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LOOP GROUPS $SL_2(\mathbf{F}[X, X^{-1}])$, UNIVERSAL CENTRAL EXTENSIONS AND ADDITIVE STEINBERG SYMBOLS

By

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Abstract. We determine the group presentations of universal central extensions derived from loop groups, where loop groups are Chevalley groups over Laurent polynomial rings. We also show that the universal central extensions have Tits systems. For our purpose, we introduce additive Steinberg symbols.

0. Introduction

Let A be an $n \times n$ Cartan matrix of finite type and Φ the root system defined from A (cf. [6]). We can construct $G_{sc}(A, -)$, the Chevalley-Demazure group scheme of universal type associated to A, i.e., the representable covariant functor from the category of commutative rings with 1 to that of groups (cf. [1], [4], [5]). It is known that if $A = \mathrm{A}_n$, $G_{sc}(\mathrm{A}_n, -)$ is isomorphic to $SL_{n+1}(-)$.

For $\mathbf{F}[X, X^{-1}]$, a Laurent polynomial ring over an arbitrary field $\mathbf{F}$, we call $G_{sc}(A, \mathbf{F}[X, X^{-1}])$ the loop group of type A. It is known that if R is a field, a polynomial ring, a Laurent polynomial ring, the ring of integers or a semi-local ring all of whose residue fields are infinite, $G_{sc}(A, R)$ is isomorphic to the group generated by $x_\gamma(t)$ ($\gamma \in \Phi$, $t \in R$) with the defining relations (A), (B), (B)$'$ and (C) (cf. [2], [8], [19]). The relation (A) represents the additivity of each root subgroups and the relation (C) represents the multiplicativity of each one dimensional tori.

Let R be an arbitrary commutative ring with 1. We define $St(A, R)$, the Steinberg group over R, to be a group generated by $\bar{x}_\gamma(t)$ ($\gamma \in \Phi$, $t \in R$) with the defining relations (A), (B) and (B)$'$ (cf. [14]). In 1960's, R. Steinberg has shown

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that if R is a field with sufficiently many elements, $St(A, R)$ is centrally closed, i.e., the identity map of $St(A, R)$ is its universal central extension (cf. [15], [16]). In general, M. R. Stein has shown that $St(A, R)$ is centrally closed provided that Rank $A \geq 4$ and $A \neq \mathrm{D}_4, \mathrm{F}_4$ (cf. [14]). In the case of Rank A being small, we can also prove that $St(A, R)$ is centrally closed provided that R has many units since the R. Steinberg's arguments can be applied to them.

We can construct a canonical homomorphism from $St(A, R)$ to $G_{sc}(A, R)$ and we denote the kernel of the homomorphism by $K_2(A, R)$. From the facts mentioned in the previous paragraph, $St(A, R)$ is the universal central extension of $G_{sc}(A, R)$ and $K_2(A, R)$ is the Schur multiplier of $G_{sc}(A, R)$ provided that R is a field, a polynomial ring, a Laurent polynomial ring or a semi-local ring all of whose residue fields are infinite, and has many units. For the case of R being a field, the group presentation of $K_2(A, R)$ has been given by H. Matsumoto [10]. In his presentation (called a Matsumoto type presentation), he uses the Steinberg symbols, which come from the relation (C). For the case of R being a Laurent polynomial ring, the structures of $K_2(A, R)$ have been studied by J. Morita [12] and its Matsumoto type presentation has been given by M. Tomie [18]. W. van der Kallen has given a Matsumoto type presentation for the case of R being U-irreducible (cf. [19]).

Let $\mathbf{F}$ be a field and let G' be a group and κ a central extension from G' to $St(A, \mathbf{F})$. To prove the universality of $St(A, \mathbf{F})$, R. Steinberg has detected $x'_\gamma(t)$ ($\gamma \in \Phi$, $t \in \mathbf{F}$) in G' and he has shown that these elements satisfy the relations (A), (B) and (B)$'$. If Rank Φ and the cardinality of $\mathbf{F}$ are small, however, there are several cases where his method cannot be applied. For example, if G' is a universal central extension of $St(\mathrm{A}_1, \mathbf{F}_9)$, the relation (A) doesn't hold since $x'_\gamma(t)x'_\gamma(t')x'_\gamma(t+t')^{-1} \neq 1$ for some $t, t' \in \mathbf{F}$. To treat the universal central extensions comprehensively, we introduce additive Steinberg symbols $\hat{\theta}_\gamma(t, t')$, which correspond to $x'_\gamma(t)x'_\gamma(t')x'_\gamma(t+t')^{-1}$. Using the additive Steinberg symbols, we will show that the universal central extension of $G_{sc}(\mathrm{A}_1, \mathbf{F}[X, X^{-1}])$ is isomorphic to the group generated by $\hat{x}_\pm(t)$ ($t \in \mathbf{F}[X, X^{-1}]$) with the following defining relations:

(B)$'$ $\hat{n}_\pm(v)\hat{x}_\pm(t)\hat{n}_\pm(v)^{-1} = \hat{x}_\mp(-v^{-2}t)$,
($\hat{\theta}1$) $\hat{\theta}_\pm(t, t')$ is central,
($\hat{\theta}2$) $\hat{\theta}$ is biadditive,
($\hat{\theta}3$) $\hat{\theta}_\pm(t, t') = \hat{\theta}_\pm(\alpha^2 t', t)$,

where $t, t' \in \mathbf{F}[X, X^{-1}]$, $v \in \mathbf{F}[X, X^{-1}]^\times$, $\hat{\theta}_\pm(t, t') = \hat{x}_\pm(t)\hat{x}_\pm(t')\hat{x}_\pm(t+t')^{-1}$, $\hat{n}_\pm(v) = \hat{x}_\pm(v)\hat{x}_\mp(v^{-1})^{-1}\hat{x}_\pm(v)$ and α is a prefixed element of $\mathbf{F}$.

We will also show that the universal central extension of $G_{sc}(\mathrm{A}_1, \mathbf{F}[X, X^{-1}])$ has a Tits system.

Notations used here are as follows. For elements g, g' of a group G, we denote the commutator $gg'g^{-1}g'^{-1}$ by $[g, g']$, the commutator group of G by $[G, G]$ and the center of G by $Z(G)$. The symbol $\langle \cdots \rangle$ means that the group is generated by $\cdots$. For a set D, we denote the cardinality of D by $\#D$.

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1. Central Extensions

Let G and G' be groups and κ a group homomorphism from G' onto G. A pair (G', κ) is called a central extension of G if $\mathrm{Ker}\,\kappa$ is included in $Z(G')$. We abbreviate (G', κ) to G' provided that no confusion occurs.

Notations are as above. The pair (G', κ) is called the universal central extension of G if the following conditions are satisfied:

(UC1) $G' = [G', G']$,

(UC2) Let (G'', κ') be an another central extension of G. Then, there exist a group homomorphism $\kappa'' : G' \to G''$ such that $\kappa = \kappa' \circ \kappa''$.

It is well-known that a universal central extension of G is uniquely determined up to isomorphism. It is easily shown that κ'' of (UC2) is uniquely determined.

From the definition, G must be perfect if a universal central extension of G exists. The next proposition indicates that the opposite direction also holds.

Proposition 1.1 ([16], [17]). *Let G be a perfect group. Then, its universal central extension exists.*

Let G be a group. We call G centrally closed if (G, id) is a universal central extension of G. If G is centrally closed, G must be perfect.

Proposition 1.2 ([16], [17]). *Let G be a group and (G', κ) a central extension of G. Then, the following conditions are equivalent*:

(a) *(G', κ) is a universal central extension of G,*

(b) *G' is centrally closed,*

(c) *Let (G'', κ') be a central extension of G'. Then, (G'', κ') is split, i.e., there exists a group homomorphism $\kappa'' : G' \to G''$ such that $\kappa' \circ \kappa'' = \mathrm{id}$.*

Let G be a group, (G', κ) a central extension of G and (G'', κ') a central extension of G'. Then, is $(G'', \kappa \circ \kappa')$ a central extension of G? In general, it doesn't hold. For example, set $G = \{1\}$ and

$$G'' = \left\{ \begin{pmatrix} 1 & d & g \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \middle| d, g, k \in \mathbf{F}_2 \right\}, \quad L = \left\{ \begin{pmatrix} 1 & 0 & g \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| g \in \mathbf{F}_2 \right\}, \quad G' = G''/L.$$

But the next proposition holds.

