TSUKUBA J. MATH.
Vol. 36 No. 1 (2012), 43–52

THE CONSTRUCTION OF ROTATION SURFACES OF CONSTANT MEAN CURVATURE AND THE CORRESPONDING LAGRANGIANS

By

Keiichi KIKUCHI*

(dedicated to Prof. Tominosuke Otsuki)

Abstract. A family of S^1-equivariant hypersurfaces of constant mean curvature can be obtained by using the Lagrangians with suitable potentials in the unit 3-sphere equipped with a certain parameterized metric. The conservation law is effectively applied to the construction of S^1-equivariant hypersurfaces of constant mean curvature.

1. Introduction

W.-Y. Hsiang [6] investigated the rotation hypersurfaces of constant mean curvature in the hyperbolic or spherical n-space. In [2], Eells and Ratto have constructed the rotation (S^1-equivariant) minimal hypersurfaces in the unit 3-sphere with standard metric by using a certain first integral, which is invariant with respect to the rotation angle of generating curves on the orbit space. It is cleared that the first integral (conserved quantity) can be obtained by using the Lagrangian of the corresponding dynamical system with respect to the Hsiang-Lawson metric [6] [7] on the orbit space via the Hamilton's equation when we consider the rotation angle of generating curves as "time". We should remark that the corresponding Lagrangian has the vanishing potential when we construct the rotation minimal hypersurfaces. However, in case that we construct the rotation non-minimal CMC-hypersurface in the unit 3-sphere, the potential of the Lagrangian is a nonvanishing function. In the section 4, we determine the

* 2000 *MSC*: 53C43, 70H03.
Keywords: CMC-rotation surfaces, Lagrangians, conservation laws.
E-mail: kikuchi@jewel.ocn.ne.jp
Received April 22, 2010.
Revised October 24, 2011.

potential function of the Lagrangian which corresponds to the rotation CMC-surfaces immersed in the unit 3-sphere. As a result we can see that the corresponding potential depends on the constant mean curvature itself (Theorem 4.3). In Appendix, we find the Lagrangians which correspond to the rotation CMC-surfaces of variable curvature such as the unduloid and nodoid immersed in the 3-dimensional Euclidean space.

2. Preliminaries

Let $g_{\alpha,\beta}$ be a generalized inner product on the unit 3-sphere $S^3 \subset \mathbf{C} \times \mathbf{C}$ defined by

$$(g_{\alpha,\beta})_z(v, w) = \alpha\langle v, w\rangle + \beta\langle v, iz\rangle\langle w, iz\rangle,$$

where $v = (v_1, v_2)$, $w = (w_1, w_2) \in T_z S^3$ and $\langle v, w\rangle = \Re(v_1 \overline{w}_1 + v_2 \overline{w}_2)$. We assume that α and β denote positive and nonnegative parameters, respectively. The Cartan hypersurface $SO(3)/\mathbf{Z}_2 \times \mathbf{Z}_2$ in the unit 4-sphere is covered by S^3 (via an 8-fold covering), whose metric is rescaled along the Hopf fibres and its metric on S^3 coincides with $g_{4,12}$ ($\alpha = 4$, $\beta = 12$) [5] [9]. The family of metrics $g_{\alpha,\beta}$ defined on S^3 contains this one as a special case.

Here we summarize the notations which are used in the paper.

X denotes the orbit space by $g_{\alpha,\beta}$-isometric S^1-action $r_t : S^3 \to S^3$ as follows.

$$r_t(z) = (z_1, e^{it} z_2), \quad z = (z_1, z_2).$$

As the parametrization of X we use the following map:

$$(\theta, \phi) \to (e^{i\phi} \cos\theta, \sin\theta), \quad 0 \le \phi \le 2\pi,\ 0 \le \theta \le \frac{\pi}{2}.$$

$h_{\alpha,\beta} := (h_{\alpha,\beta})_1\, d\theta^2 + (h_{\alpha,\beta})_2\, d\phi^2$ stands for the orbital metric on $X^* = X \backslash (\partial X \cup \{pole\})$ and

$$(h_{\alpha,\beta})_1 = \alpha, \quad (h_{\alpha,\beta})_2 = \frac{\alpha(\alpha+\beta)\cos^2\theta}{\alpha + \beta\sin^2\theta}.$$

$V = 2\pi \sin\theta\sqrt{\alpha + \beta\sin^2\theta}$ is the volume function of orbits and $\hat{h}_{\alpha,\beta} = V^2 h_{\alpha,\beta}$ is the Hsiang-Lawson metric on X^*.

