)A$ for any X and Y tangent to M, α being a function, then M is a hypersurface of totally geodesic $M^{(n+1)/2}(c)$ in $M^m(c)$.*

Proof. We prove that the first normal space has constant dimension and it is parallel with respect to the normal connection.

If $A_V = 0$ for $V \in FH_x^{\perp}$, then (3.1) implies that M is totally geodesic. This contradicts $c \neq 0$. Thus we have $A_V \neq 0$. We see that $N_0(x) = N_x$ and the first normal space $N_1(x) = FH_x^{\perp}$ is of dimension 1. For $V \in FH_x^{\perp}$ and $U \in N_x$, we have

$$g(D_XV, fU) = -g(V, (\nabla_X f)U) = -g(V, -FA_UX - B(X, tU)) = 0.$$

Thus we see that $D_X V \in FH_x^{\perp}$. So the first normal space is parallel with respect to the normal connection.

Thus we see that M is a hypersurface of totally geodesic $M^{(n+1)/2}(c)$ in $M^m(c)$ (see [9; p. 77]). □

To prove our main theorem, we use the following theorem given by Lobos and Ortega [5].

Theorem B. *Let M be a connected real hypersurface in $M^n(c)$, $n \geq 2$, $c = \pm 1$ which satisfies $R(X, Y)A = \alpha(X \wedge Y)A$ for any X and Y tangent to M, α being a function. Then α is constant and positive, and M is one of the following real hypersurfaces;*

- i) *If $c = +1$, then $\alpha = \cot^2(r)$, for $0 < r < \pi/2$, and M is an open subset of a geodesic hypersphere of radius r.*
- ii) *If $c = -1$, then*
  - a) *$1 < \alpha = \coth^2(r)$, for $r > 0$, and M is an open subset of a geodesic hypersphere of radius r;*
  - b) *$\alpha = 1$, and M is an open subset of a horosphere;*
  - c) *$0 < \alpha = \tanh^2(r) < 1$, for $r > 0$, and M is an open subset of a tube of radius r over a totally geodesic $\mathbf{C}H^{n-1}$.*

From Lemma 3.4 and Theorem B, we obtain our main theorem.

Theorem 3.5. *Let M be an n-dimensional proper CR submanifold of a complex space form $M^m(c)$, $c = \pm 1$, with semi-flat normal connection. We suppose that the dimension h of the holomorphic tangent space > 2. If the second fundamental form A satisfies $R(X, Y)A = \alpha(X \wedge Y)A$ for any X and Y tangent to M, α being a function, then α is constant and M is one of the following hypersurfaces of totally geodesic $M^{(n+1)/2}(c)$ in $M^m(c)$;*

- i) *If $c = +1$, then $\alpha = \cot^2(r)$, for $0 < r < \pi/2$, and M is an open subset of a geodesic hypersphere of radius r.*
- ii) *If $c = -1$, then*
  - a) *$1 < \alpha = \coth^2(r)$, for $r > 0$, and M is an open subset of a geodesic hypersphere of radius r;*
  - b) *$\alpha = 1$, and M is an open subset of a horosphere;*
  - c) *$0 < \alpha = \tanh^2(r) < 1$, for $r > 0$, and M is an open subset of a tube of radius r over a totally geodesic $\mathbf{C}H^{(n-1)/2}$.*

References

[1] T. Hamada, On real hypersurfaces of a complex projective space with recurrent second fundamental form, Ramanujan Math. Soc. **11** (1996), 103–107.
[2] I. Ishihara, Kaehler submanifolds satisfying a certain condition on normal bundle, Atti della Accademia Nazionale dai Lincei LXII (1977), 30–35.
[3] M. Kon, Ricci recurrent *CR* submanifolds of a complex space form, Tsukuba J. Math. **31** (2007), 233–252.
[4] M. Kon, Semi-parallel *CR* submanifolds in a complex space form, Colloquium Mathematicum **124** (2011), 237–246.
[5] G. A. Lobos and M. Ortega, Pseudo-parallel real hypersurfaces in complex space form, Bull. Korean Math. Soc. **41** (2004), 609–618.
[6] S. Maeda, Real hypersurfaces of complex projective spaces, Math. Ann. **263** (1983), 473–478.
[7] R. Niebergall and P. J. Ryan, Semi-parallel and semi-symmetric real hypersurfaces in complex space forms, Kyungpook Math. J. **38** (1998), 227–234.
[8] K. Yano and M. Kon, *CR* submanifolds of a complex projective space, J. Diff. Geom. **16** (1981), 431–444.
[9] K. Yano and M. Kon, Structures on manifolds, World Scientific Publishing, Singapore, 1984.

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