Proposition 1.3. *Let G be a group, (G', κ) a central extension of G and (G'', κ') a central extension of G'. Assume that G' is perfect (hence, G is also perfect). Then, $(G'', \kappa' \circ \kappa)$ is a central extension of G.*

To prove Proposition 1.3, we use the next lemmas.

Lemma 1.4 ([16]). *Let G be a perfect group and (G', κ) a central extension of G. Then, $G' = (\mathrm{Ker}\, \kappa) \cdot [G', G']$ and $[G', G']$ is a perfect group.*

Lemma 1.5. *Let G be a group and g an element of G. Assume that $[g, g']$ is included in $Z(G)$ for all $g' \in G$. Define a map κ from G to $Z(G)$ by $g' \mapsto [g, g']$. Then, κ is a group homomorphism. In particular, if G is perfect, g must be central in G.*

Proof. for $g', g'' \in G$,

$$[g, g'g''] = gg'g^{-1}(gg''g^{-1}g''^{-1})g'^{-1} = gg'g^{-1}g'^{-1}[g, g''] = [g, g'][g, g'']. \qquad \square$$

Proof of Proposition 1.3. Let g be an element of $\mathrm{Ker}(\kappa \circ \kappa')$. From Lemma 1.4, there exists $g' \in \mathrm{Ker}\, \kappa'$ and $g'' \in [G'', G'']$ such that $g = g'g''$. To prove Proposition 1.3, it is enough to check that g'' is central in $[G'', G'']$.

For all $d \in G''$, $[g'', d] = [g, d] \in Z(G'')$ because $\kappa'(g)$ is central in G'. The restriction of κ' to $[G'', G'']$ is also a central extension of G' since G' is perfect. Applying Lemma 1.5, we can prove Proposition 1.3. $\square$

Using Proposition 1.2 and Proposition 1.3, we have the next corollaries.

Corollary 1.6. *Let G be a perfect group and (G', κ) be a central extension of G with G' being perfect. Let (G'', κ') be a universal central extension of G'. Then, $(G'', \kappa' \circ \kappa)$ is a universal central extension of G.*

Corollary 1.7. *Let G be a perfect group and (G', κ) a central extension of G. Then,*

(a) $\kappa^{-1}(Z(G)) = Z(G')$.
(b) *Let* $g, g', d, d' \in G'$ *and assume that* $\kappa(g)\kappa(g')^{-1}, \kappa(d)\kappa(d')^{-1} \in Z(G)$. *Then,* $[g, d] = [g', d']$.

Proof. (a) $\kappa^{-1}(Z(G)) \supset Z(G')$ is obvious. Let κ' be the canonical group homomorphism from G to $G/Z(G)$. Then, (G, κ') is a central extension of $G/Z(G)$ and $\kappa^{-1}(Z(G)) = \mathrm{Ker}(\kappa' \circ \kappa)$. We have $\kappa^{-1}(Z(G)) \subset Z(G')$ from Proposition 1.3.

(b) is obvious from (a). □

2. The Loop Groups and the Steinberg Groups

Let $\mathbf{F}$ be an arbitrary field. The group $SL_{n+1}(\mathbf{F}[X, X^{-1}])$ is called the loop group over $\mathbf{F}$ (of type A_n). In this article, we treat only the case where $n = 1$, i.e., $SL_2(\mathbf{F}[X, X^{-1}])$. In general, for an arbitrary root system of finite type, we can construct the loop group using Chevalley-Demazure group scheme (cf. [4], [5]).

It is known that $SL_2(\mathbf{F}[X, X^{-1}])$ (resp. $SL_2(\mathbf{F})$) is isomorphic to the group generated by $x_\pm(t)$ $(t \in \mathbf{F}[X, X^{-1}])$ (resp. $t \in \mathbf{F}$) with the following defining relations ([2], [8]):

(A) $x_\pm(t)x_\pm(t') = x_\pm(t + t')$,
(B)$'$ $n_\pm(v)x_\pm(t)n_\pm(v)^{-1} = x_\mp(-v^{-2}t)$,
(C) $h_\pm(v)h_\pm(v') = h_\pm(vv')$,

where $t, t' \in \mathbf{F}[X, X^{-1}]$ (resp. $t, t' \in \mathbf{F}$), $v, v' \in \mathbf{F}[X, X^{-1}]^\times$ (resp. $v, v' \in \mathbf{F}^\times$), $n_\pm(v) = x_\pm(v)x_\mp(v^{-1})^{-1}x_\pm(v)$ and $h_\pm(v) = n_\pm(v)n_\pm(1)^{-1}$. We denote the multiplicative group of $\mathbf{F}[X, X^{-1}]$ (resp. $\mathbf{F}$) by $\mathbf{F}[X, X^{-1}]^\times$ (resp. $\mathbf{F}^\times$). For $t \in \mathbf{F}[X, X^{-1}]$ (resp. $t \in \mathbf{F}$), the generators $x_+(t)$, $x_-(t)$ correspond to:

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

respectively in $SL_2(\mathbf{F}[X, X^{-1}])$ (resp. $SL_2(F)$).

Let R be a commutative ring with 1. We define the Steinberg group $St_2(R)$ over R (of type A_1) to be the group generated by $\bar{x}_\pm(t)$ $(t \in R)$ with the defining relations (A) and (B)$'$, where $x_\pm$ is replaced by $\bar{x}_\pm$. In this article, we denote the generators of $SL_2(\mathbf{F}[X, X^{-1}])$ by $x_\pm(t)$ $(t \in \mathbf{F}[X, X^{-1}])$ and the generators of $St_2(R)$ by $\bar{x}_\pm(t)$ $(t \in R)$. The following propositions are easily proven.

Proposition 2.1 ([18]). *Let* $\mathbf{F}$ *be an arbitrary field. The following formulas hold in* $SL_2(\mathbf{F}[X, X^{-1}])$:

(a) $x_\pm(0) = 1,\ x_\pm(t)^{-1} = x_\pm(-t),\ n_\pm(v)^{-1} = n_\pm(-v)$,
(b) $h_\pm(1) = 1,\ h_\pm(v)^{-1} = h_\pm(v^{-1})$,
(c) $n_\pm(v) = n_\mp(-v^{-1}),\ h_\pm(v) = h_\mp(v^{-1})$,
(d) $n_\pm(v)n_\pm(v')n_\pm(v)^{-1} = n_\mp(-v^{-2}v')$,
(e) $n_\pm(v)h_\pm(v')n_\pm(v)^{-1} = h_\mp(v')$,
(f) $h_\pm(v)x_\pm(t)h_\pm(v)^{-1} = x_\pm(v^2t)$,
(g) $h_\pm(v)n_\pm(v')h_\pm(v)^{-1} = n_\pm(v^2v')$,
(h) $h_\pm(v)h_\pm(v')h_\pm(v)^{-1} = h_\pm(v')$,

where $t \in \mathbf{F}[X, X^{-1}]$ *and* $v, v' \in \mathbf{F}[X, X^{-1}]^\times$.

Proposition 2.2 ([18]). *Let* R *be an arbitrary commutative ring with* 1. *For* $v, v' \in R^\times$, *we define*

$$\bar{n}_\pm(v) = \bar{x}_\pm(v)\bar{x}_\mp(v^{-1})^{-1}\bar{x}_\pm(v),$$

$$\bar{h}_\pm(v) = \bar{n}_\pm(v)\bar{n}_\pm(1)^{-1},$$

$$\bar{c}_\pm(v, v') = \bar{h}_\pm(v)\bar{h}_\pm(v')\bar{h}_\pm(vv')^{-1}.$$

Then, the following formulas hold in $St_2(R)$:

(a) $\bar{x}_\pm(0) = 1,\ \bar{x}_\pm(t)^{-1} = \bar{x}_\pm(-t),\ \bar{n}_\pm(v)^{-1} = \bar{n}_\pm(-v)$,
(b) $\bar{h}_\pm(1) = 1,\ \bar{h}_\pm(v)^{-1} = \bar{h}_\pm(v^{-1})\bar{c}_\pm(v, v^{-1})^{-1}$,
(c) $\bar{n}_\pm(v) = \bar{n}_\mp(-v^{-1}),\ \bar{h}_\pm(v) = \bar{h}_\mp(v^{-1})\bar{c}_\mp(v, -1)^{-1}$,
(d) $\bar{n}_\pm(v)\bar{n}_\pm(v')\bar{n}_\pm(v)^{-1} = \bar{n}_\mp(-v^{-2}v')$,
(e) $\bar{n}_\pm(v)\bar{h}_\pm(v')\bar{n}_\pm(v)^{-1} = \bar{h}_\mp(v')\bar{c}_\mp(v', -v^{-2})^{-1}$,
(f) $\bar{h}_\pm(v)\bar{x}_\pm(t)\bar{h}_\pm(v)^{-1} = \bar{x}_\pm(v^2t)$,
(g) $\bar{h}_\pm(v)\bar{n}_\pm(v')\bar{h}_\pm(v)^{-1} = \bar{n}_\pm(v^2v')$,
(h) $\bar{h}_\pm(v)\bar{h}_\pm(v')\bar{h}_\pm(v)^{-1} = \bar{h}_\pm(v')\bar{c}_\pm(v', v^2)^{-1}$,
(i) $\bar{c}_\pm(v, v') \in Z(St(R))$,

where $t \in R$ *and* $v, v' \in R^\times$.