$\gamma : J \subset \mathbf{R} \to (X^*, h_{\alpha,\beta})$ denotes a curve parametrized by arclength s. And also $\tau(\gamma) := \nabla_{\dot\gamma}\dot\gamma$ and $\hat\tau(\gamma) := \hat\nabla_{\dot\gamma}\dot\gamma$ stand for the tension fields of γ with respect to the metrics $h_{\alpha,\beta}$ and $\hat{h}_{\alpha,\beta}$, respectively. The geodesic curvature $\kappa(\gamma)$ at $\gamma(s)$ is defined by $\kappa(\gamma) := h_{\alpha,\beta}(\tau(\gamma), \eta)$ where η denotes the unit normal vector field to γ.

3. S^1-Equivariant CMC-Immersion

For a curve $\gamma : J \to X^*$, we consider an S^1-equivariant map $\mu : M = \gamma^{-1}((S^3, g_{\alpha,\beta})) \to (S^3, g_{\alpha,\beta})$ such that $\gamma \circ \pi = \sigma \circ \mu$, where $\pi : M \to J$ and $\sigma : (S^3, g_{\alpha,\beta}) \to X^*$ are Riemannian submersions. Throughout the paper, we assume that μ is an S^1-equivariant constant mean curvature H immersion. Then we have

$$\kappa(\gamma) - \eta(\log V) = 2H, \tag{1}$$

since

$$h_{\alpha,\beta}(\tau(\gamma), \eta) - \eta(\log V) = h_{\alpha,\beta}(\hat{\tau}(\gamma), \eta).$$

On the orbit space $(X^*, h_{\alpha,\beta})$, the velocity vector field of a curve $\gamma(s) = (\theta(s), \phi(s))$ is given by the following component functions.

$$\theta'(s) = \frac{1}{\sqrt{\alpha}} \cos \lambda(s), \quad \phi'(s) = \frac{\sqrt{\alpha + \beta \sin^2 \theta(s)} \sin \lambda(s)}{\sqrt{\alpha(\alpha + \beta)} \cos \theta(s)}.$$

Lemma 3.1. *The following formulas hold on* $(X^*, h_{\alpha,\beta})$.

$$\eta(s) = -\frac{1}{\sqrt{\alpha}} \sin \lambda(s) \frac{\partial}{\partial \theta} + \frac{\sqrt{\alpha + \beta \sin^2 \theta(s)} \cos \lambda(s)}{\sqrt{\alpha(\alpha + \beta)} \cos \theta(s)} \frac{\partial}{\partial \phi}. \tag{2}$$

$$\tau(\gamma) = \tau(\gamma)_1 \frac{\partial}{\partial \theta} + \tau(\gamma)_2 \frac{\partial}{\partial \phi}, \tag{3}$$

where

$$\tau(\gamma)_1 = -\frac{1}{\sqrt{\alpha}}(\sin \lambda(s))\lambda'(s) + \frac{(\alpha + \beta) \tan \theta(s) \sin^2 \lambda(s)}{\alpha(\alpha + \beta \sin^2 \theta(s))}.$$

and

$$\tau(\gamma)_2 = -\frac{\cos \lambda(s)}{\sqrt{\alpha(\alpha + \beta)}} \left\{ \frac{(\alpha + \beta) \sin \theta(s) \sin \lambda(s)}{\sqrt{\alpha(\alpha + \beta \sin^2 \theta(s))} \cos^2 \theta(s)} - \frac{\sqrt{\alpha + \beta \sin^2 \theta(s)} \lambda'(s)}{\cos \theta(s)} \right\}.$$

Then using the formula (1) we have the following differential equation (4) of generating curves which corresponds to the CMC-rotation hypersurfaces immersed in $(S^3, g_{\alpha,\beta})$, since using Lemma 3.1 the geodesic curvature $\kappa(\gamma)$ is given by

$$\kappa(\gamma) = \lambda'(s) - \frac{(\alpha+\beta)\tan\theta(s)\sin\lambda(s)}{\sqrt{\alpha}(\alpha+\beta\sin^2\theta(s))},$$

$$\lambda'(s) + \frac{1}{\sqrt{\alpha}}(\cot\theta(s) - \tan\theta(s))\sin\lambda(s) - 2H = 0. \tag{4}$$

4. Conservation Laws

We consider a generating curve $\gamma(s) = (\theta(s), \phi(s))$ on X^* such that $\theta = \theta(\phi)$ and $\phi'(s) > 0$. Then we can consider the space $\Xi(\theta, \theta^{\#})$ of motion with $\theta^{\#} = \frac{d\theta}{d\phi}$ and time ϕ. Let $\mathscr{L} = \mathscr{L}(\theta, \theta^{\#})$ be a Lagrangian on $\Xi(\theta, \theta^{\#})$. Via the Legendre transformation we have the Hamiltonian $\mathscr{H}$ on the phase space $\Xi^*(\theta, p)$:

$$\mathscr{H} = \theta^{\#}p - \mathscr{L}, \quad p = \frac{\partial\mathscr{L}}{\partial\theta^{\#}}.$$

The conservation laws of our system imply the following.

PROPOSITION 4.1. *Let the Lagrangian $\mathscr{L}$ on $\Xi(\theta, \theta^{\#})$ be the following form*:

$$\mathscr{L} = \sqrt{(\hat{h}_{\alpha,\beta})_1(\theta^{\#})^2 + (\hat{h}_{\alpha,\beta})_2} + G(\theta),$$

where $\hat{h}_{\alpha,\beta}$ is the Hsiang-Lawson metric on X^* and $G(\theta)$ is a potential function on the configuration space.

Then we have

$$\frac{d}{d\phi}\left\{\frac{(\hat{h}_{\alpha,\beta})_2}{\sqrt{(\hat{h}_{\alpha,\beta})_1(\theta^{\#})^2 + (\hat{h}_{\alpha,\beta})_2}} + G(\theta)\right\} = 0, \tag{5}$$

where the conserved quantity in the formula represents the Hamiltonian of our system.

By means of the Hamilton's equation (5), we shall determine the potential $G(\theta)$ which corresponds to the CMC-rotation hypersurfaces immersed in $(S^3, g_{\alpha,\beta})$ via the differential equation (4) of generating curves on the orbit space X^*.

The direct computation yields the following.

Lemma 4.2. *Assume that* θ *and* λ *are functions of* ϕ *and* $\frac{d\lambda}{d\phi} = \frac{\lambda'(s)}{\phi'(s)}$. *Then we have*

$$\frac{d}{d\phi}\frac{(\hat{h}_{\alpha,\beta})_2}{\sqrt{(\hat{h}_{\alpha,\beta})_1(\theta^{\#})^2 + (\hat{h}_{\alpha,\beta})_2}} = \Psi\left(\lambda'(s) + \frac{2}{\sqrt{\alpha}}\cot 2\theta(s)\sin\lambda(s)\right), \tag{6}$$

where

$$\Psi = \frac{2\alpha(\alpha+\beta)\pi\sin\theta(s)\cos^2\theta(s)\cot\lambda(s)}{\sqrt{\alpha+\beta\sin^2\theta(s)}}.$$

As a consequence, we have the following.

Theorem 4.3. *On our system, the Lagrangian* $\mathscr{L}$ *and the Hamiltonian* $\mathscr{H}$ *which correspond to the* S^1*-equivariant CMC-H hypersurface immersed in* $(S^3, g_{\alpha,\beta})$ *can be determined as follows*:

$$\mathscr{L} = \sqrt{(\hat{h}_{\alpha,\beta})_1(\theta^{\#})^2 + (\hat{h}_{\alpha,\beta})_2} + \alpha\sqrt{\alpha+\beta}\pi H\cos 2\theta,$$

$$\mathscr{H} = -\left\{\frac{(\hat{h}_{\alpha,\beta})_2}{\sqrt{(\hat{h}_{\alpha,\beta})_1(\theta^{\#})^2 + (\hat{h}_{\alpha,\beta})_2}} + \alpha\sqrt{\alpha+\beta}\pi H\cos 2\theta\right\}.$$