There is a group homomorphism $\bar{\varphi}$ from $St_2(\mathbf{F}[X, X^{-1}])$ to $SL_2(\mathbf{F}[X, X^{-1}])$ given by $\bar{\varphi}(\bar{x}_\pm(t)) = x_\pm(t)$ $(t \in \mathbf{F}[X, X^{-1}])$. The pair $(St_2(\mathbf{F}[X, X^{-1}]), \bar{\varphi})$ is obviously a central extension of $SL_2(\mathbf{F}[X, X^{-1}])$.

If $\mathbf{F}$ is $\mathbf{F}_2$ or $\mathbf{F}_3$, $SL_2(\mathbf{F}[X, X^{-1}])$ is not perfect since there is a group homomorphism from $SL_2(\mathbf{F}[X, X^{-1}])$ onto $SL_2(\mathbf{F})$ and $SL_2(\mathbf{F})$ is not perfect.

On the other hand, if $\mathbf{F}$ has more than 4 elements, we can choose a element $u \in \mathbf{F}^\times$ such that $u^2 - 1 \in \mathbf{F}^\times$. For all $t \in \mathbf{F}[X, X^{-1}]$,

$$\begin{aligned}[h_\pm(u), x_\pm(t(u^2-1)^{-1})] &= h_\pm(u)x_\pm(t(u^2-1)^{-1})h_\pm(u)^{-1}x_\pm(t(u^2-1)^{-1})^{-1} \\ &= x_\pm(tu^2(u^2-1)^{-1})x_\pm(-t(u^2-1)^{-1}) \\ &= x_\pm(t).\end{aligned}$$

Thus, $SL_2(\mathbf{F}[X, X^{-1}])$ is perfect if $\mathbf{F}$ has more than 4 elements and its universal central extension exists (proposition 1.1). We will give a group presentation of the universal central extension in Section 3.

Next, we introduce Tits systems. Let G' be a group and B', N' subgroups of G'. Let S' be a subset of all left cosets set $N'/(B' \cap N')$. We call (G', B', N', S') a Tits system if the following conditions are satisfied (cf. [3], [13], [17]):

(T1) G' is generated by B', N', and $T' = B' \cap N' \lhd N'$,
(T2) $N'/T' = \langle S \rangle$, the order of s is 2 for all $s \in S$,
(T3) $s'B'w' \subset B's'w'B' \cup B'w'B'$ for all $s' \in S'$, $w' \in N'/T'$,
(T4) $s'B's' \not\subset B'$.

We call N'/T' the Weyl group of the Tits system. It is known that N'/T' is a Coxeter group (cf. [7]).

Proposition 2.3 (Bruhat decomposition). *Let (G', B', N', S') be a Tits system. Then, we have*

$$G' = \bigcup_{\overline{w'} \in N'/T'} B'w'B'.$$

This is a disjoint union and N'/T' is in bijective correspondence with the double cosets $B' \backslash G'/B'$.

We define the following subgroups of $SL_2(\mathbf{F}[X, X^{-1}])$ by:

$$\begin{aligned} U_{(\pm,m)} &= \langle x_\pm(sX^m) \mid s \in \mathbf{F} \rangle \quad (\text{for all } m \in \mathbf{Z}), \\ U &= \langle U_{(+,k)}, U_{(-,k')} \mid k \in \mathbf{Z}_{\geq 0}, k' \in \mathbf{Z}_{\geq 1} \rangle, \\ T &= \langle h_\pm(u) \mid u \in \mathbf{F}^\times \rangle, \\ B &= \langle U, T \rangle, \\ N &= \langle n_\pm(v) \mid v \in \mathbf{F}[X, X^{-1}]^\times \rangle. \end{aligned}$$

Set $S = \{n_+(1), n_-(X) \text{ Mod } T\}$. Then,

Proposition 2.4 ([11]). *The quadruplet* $(SL_2(\mathbf{F}[X, X^{-1}]), B, N, S)$ *is a Tits system,* $B \cap N = T$ *and* N/T *is isomorphic to the infinite dihedral group.*

3. The Universal Central Extension of the Loop Group $SL_2(\mathbf{F}[X, X^{-1}])$

Let $\mathbf{F}$ be a field with more than 4 elements. As mentioned in the last of Section 2, $SL_2(\mathbf{F}[X, X^{-1}])$ is perfect and its universal central extension exists. If $\#\mathbf{F}$ is large enough, we can prove that $St_2(\mathbf{F}[X, X^{-1}])$ is a universal central extension of $SL_2(\mathbf{F}[X, X^{-1}])$ as in R. Steinberg's paper [15], in which he proved that if $\mathbf{F} \neq \mathbf{F}_4, \mathbf{F}_9$, $St_2(\mathbf{F})$ is a universal central extension of $SL_2(\mathbf{F})$. On the other hand, if $\mathbf{F} = \mathbf{F}_4$ or $\mathbf{F}_9$, $St_2(\mathbf{F})$ is not a universal central extension of $SL_2(\mathbf{F})$.

In this section, we assume that R is a commutative ring with 1 such that there exists $\alpha \in R^\times$ satisfying $\alpha^2 - 1 \in R^\times$. We fix such an element α for each R. Define $\widehat{St}_2(R)$ to be the group generated by $\hat{x}_\pm(t)$ $(t \in R)$ with the following defining relations:

(B)$'$ $\hat{n}_\pm(v)\hat{x}_\pm(t)\hat{n}_\pm(v)^{-1} = \hat{x}_\mp(-v^{-2}t)$,
($\hat{\theta}1$) $\hat{\theta}_\pm(t, t')$ is central,
($\hat{\theta}2$) $\hat{\theta}$ is biadditive, i.e.,
$\hat{\theta}_\pm(t, t')\hat{\theta}_\pm(t, t'') = \hat{\theta}_\pm(t, t' + t'')$,
$\hat{\theta}_\pm(t, t'')\hat{\theta}_\pm(t', t'') = \hat{\theta}_\pm(t + t', t'')$,
($\hat{\theta}3$) $\hat{\theta}_\pm(t, t') = \hat{\theta}_\pm(\alpha^2 t', t)$,

where $t, t' \in R$, $v \in R^\times$, $\hat{n}_\pm(v) = \hat{x}_\pm(v)\hat{x}_\mp(v^{-1})^{-1}\hat{x}_\pm(v)$ and $\hat{\theta}_\pm(t, t') = \hat{x}_\pm(t)\hat{x}_\pm(t')\hat{x}_\pm(t + t')^{-1}$. We define:

$$\hat{h}_\pm(v) = \hat{n}_\pm(v)\hat{n}_\pm(1)^{-1},$$

$$\hat{c}_\pm(v, v') = \hat{h}_\pm(v)\hat{h}_\pm(v')\hat{h}_\pm(vv')^{-1}.$$

We can construct the canonical group homomorphism $\hat{\varphi}$ from $\widehat{St}_2(R)$ onto $St_2(R)$ given by $\hat{\varphi}(\hat{x}_\pm(t)) = \bar{x}_\pm(t)$ $(t \in R)$. Obviously, $(\widehat{St}_2(R), \hat{\varphi})$ is a central extension of $St_2(R)$, whose kernel is the subgroup of $\widehat{St}_2(R)$ generated by $\hat{\theta}_\pm(t, t')$ $(t, t' \in R)$. As in the last of Section 2, we can prove that $St_2(R)$ is perfect and its universal central extension exists. We will show that $\widehat{St}_2(R)$ is a universal central extension of $St_2(R)$, later.