Proof. Using Lemma 4.2 and the differential equation of generating curves (4) we have

$$\frac{d}{d\phi}G(\theta) = -2H\Psi,$$

from which we obtain

$$\frac{d}{d\theta}G(\theta) = -\frac{4H\pi\alpha(\alpha+\beta)\sin\theta(s)\cos^2\theta(s)\cot\lambda(s)\phi'(s)}{\sqrt{\alpha+\beta\sin^2\theta(s)}\theta'(s)}.$$

Since H is a constant mean curvature and

$$\frac{\phi'(s)}{\theta'(s)} = \frac{\sqrt{\alpha+\beta\sin^2\theta(s)}\tan\lambda(s)}{\sqrt{\alpha+\beta}\cos\theta(s)},$$

we can choose such as $G(\theta) = \alpha\sqrt{\alpha+\beta}\pi H\cos 2\theta$. □

5. Generating Curves for CMC-H Rotation Hypersurfaces

Let $\gamma(s) = (\theta(s), \phi(s))$ be a generating curve on X^* such that $\theta = \theta(\phi)$ and $\phi'(s) > 0$ with the arc length s. Then we set the following initial conditions:

$$\theta_0 := \theta(0), \quad \phi(0) = 0, \quad \theta'(0) = 0, \quad \lambda(0) = \frac{\pi}{2}.$$

The Hamilton's equation $\frac{d\mathscr{H}}{d\phi} = 0$ (Theorem 4.3) implies that

$$\frac{d}{d\phi}(\sqrt{\alpha(\alpha+\beta)}\pi \sin 2\theta(s) \sin \lambda(s) + \alpha\sqrt{\alpha+\beta}\pi H \cos 2\theta(s)) = 0,$$

from which we have

$$\sqrt{\alpha(\alpha+\beta)}\pi \sin 2\theta(s) \sin \lambda(s) + \alpha\sqrt{\alpha+\beta}\pi H \cos 2\theta(s) = K,$$

where

$$K = \sqrt{\alpha(\alpha+\beta)}\pi \sin 2\theta_0 + \alpha\sqrt{\alpha+\beta}\pi H \cos 2\theta_0.$$

On the other hand, using the formulas

$$\left(\frac{d\theta}{d\phi}\right)^2 = \left\{\frac{(\hat{h}_{\alpha,\beta})_2}{(K - G(\theta))^2} - 1\right\}\frac{(\hat{h}_{\alpha,\beta})_2}{(\hat{h}_{\alpha,\beta})_1}$$

and

$$\left(\frac{d^2\theta}{d\phi^2}\right)_{s=0} = \frac{1}{2}\left(\frac{d}{d\theta}\right)_{s=0}\left(\frac{d\theta}{d\phi}\right)^2, \quad (K - G(\theta_0))^2 = (\hat{h}_{\alpha,\beta})_2(\theta_0),$$

we have

$$\begin{aligned}\left(\frac{d^2\theta}{d\phi^2}\right)_{s=0} &= \frac{1}{2}\left(\frac{d}{d\theta}\right)_{s=0}\left\{\frac{(\hat{h}_{\alpha,\beta})_2(\theta)}{(K - G(\theta))^2}\right\}\frac{(\hat{h}_{\alpha,\beta})_2(\theta_0)}{(\hat{h}_{\alpha,\beta})_1(\theta_0)} \\ &= \frac{1}{(\hat{h}_{\alpha,\beta})_1(\theta_0)}\left\{\left(\frac{d}{d\theta}\right)_{s=0} G(\theta(s))(K - G(\theta_0)) + \frac{1}{2}\left(\frac{d}{d\theta}\right)_{s=0}(\hat{h}_{\alpha,\beta})_2(\theta(s))\right\}.\end{aligned}$$