The group presentation of $\widehat{St}_2(R)$ depends on the fixed element α. Therefore, $\widehat{St}_2(R)$ should be written as $\widehat{St}{}_2^\alpha(R)$. Let α' be an another such element. A universal central extension defined from α' is isomorphic to the first one since a universal central extension is uniquely determined up to isomorphism. There is a direct proof of $\widehat{St}_2^{\alpha}(R) \simeq \widehat{St}_2^{\alpha'}(R)$. See Remark 3.9 for the proof.

Proposition 3.1. *The following relations hold in* $\widehat{St}_2(R)$:

$$\text{(a)}\ \hat{h}_{\pm}(1) = 1, \tag{1}$$

$$\text{(b)}\ \hat{\theta}_{\pm}(t, 0) = \hat{\theta}_{\pm}(0, t) = 1, \tag{2}$$

$$\text{(c)}\ \hat{x}_{\pm}(0) = 1, \tag{3}$$

$$\text{(d)}\ \hat{\theta}_{\pm}(t, t')^{-1} = \hat{\theta}_{\pm}(-t, t') = \hat{\theta}_{\pm}(t, -t'), \tag{4}$$

$$\text{(e)}\ \hat{\theta}_{\pm}(t, t') = \hat{\theta}_{\pm}(-t, -t'), \tag{5}$$

$$\text{(f)}\ \hat{\theta}_{\pm}(t, t') = \hat{\theta}_{\mp}(t, t'), \tag{6}$$

$$\text{(g)}\ \hat{\theta}_{\pm}(t, t') = \hat{\theta}_{\pm}(v^2 t, v^2 t'), \tag{7}$$

$$\text{(h)}\ [\hat{x}_{\pm}(t), \hat{x}_{\pm}(t')] = \hat{\theta}_{\pm}((\alpha^2 - 1)t', t), \tag{8}$$

where $t, t' \in R$ *and* $v \in R^{\times}$.

Proof. (a): $\hat{h}_{\pm}(1) = \hat{n}_{\pm}(1)\hat{n}_{\pm}(1)^{-1} = 1$.
(b): By $(\hat{\theta}2)$, $\hat{\theta}(t, 0)^2 = \hat{\theta}(t, 0)$ and $\hat{\theta}(0, t)^2 = \hat{\theta}(0, t)$.
(c): By (2), $\hat{x}_{\pm}(0) = \hat{x}_{\pm}(0)\hat{x}_{\pm}(0)\hat{x}_{\pm}(0)^{-1} = \hat{\theta}(0, 0) = 1$.
(d): By $(\hat{\theta}2)$ and (2),

$$\hat{\theta}_{\pm}(t, t')\hat{\theta}_{\pm}(-t, t') = \hat{\theta}_{\pm}(0, t') = 1,$$

$$\hat{\theta}_{\pm}(t, t')\hat{\theta}_{\pm}(t, -t') = \hat{\theta}_{\pm}(t, 0) = 1.$$

(e) is obvious from (4).
(f) is proven by:

$$\begin{aligned}\hat{\theta}_{\pm}(t, t') &= \hat{n}_{\pm}(1)\hat{x}_{\pm}(t)\hat{x}_{\pm}(t')\hat{x}_{\pm}(t + t')^{-1}\hat{n}_{\pm}(1)^{-1} \quad \text{by } (\hat{\theta}1)\\ &= \hat{x}_{\mp}(-t)\hat{x}_{\mp}(-t')\hat{x}_{\mp}(-(t + t'))^{-1} \quad \text{by (B)}'\\ &= \hat{\theta}_{\mp}(-t, -t')\\ &= \hat{\theta}_{\mp}(t, t') \quad \text{by (5)}.\end{aligned}$$

(g) is proven by:

$$\begin{aligned}\hat{\theta}_{\pm}(t,t') &= \hat{n}_{\pm}(v^{-1})\hat{x}_{\pm}(t)\hat{x}_{\pm}(t')\hat{x}_{\pm}(t+t')^{-1}\hat{n}_{\pm}(v^{-1})^{-1} \quad \text{by } (\hat{\theta}1)\\ &= \hat{x}_{\mp}(-v^2t)\hat{x}_{\mp}(-v^2t')\hat{x}_{\mp}(-v^2(t+t'))^{-1} \quad \text{by } (\mathrm{B})'\\ &= \hat{\theta}_{\mp}(-v^2t,-v^2t')\\ &= \hat{\theta}_{\pm}(v^2t,v^2t') \quad \text{by } (5)(6).\end{aligned}$$

(h) is proven by:

$$\begin{aligned}[\hat{x}_{\pm}(t),\hat{x}_{\pm}(t')] &= (\hat{x}_{\pm}(t)\hat{x}_{\pm}(t')\hat{x}_{\pm}(t+t')^{-1})(\hat{x}_{\pm}(t+t')\hat{x}_{\pm}(t)^{-1}\hat{x}_{\pm}(t')^{-1})\\ &= \hat{\theta}_{\pm}(t,t')\hat{\theta}_{\pm}(t',t)^{-1}\\ &= \hat{\theta}_{\pm}(\alpha^2t',t)\hat{\theta}_{\pm}(-t',t) \quad \text{by } (\hat{\theta}3)(4)\\ &= \hat{\theta}_{\pm}((\alpha^2-1)t',t) \quad \text{by } (\hat{\theta}2).\end{aligned}$$

□

Proposition 3.2. *The following relations hold in* $\widehat{St}_2(R)$:

$$\text{(a)}\ \hat{x}_{\pm}(t)^{-1} = \hat{x}_{\pm}(-t)\hat{\theta}_{\pm}(t,t), \tag{9}$$

$$\text{(b)}\ \hat{n}_{\pm}(v)^{-1} = \hat{n}_{\pm}(-v)\hat{\theta}_{\pm}(v,v), \tag{10}$$

$$\text{(c)}\ \hat{h}_{\pm}(v)^{-1} = \hat{h}_{\pm}(v^{-1})\hat{c}_{\pm}(v,v^{-1})^{-1}, \tag{11}$$

$$\text{(d)}\ \hat{n}_{\pm}(v) = \hat{n}_{\mp}(-v^{-1}), \tag{12}$$

$$\text{(e)}\ \hat{h}_{\pm}(v) = \hat{c}_{\mp}(v^{-1},-1)^{-1}\hat{h}_{\mp}(v^{-1}), \tag{13}$$

$$\text{(f)}\ \hat{x}_{\pm}(v) = \hat{x}_{\mp}(v^{-1})\hat{n}_{\mp}(-v^{-1})\hat{x}_{\mp}(v^{-1})\hat{\theta}_{\mp}(v,v), \tag{14}$$

where $t \in R$ *and* $v \in R^{\times}$.

Proof. (a) is proven by:

$$\begin{aligned}\hat{x}_{\pm}(t)(\hat{x}_{\pm}(-t)\hat{\theta}_{\pm}(t,t)) &= \hat{\theta}_{\pm}(t,-t)\hat{\theta}_{\pm}(t,t) \quad \text{by } (3)\\ &= 1 \quad \text{by } (\hat{\theta}2)(2).\end{aligned}$$

(b) is proven by:

$$\begin{aligned}\hat{n}_{\pm}(v)^{-1} &= \hat{x}_{\pm}(v)^{-1}\hat{x}_{\mp}(v^{-1})\hat{x}_{\pm}(v)^{-1}\\ &= \hat{x}_{\pm}(-v)\hat{x}_{\mp}(-v^{-1})^{-1}\hat{x}_{\pm}(-v)\hat{\theta}_{\mp}(-v^{-1},-v^{-1})^{-1}\hat{\theta}_{\pm}(v,v)^2 \quad \text{by } (\hat{\theta}1)(9)\end{aligned}$$

$$= \hat{n}_\pm(-v)\hat{\theta}_\mp(-v,-v)^{-1}\hat{\theta}_\pm(v,v)^2 \quad \text{by (7)}$$

$$= \hat{n}_\pm(-v)\hat{\theta}_\pm(v,v) \quad \text{by (5)(6).}$$

(c): By (1), $\hat{h}_\pm(v)\hat{h}_\pm(v^{-1})\hat{c}_\pm(v,v^{-1})^{-1} = \hat{c}_\pm(v,v^{-1})\hat{c}_\pm(v,v^{-1})^{-1} = 1$.
(d) is proven by:

$$\hat{n}_\pm(v) = \hat{n}_\pm(v)\hat{n}_\pm(v)\hat{n}_\pm(v)^{-1} = \hat{n}_\pm(v)\hat{x}_\pm(v)\hat{x}_\mp(v^{-1})^{-1}\hat{x}_\pm(v)\hat{n}_\pm(v)^{-1}$$

$$= \hat{x}_\mp(-v^{-1})\hat{n}_\pm(v)\hat{x}_\mp(v^{-1})^{-1}\hat{n}_\pm(v)^{-1}\hat{x}_\mp(-v^{-1}) \quad \text{by (B)}'$$

$$= \hat{x}_\mp(-v^{-1})\hat{n}_\pm(-v)^{-1}\hat{x}_\mp(v^{-1})^{-1}\hat{n}_\pm(-v)\hat{x}_\mp(-v^{-1}) \quad \text{by } (\hat{\theta}1)(10)$$

$$= \hat{x}_\mp(-v^{-1})\hat{x}_\pm(-v)^{-1}\hat{x}_\mp(-v^{-1}) = \hat{n}_\mp(-v^{-1}) \quad \text{by (B)}'.$$

(e): By (12),

$$\hat{h}_\pm(v) = \hat{n}_\pm(v)\hat{n}_\pm(1)^{-1} = \hat{n}_\mp(-v^{-1})\hat{n}_\mp(-1)^{-1} = \hat{h}_\mp(-v^{-1})\hat{h}_\mp(-1)^{-1}$$

$$= \hat{h}_\mp(-v^{-1})\hat{h}_\mp(-1)^{-1}\hat{h}_\mp(v^{-1})^{-1}\hat{h}_\mp(v^{-1}) = \hat{c}_\mp(v^{-1},-1)^{-1}\hat{h}_\mp(v^{-1}).$$

(f) is proven by:

$$\hat{x}_\mp(v^{-1})\hat{n}_\mp(-v^{-1})\hat{x}_\mp(v^{-1})\hat{\theta}_\mp(v,v)$$

$$= \hat{x}_\mp(v^{-1})(\hat{x}_\mp(-v^{-1})\hat{x}_\pm(-v)^{-1}\hat{x}_\mp(-v^{-1}))\hat{x}_\mp(v^{-1})\hat{\theta}_\mp(v,v)$$

$$= \hat{\theta}_\mp(v^{-1},-v^{-1})\hat{x}_\pm(-v)^{-1}\hat{\theta}_\mp(-v^{-1},v^{-1})\hat{\theta}_\mp(v,v) \quad \text{by (3)}$$

$$= \hat{\theta}_\mp(v,-v)\hat{x}_\pm(v)\hat{\theta}_\pm(-v,-v)\hat{\theta}_\mp(-v,v)\hat{\theta}_\mp(v,v) \quad \text{by (7)(9)}$$

$$= \hat{x}_\pm(v) \quad \text{by } (\hat{\theta}1)(\hat{\theta}2)(2)(6).$$

□

Proposition 3.3. *The following relations hold in* $\widehat{St}_2(R)$:

$$\text{(a)}\ \hat{x}_\pm(t)\hat{x}_\pm(t')\hat{x}_\pm(t)^{-1} = \hat{x}_\pm(t')\hat{\theta}_\pm((\alpha^2-1)t',t), \tag{15}$$

$$\text{(b)}\ \hat{n}_\varepsilon(v)\hat{x}_\delta(t)\hat{n}_\varepsilon(v)^{-1} = \hat{x}_{-\delta}(-v^{-2\varepsilon\delta}t), \tag{16}$$

$$\text{(c)}\ \hat{h}_\varepsilon(v)\hat{x}_\delta(t)\hat{h}_\varepsilon(v)^{-1} = \hat{x}_\delta(v^{2\varepsilon\delta}t), \tag{17}$$

where $t, t' \in R$, $v \in R^\times$ *and* $\varepsilon, \delta \in \{+,-\}$. *Notice that* $\varepsilon\delta = +$ *if* $\varepsilon = \delta$, *and* $\varepsilon\delta = -$ *if* $\varepsilon \neq \delta$.

Proof. (a) is obvious from $(\hat{\theta}1)$ and (8).
(b) is obvious from (B)′ and (12).

(c) is proven by:

$$\begin{aligned}\hat{h}_\varepsilon(v)\hat{x}_\delta(t)\hat{h}_\varepsilon(v)^{-1} &= \hat{n}_\varepsilon(v)\hat{n}_\varepsilon(1)^{-1}\hat{x}_\delta(t)\hat{n}_\varepsilon(1)\hat{n}_\varepsilon(v)^{-1}\\ &= \hat{n}_\varepsilon(v)\hat{x}_{-\delta}(-t)\hat{n}_\varepsilon(v)^{-1} \quad \text{by (16)}\\ &= \hat{x}_\delta(v^2 t) \quad \text{by (16).}\end{aligned}$$

□

Proposition 3.4. *The following relations hold in* $\widehat{St}_2(R)$:

$$\text{(a)}\ \hat{n}_\varepsilon(v)\hat{n}_\delta(v')\hat{n}_\varepsilon(v)^{-1} = \hat{n}_{-\delta}(-v^{-2\varepsilon\delta}v'), \tag{18}$$

$$\text{(b)}\ \hat{h}_\varepsilon(v)\hat{n}_\delta(v')\hat{h}_\varepsilon(v)^{-1} = \hat{n}_\delta(v^{2\varepsilon\delta}v'), \tag{19}$$

where $v, v' \in R^\times$ *and* $\varepsilon, \delta \in \{+,-\}$. *Notice that* $\varepsilon\delta = +$ *if* $\varepsilon = \delta$, *and* $\varepsilon\delta = -$ *if* $\varepsilon \neq \delta$.

Proof. (a) and (b) are obvious from (16) and (17). □

Proposition 3.5. *The following relations hold in* $\widehat{St}_2(R)$:

$$\text{(a)}\ \hat{c}_\pm(v,v') \text{ is central in } \widehat{St}_2(R), \tag{20}$$

$$\text{(b)}\ \hat{n}_\varepsilon(v)\hat{h}_\delta(v')\hat{n}_\varepsilon(v)^{-1} = \hat{h}_{-\delta}(v')\hat{c}_{-\delta}(v', -v^{-2\varepsilon\delta})^{-1}, \tag{21}$$

$$\text{(c)}\ \hat{n}_\varepsilon(v)^{-1}\hat{h}_\delta(v')\hat{n}_\varepsilon(v) = \hat{h}_{-\delta}(v')\hat{c}_{-\delta}(v', -v^{-2\varepsilon\delta})^{-1}, \tag{22}$$

$$\text{(d)}\ \hat{h}_\varepsilon(v)\hat{h}_\delta(v')\hat{h}_\varepsilon(v)^{-1} = \hat{h}_\delta(v')\hat{c}_\delta(v', v^{2\varepsilon\delta})^{-1}, \tag{23}$$

$$\text{(e)}\ \hat{h}_\varepsilon(v)^{-1}\hat{h}_\delta(v')\hat{h}_\varepsilon(v) = \hat{h}_\delta(v')\hat{c}_\delta(v', v^{-2\varepsilon\delta})^{-1}, \tag{24}$$

where $v, v' \in R^\times$ *and* $\varepsilon, \delta \in \{+,-\}$. *Notice that* $\varepsilon\delta = +$ *if* $\varepsilon = \delta$, *and* $\varepsilon\delta = -$ *if* $\varepsilon \neq \delta$.

Proof. (a) is obvious from (17).
(b) is proven by:

$$\begin{aligned}\hat{n}_\varepsilon(v)\hat{h}_\delta(v')\hat{n}_\varepsilon(v)^{-1} &= \hat{n}_\varepsilon(v)\hat{n}_\delta(v')\hat{n}_\delta(1)^{-1}\hat{n}_\varepsilon(v)^{-1}\\ &= \hat{n}_\varepsilon(v)\hat{n}_\delta(v')\hat{n}_\varepsilon(v)^{-1}\hat{n}_\varepsilon(v)\hat{n}_\delta(1)^{-1}\hat{n}_\varepsilon(v)^{-1}\\ &= \hat{n}_{-\delta}(-v^{-2\varepsilon\delta}v')\hat{n}_{-\delta}(-v^{-2\varepsilon\delta})^{-1} \quad \text{by (18)}\\ &= \hat{h}_{-\delta}(-v^{-2\varepsilon\delta}v')\hat{h}_{-\delta}(-v^{-2\varepsilon\delta})^{-1}\hat{h}_{-\delta}(v')^{-1}\hat{h}_{-\delta}(v')\\ &= \hat{h}_{-\delta}(v')\hat{c}_{-\delta}(v', -v^{-2\varepsilon\delta})^{-1} \quad \text{by (20).}\end{aligned}$$

(c) is obvious from ($\hat{\theta}1$), (10) and (21).

(d) is proven by:

$$\begin{aligned}\hat{h}_\varepsilon(v)\hat{h}_\delta(v')\hat{h}_\varepsilon(v)^{-1} &= \hat{h}_\varepsilon(v)\hat{n}_\delta(v')\hat{n}_\delta(1)^{-1}\hat{h}_\varepsilon(v)^{-1} \\ &= \hat{h}_\varepsilon(v)\hat{n}_\delta(v')\hat{h}_\varepsilon(v)^{-1}\hat{h}_\varepsilon(v)\hat{n}_\delta(1)^{-1}\hat{h}_\varepsilon(v)^{-1} \\ &= \hat{n}_\delta(v^{2\varepsilon\delta}v')\hat{n}_\delta(v^{2\varepsilon\delta})^{-1} \quad \text{by (19)} \\ &= \hat{h}_\delta(v^{2\varepsilon\delta}v')\hat{h}_\delta(v^{2\varepsilon\delta})^{-1}\hat{h}_\delta(v')^{-1}\hat{h}_\delta(v') \\ &= \hat{h}_\delta(v')\hat{c}_\delta(v', v^{2\varepsilon\delta})^{-1} \quad \text{by (20)}.\end{aligned}$$

(e) is obvious from (11), (20) and (23). □

Using the relations of $\widehat{St}_2(R)$ and that of $St_2(R)$, we can prove the universality of $\widehat{St}_2(R)$.

Theorem 3.6. *$(\widehat{St}_2(R), \hat{\varphi})$ is the universal central extension of $St_2(R)$.*

The idea of the proof is in Steinberg's lecture notes [16]. We will check that $(\widehat{St}_2(R), \hat{\varphi})$ satisfies (UC1) and (UC2).

Lemma 3.7. *The group $\widehat{St}_2(R)$ is perfect.*

Proof. We have:

$$[\hat{x}_\pm(t'), \hat{x}_\pm((\alpha^2 - 1)^{-1}t)] = \hat{\theta}_\pm(t, t') \quad \text{by (8)}.$$

Hence, $\hat{\theta}_\pm(t, t') \in [\widehat{St}_2(R), \widehat{St}_2(R)]$ for all $t, t' \in R$. Then, for all $t \in R$,

$$\begin{aligned}[\hat{h}_\pm(\alpha), \hat{x}_\pm(t)] &= \hat{x}_\pm(\alpha^2 t)\hat{x}_\pm(t)^{-1} \quad \text{by (17)} \\ &= \hat{x}_\pm(\alpha^2 t)\hat{x}_\pm(-t)\hat{\theta}_\pm(t, t) \quad \text{by (9)} \\ &= \hat{x}_\pm((\alpha^2 - 1)t)\hat{\theta}_\pm(\alpha^2 t, -t)\hat{\theta}_\pm(t, t).\end{aligned}$$

Thus, $\hat{x}_\pm(t) \in [\widehat{St}_2(R), \widehat{St}_2(R)]$ for all $t \in R$. □

Let (G', κ) be a central extension of $St_2(R)$. We have to show there exists a group homomorphism $\kappa' : \widehat{St}_2(R) \to G'$ such that $\kappa \circ \kappa' = \hat{\varphi}$.

Define the element $x'_\pm(t)$ of G' to be $[\kappa^{-1}(\bar{h}_\pm(\alpha)), \kappa^{-1}(\bar{x}_\pm((\alpha^2 - 1)^{-1}t))]$ for all $t \in R$. The element $x'_\pm(t) \in G'$ is uniquely determined by Corollary 1.7. We also define:

$$n'_{\pm}(v) = x'_{\pm}(v)x'_{\mp}(v^{-1})^{-1}x'_{\pm}(v),$$

$$h'_{\pm}(v) = n'_{\pm}(v)n'_{\pm}(1)^{-1},$$

$$c'_{\pm}(v, v') = h'_{\pm}(v)h'_{\pm}(v')h'_{\pm}(vv')^{-1}$$

$$\theta'_{\pm}(t, t') = x'_{\pm}(t)x'_{\pm}(t')x'_{\pm}(t + t')^{-1},$$

where, $t, t' \in R$, $v \in R^{\times}$. We remark that $\kappa(x'_{\pm}(t)) = \bar{x}_{\pm}(t)$.

LEMMA 3.8. *Notations are as above.*

(a) $n'_{\pm}(v)x'_{\pm}(t)n'_{\pm}(v)^{-1} = x'_{\mp}(-v^{-2}t)$,
(b) $h'_{\pm}(v)x'_{\pm}(t)h'_{\pm}(v)^{-1} = x'_{\pm}(v^2 t)$,
(c) $x'_{\pm}(0) = 1$,
(d) $\theta'_{\pm}(t, t')$, $[x'_{\pm}(t), x'_{\pm}(t')]$ *is central in* G',
(e) $[x'_{\pm}(t), x'_{\pm}(t')x'_{\pm}(t'')] = [x'_{\pm}(t), x'_{\pm}(t')][x'_{\pm}(t), x'_{\pm}(t'')]$,
(f) $[x'_{\pm}(t)x'_{\pm}(t'), x'_{\pm}(t'')] = [x'_{\pm}(t), x'_{\pm}(t'')][x'_{\pm}(t'), x'_{\pm}(t'')]$,
(g) $[x'_{\pm}(t), x'_{\pm}(t')] = \theta'_{\pm}((\alpha^2 - 1)t', t)$,

where, $t, t', t'' \in R$ *and* $v \in R^{\times}$.

PROOF. (a): In $St_2(R)$, we have by Proposition 2.2:

$$\bar{n}_{\pm}(v)\bar{h}_{\pm}(\alpha)\bar{n}_{\pm}(v)^{-1} = \bar{h}_{\mp}(\alpha)\bar{c}_{\mp}(\alpha, -v^{-2})^{-1};$$

$$\bar{n}_{\pm}(v)\bar{x}_{\pm}((\alpha^2 - 1)^{-1}t)\bar{n}_{\pm}(v)^{-1} = \bar{x}_{\mp}(-v^{-2}(\alpha^2 - 1)^{-1}t).$$

From Proposition 2.2 (i) and Corollary 1.7,

$$\begin{aligned} &n'_{\pm}(v)x'_{\pm}(t)n'_{\pm}(v)^{-1} \\ &\quad = [n'_{\pm}(v)h'_{\pm}(\alpha)n'_{\pm}(v)^{-1}, n'_{\pm}(v)x'_{\pm}((\alpha^2 - 1)^{-1}t)n'_{\pm}(v)^{-1}] \\ &\quad = [\kappa^{-1}(\bar{n}_{\pm}(v)\bar{h}_{\pm}(\alpha)\bar{n}_{\pm}(v)^{-1}), \kappa^{-1}(\bar{n}_{\pm}(v)\bar{x}_{\pm}((\alpha^2 - 1)^{-1}t)\bar{n}_{\pm}(v)^{-1})] \\ &\quad = [\kappa^{-1}(\bar{h}_{\mp}(\alpha)\bar{c}_{\mp}(\alpha, -v^{-2})^{-1}), \kappa^{-1}(\bar{x}_{\mp}(-v^{-2}(\alpha^2 - 1)^{-1}t))] \\ &\quad = [h'_{\mp}(\alpha)c'_{\mp}(\alpha, -v^{-2})^{-1}, x'_{\mp}(-v^{-2}(\alpha^2 - 1)^{-1}t)] \\ &\quad = [h'_{\mp}(\alpha), x'_{\mp}(-v^{-2}(\alpha^2 - 1)^{-1}t)] \\ &\quad = x'_{\mp}(-v^{-2}t). \end{aligned}$$

(b): In $St_2(R)$, we have by Proposition 2.2:

$$\bar{h}_\pm(v)\bar{h}_\pm(\alpha)\bar{h}_\pm(v)^{-1} = \bar{h}_\pm(\alpha)\bar{c}_\pm(\alpha, v^2)^{-1}$$

$$\bar{h}_\pm(v)\bar{x}_\pm((\alpha^2-1)^{-1}t)\bar{h}_\pm(v)^{-1} = \bar{x}_\pm(v^2(\alpha^2-1)^{-1}t).$$

From Proposition 2.2 and Corollary 1.7, we have:

$$\begin{aligned} &h'_\pm(v)x'_\pm(t)h'_\pm(v)^{-1} \\ &\quad= [h'_\pm(v)h'_\pm(\alpha)h'_\pm(v)^{-1}, h'_\pm(v)x'_\pm((\alpha^2-1)^{-1}t)h'_\pm(v)^{-1}] \\ &\quad= [\kappa^{-1}(\bar{h}_\pm(v)\bar{h}_\pm(\alpha)\bar{h}_\pm(v)^{-1}), \kappa^{-1}(\bar{h}_\pm(v)\bar{x}_\pm((\alpha^2-1)^{-1}t)\bar{h}_\pm(v)^{-1})] \\ &\quad= [\kappa^{-1}(\bar{h}_\pm(\alpha)\bar{c}_\pm(\alpha, v^2)^{-1}), \kappa^{-1}(\bar{x}_\pm(v^2(\alpha^2-1)^{-1}t))] \\ &\quad= [h'_\pm(\alpha)c'_\pm(\alpha, v^2)^{-1}, x'_\pm(v^2(\alpha^2-1)^{-1}t)] \\ &\quad= [h'_\pm(\alpha), x'_\pm(v^2(\alpha^2-1)^{-1}t)] \\ &\quad= x'_\pm(v^2t). \end{aligned}$$

(c): In $St_2(R)$, $\bar{x}_\pm(0) = 1$. Then, $\kappa^{-1}(\bar{x}_\pm(0))$ is central in G'. Thus,

$$x'_\pm(0) = [\kappa^{-1}(\bar{h}_\pm(\alpha)), \kappa^{-1}(\bar{x}_\pm(0))] = 1.$$

(d) is clear from $\kappa(\theta'_\pm(t, t')) = \kappa([x'_\pm(t), x'_\pm(t')]) = 1$.
(e): From (d),

$$\begin{aligned} [x'_\pm(t), x'_\pm(t')x'_\pm(t'')] &= [x'_\pm(t), x'_\pm(t')]x'_\pm(t')[x'_\pm(t), x'_\pm(t'')]x'_\pm(t')^{-1} \\ &= [x'_\pm(t), x'_\pm(t')][x'_\pm(t), x'_\pm(t'')]. \end{aligned}$$

(f): From (d),

$$\begin{aligned} [x'_\pm(t)x'_\pm(t'), x'_\pm(t'')] &= x'_\pm(t)[x'_\pm(t'), x'_\pm(t'')]x'_\pm(t)^{-1}[x'_\pm(t), x'_\pm(t'')] \\ &= [x'_\pm(t), x'_\pm(t'')][x'_\pm(t'), x'_\pm(t'')]. \end{aligned}$$

(g): From the definition of x' and (d),

$$\begin{aligned} \theta'_\pm(t, t') &= h'_\pm(\alpha)\theta'_\pm(t, t')h'_\pm(\alpha)^{-1} \\ &= [h'_\pm(\alpha), x'_\pm(t)]x'_\pm(t)[h'_\pm(\alpha), x'_\pm(t')]x'_\pm(t')x'_\pm(t+t')^{-1}[h'_\pm(\alpha), x'_\pm(t+t')]^{-1} \end{aligned}$$

$$= x'_\pm((\alpha^2-1)t)x'_\pm(t)x'_\pm((\alpha^2-1)t')x'_\pm(t')x'_\pm(t+t')^{-1}x'_\pm((\alpha^2-1)(t+t'))^{-1}$$

$$= x'_\pm((\alpha^2-1)t)x'_\pm(t)x'_\pm((\alpha^2-1)t')x'_\pm(t)^{-1}\theta'_\pm(t,t')x'_\pm((\alpha^2-1)(t+t'))^{-1}$$

$$= x'_\pm((\alpha^2-1)t)[x'_\pm(t),x'_\pm((\alpha^2-1)t')]x'_\pm((\alpha^2-1)t')\theta'_\pm(t,t')$$

$$\times\, x'_\pm((\alpha^2-1)(t+t'))^{-1}$$

$$= \theta'_\pm((\alpha^2-1)t,(\alpha^2-1)t')[x'_\pm(t),x'_\pm((\alpha^2-1)t')]\theta'_\pm(t,t').$$

Replacing t, t' by t', $(a^2-1)^{-1}t$ respectively, we can get

$$\theta'_\pm((\alpha^2-1)t',t) = [x'_\pm(t'),x'_\pm(t)]^{-1} = [x'_\pm(t),x'_\pm(t')]. \qquad \square$$

To prove Theorem 3.6, we must show that $x'_\pm(t), n'_\pm(u), \theta_\pm(t,t')$ satisfy the relation (B)$'$, $(\hat\theta 1)$, $(\hat\theta 2)$, $(\hat\theta 3)$, where (B)$'$, $(\hat\theta 1)$, $(\hat\theta 2)$ are obvious from Lemma 3.8.

$(\hat\theta 3)$: By Lemma 3.8 and the biadditivity of $\theta'_\pm$,

$$\begin{aligned}\theta'_\pm(t,t') &= [x'_\pm(t),x'_\pm(t')]\theta'_\pm(t',t)\\ &= \theta'_\pm((\alpha^2-1)t',t)\theta'_\pm(t',t)\\ &= \theta'_\pm(\alpha^2 t',t).\end{aligned}$$

Then, we can construct a group homomorphism $\kappa' : \widehat{St}_2(R) \to G'$ such that $\kappa'(\hat x_\pm(t)) = x'_\pm(t)$ and it is easily shown that $\kappa \circ \kappa' = \hat\varphi$.

Remark 3.9. Let α, α' be elements of $R^\times$ satisfying α^2-1, $\alpha'^2-1 \in R^\times$ respectively, and denote by $\widehat{St}{}_2^{\alpha}(R)$, $\widehat{St}{}_2^{\alpha'}(R)$, the groups defined by the relations (B)$'$, $(\hat\theta 1)$, $(\hat\theta 2)$, $(\hat\theta 3)$ respectively. Then, we can prove $\widehat{St}{}_2^{\alpha}(R) \simeq \widehat{St}{}_2^{\alpha'}(R)$ directly, as follows.

From the proof of Theorem 3.6, we can construct a group homomorphism $\pi_{\alpha\alpha'}$ from $\widehat{St}{}_2^{\alpha}(R)$ to $\widehat{St}{}_2^{\alpha'}(R)$ sending $\hat x_\pm(t)$ $(t \in R)$ to:

$$[\hat h_\pm(\alpha), \hat x_\pm((\alpha^2-1)^{-1}t)]$$

$$= \hat x_\pm(\alpha^2(\alpha^2-1)^{-1}t)\hat x_\pm(-(\alpha^2-1)^{-1}t)\hat\theta_\pm((\alpha^2-1)^{-1}t,(\alpha^2-1)^{-1}t) \quad \text{by (9)(17)}$$

$$= \hat x_\pm(t)\hat\theta_\pm(\alpha^2(\alpha^2-1)^{-1}t,-(\alpha^2-1)^{-1}t)\hat\theta_\pm((\alpha^2-1)^{-1}t,(\alpha^2-1)^{-1}t) \quad \text{by } (\hat\theta 1)$$

$$= \hat x_\pm(t)\hat\theta_\pm(t,(\alpha^2-1)^{-1}t)^{-1} \quad \text{by } (\hat\theta 2)(4)(5).$$

Similarly, we can construct a group homomorphism $\pi_{\alpha'\alpha}$ from $\widehat{St}_2^{\alpha'}(R)$ to $\widehat{St}_2^{\alpha}(R)$ sending $\hat{x}_\pm(t)$ $(t \in R)$ to $\hat{x}_\pm(t)\hat{\theta}_\pm(t, (\alpha'^2 - 1)^{-1}t)^{-1}$. Then,

$$\begin{aligned}
&\pi_{\alpha'\alpha} \circ \pi_{\alpha\alpha'}(\hat{x}_\pm(t)) \\
&\quad = \pi_{\alpha'\alpha}(\hat{x}_\pm(t)\hat{\theta}_\pm(t, (\alpha^2 - 1)^{-1}t)^{-1}) \\
&\quad = \pi_{\alpha'\alpha}(\hat{x}_\pm(t)\hat{x}_\pm(t + (\alpha^2 - 1)^{-1}t)\hat{x}_\pm((\alpha^2 - 1)^{-1}t)^{-1}\hat{x}_\pm(t)^{-1}) \\
&\quad = \hat{x}_\pm(t)\hat{\theta}_\pm(t, (\alpha^2 - 1)^{-1}t)^{-1}\hat{\theta}_\pm(t + (\alpha^2 - 1)^{-1}t, (\alpha'^2 - 1)^{-1}(t + (\alpha^2 - 1)^{-1}t))^{-1} \\
&\qquad \times \hat{\theta}_\pm((\alpha^2 - 1)^{-1}t, (\alpha^2 - 1)^{-1}(\alpha'^2 - 1)^{-1}t) \quad \text{by } (\hat{\theta}1). \qquad (\diamondsuit)
\end{aligned}$$

Using the relations $(\hat{\theta}1)$–$(\hat{\theta}3)$ and (1)–(7), we can prove that $(\diamondsuit) = \hat{x}_\pm(t)$. This means that $\pi_{\alpha'\alpha} \circ \pi_{\alpha\alpha'}$ is the identity map of $\widehat{St}_2^{\alpha}(R)$. Similarly, we can show that $\pi_{\alpha\alpha'} \circ \pi_{\alpha'\alpha}$ is the identity map of $\widehat{St}_2^{\alpha'}(R)$. Therefore, $\pi_{\alpha\alpha'}$ is an isomorphism.

In the remainder of this section, we assume that R is a Laurent polynomial ring $\mathbf{F}[X, X^{-1}]$ and $\mathbf{F}$ is a field with more than 4 elements. we choose the element $\alpha \in \mathbf{F}^\times$ satisfying $\alpha^2 - 1 \in \mathbf{F}^\times$ and fix it. Denote the canonical group homomorphism from $\widehat{St}_2(\mathbf{F}[X, X^{-1}])$ to $St_2(\mathbf{F}[X, X^{-1}])$ by $\hat{\varphi}$, again.

Let $\bar{\varphi}$ be the group homomorphism from $St_2(\mathbf{F}[X, X^{-1}])$ to $SL_2(\mathbf{F}[X, X^{-1}])$ introduced in Section 2. From Corollary 1.6, we have:

Corollary 3.10. *Notations are as above. Then,* $(\widehat{St}_2(\mathbf{F}[X, X^{-1}]), \bar{\varphi} \circ \hat{\varphi})$ *is a universal central extension of* $SL_2(\mathbf{F}[X, X^{-1}])$. *The kernel of* $\bar{\varphi} \circ \hat{\varphi}$ *is the subgroup of* $\widehat{St}_2(\mathbf{F}[X, X^{-1}])$ *generated by* $\hat{\theta}_\pm(t, t')$ $(t, t' \in \mathbf{F}[X, X^{-1}])$ *and* $\hat{c}_\pm(v, v')$ $(v, v' \in \mathbf{F}[X, X^{-1}]^\times)$.

In the remainder of this section, we will show that $\widehat{St}_2(\mathbf{F}[X, X^{-1}])$ has a Tits system.

Lemma 3.11. *Let* (G', B', N', S') *be a Tits system. Let* G'' *be a group and* B'', N'' *subgroups of* G'', S'' *a subset of left cosets set* $N''/(B'' \cap N'')$ *and* κ *a surjective group homomorphism from* G'' *to* G'. *We assume*:

(a) $B'' = \kappa^{-1}(B')$,
(b) $\kappa(N'') = N'$,
(c) $(B'' \cap N'') \lhd N''$,
(d) *Let* $\bar{\kappa}$ *be the group homomorphism from* $N''/(B'' \cap N'')$ *to* $N'/(B' \cap N')$ *canonically induced by* κ. *Then,* $\bar{\kappa}$ *is the isomorphism,*
(e) $\bar{\kappa}(S'') = S'$.

Then, (G'', B'', N'', S'') *is also a Tits system with a Weyl group isomorphic to* $N'/(B' \cap N')$.

PROOF. We check (T1)–(T4) where, (T1), (T2) are obvious.

(T3): Suppose that $s'' \in S''$ and $w'' \in N''/(B'' \cap N'')$. Then, we have (T3) since $\operatorname{Ker} \kappa \subset B''$ and

$$\kappa(s''B''w'') \subset B'\bar{\kappa}(s''w'')B' \cup B'\bar{\kappa}(w'')B' = \kappa(B''s''w''B'' \cup B''w''B'').$$

(T4) holds since $\kappa(s''B''s'') \not\subset B'$ for all $s'' \in S''$. □

We define the subgroups of $\widehat{St}_2(\mathbf{F}[X, X^{-1}])$ by:

$$\hat{C} = \langle \hat{\theta}_{\pm}(t, t') \mid t, t' \in \mathbf{F}[X, X^{-1}] \rangle,$$

$$\hat{M} = \langle \hat{c}_{\pm}(v, v'), \hat{C} \mid v, v' \in \mathbf{F}[X, X^{-1}]^{\times} \rangle,$$

$$\hat{U}_{(\pm, m)} = \langle \hat{x}_{\pm}(sX^m), \hat{M} \mid s \in \mathbf{F} \rangle \quad \text{for all } m \in \mathbf{Z},$$

$$\hat{U} = \langle \hat{U}_{(+,k)}, \hat{U}_{(-,k')} \mid k \in \mathbf{Z}_{\geq 0}, k' \in \mathbf{Z}_{\geq 1} \rangle,$$

$$\hat{T} = \langle \hat{h}_{\pm}(u), \hat{M} \mid u \in \mathbf{F}^{\times} \rangle,$$

$$\hat{B} = \langle \hat{U}, \hat{T} \rangle,$$

$$\hat{N} = \langle \hat{n}_{\pm}(v), \hat{M} \mid v \in \mathbf{F}[X, X^{-1}]^{\times} \rangle.$$

Let $\hat{S}$ be a $\{\hat{n}_+(1), \hat{n}_-(X) \operatorname{Mod}(\hat{B} \cap \hat{N})\}$. Using Lemma 3.11, we can prove the next theorem.

THEOREM 3.12. *Notations are as above. Then,* $(\widehat{St}_2(\mathbf{F}[X, X^{-1}]), \hat{B}, \hat{N}, \hat{S})$ *is a Tits system and its Weyl group is isomorphic to the infinite dihedral group.*

PROOF. By Corollary 3.10, $(\widehat{St}_2(\mathbf{F}[X, X^{-1}]), \bar{\varphi} \circ \hat{\varphi})$ is a universal central extension of $SL_2(\mathbf{F}[X, X^{-1}])$ and $\operatorname{Ker} \bar{\varphi} \circ \hat{\varphi} = \hat{M}$. Obviously, we have $(\bar{\varphi} \circ \hat{\varphi})^{-1}(B) = \hat{B}$, $(\bar{\varphi} \circ \hat{\varphi})^{-1}(N) = \hat{N}$ and $(\bar{\varphi} \circ \hat{\varphi})^{-1}(T) = \hat{T} = \hat{B} \cap \hat{N}$. By Proposition 2.4, $(SL_2(\mathbf{F}[X, X^{-1}]), B, N, S)$ is a Tits system, $B \cap N = T$ and N/T is isomorphic to the infinite dihedral group. Then, we can easily check (a)–(e) of Lemma 3.11. □

Here, this Tits system is actually of affine A_1 type, and the group $\hat{B}$ is corresponding to the so-called Iwahori subgroup (cf. [9]).

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