Consequently we have the following

Lemma 5.1. *Under the initial conditions for generating curves which correspond to the CMC-H rotation hypersurfaces, we have*

$$\left(\frac{d^2\theta}{d\phi^2}\right)_{s=0} = \frac{(\alpha+\beta)\sin^2 2\theta_0(\cot 2\theta_0 - \sqrt{\alpha}H)}{2\sin^2\theta_0(\alpha+\beta\sin^2\theta_0)},$$

and

$\theta_0 \geq \theta_{\alpha,H}$ (resp., $\leq\theta_{\alpha,H}$) *if and only if,*

$$\left(\frac{d^2\theta}{d\phi^2}\right)_{s=0} \leq 0 \quad (resp., \ \geq 0),$$

where

$$\theta_{\alpha,H} := \arctan(-\sqrt{\alpha}H + \sqrt{\alpha H^2 + 1}).$$

Assume that H is positive and sufficiently small. Then we have $0 < \theta_{\alpha,H} < \frac{\pi}{4}$, since $0 < -\sqrt{\alpha}H + \sqrt{\alpha H^2 + 1} < 1$. In Lemma 5.1 we may choose θ_0 such that $\theta_{\alpha,H} = \operatorname{arccot}(\sqrt{\alpha}H + \sqrt{\alpha H^2 + 1}) < \theta_0 < \operatorname{arccot}(\sqrt{\alpha}H) < \frac{\pi}{2}$. From Lemma 5.1, $\left(\frac{d^2\theta}{d\phi^2}\right)_{s=0} < 0$ and there exists the value ϕ_1 of ϕ such that $\theta(\phi) = \theta(\phi(s))$ decreases strictly until $\phi_1 = \phi(s_1)$, where the value of $\frac{d\theta}{d\phi}$ equals to zero at $\phi = \phi_1$, and $\theta(\phi) = \theta(\phi(s))$ takes a local minimum at $\phi = \phi_1$. In fact, if $\theta(\phi)$ does not take a local minimum, then we may assume that there exists a such that $0 \leq a < \theta_0 < \frac{\pi}{2}$ and $\lim_{s\to+\infty} \theta(s) = a$, $\lim_{s\to+\infty} \theta'(s) = 0$, $\lim_{s\to+\infty} \lambda(s) = \frac{\pi}{2}$. Then from the differential equation (4) of generating curves it follows that $a = \theta_{\alpha,H}$. On the other hand we obtain the following formula:

$$\left(\frac{d\theta}{d\phi}\right)^2 = \frac{(A(\theta) - 1)(\alpha + \beta)\cos^2\theta}{\alpha + \beta\sin^2\theta}, \tag{7}$$

where

$$A(\theta) = \frac{\sin^2 2\theta}{\{\sin 2\theta_0 + \sqrt{\alpha}H(\cos 2\theta_0 - \cos 2\theta)\}^2}.$$

The formula (7) implies $\lim_{s\to+\infty} A(\theta(s)) = 1$. Since H is sufficiently small, this yields

$$\sin(2a + B) = \sin(2\theta_0 + B), \quad \cos B = \frac{1}{\sqrt{1 + \alpha H^2}}, \quad \sin B = \frac{\sqrt{\alpha}H}{\sqrt{1 + \alpha H^2}}.$$

Moreover, we see that $2\theta_{\alpha,H} + B = \frac{\pi}{2}$. Since $\theta_{\alpha,H} = a$, by using the formula above, we have $a = \theta_0$, which is a contradiction.

Thus we can continue $\theta = \theta(\phi(s))$ as the curve satisfying the differential equation (4) by the reflection. Let $F_{\alpha,\beta,H}$ be the right hand side of (7). We can define $\Omega_{\alpha,\beta,H}$ by $F_{\alpha,\beta,H}$ as follows:

$$\Omega_{\alpha,\beta,H} = -\int_{\theta_0}^{\theta(s_1)} \frac{1}{\sqrt{F_{\alpha,\beta,H}}}\, d\theta.$$

Consequently we have the following.

THEOREM 5.2. *Let $H > 0$ be sufficiently small and choose θ_0 such that $\theta_{\alpha,H} < \theta_0 < \operatorname{arccot}(\sqrt{\alpha}H)$. If $\pi/\Omega_{\alpha,\beta,H}$ is a rational number, then the corresponding rotation hypersurface is an immersed CMC-H torus in $(S^3, g_{\alpha,\beta})$. In particular, if $\pi/\Omega_{\alpha,\beta,H}$ is an integer, then this CMC-H torus is embedded.*

THEOREM 5.3. *In the case $\theta_0 = \theta_{\alpha,H}$, let $\theta_0 = \frac{1}{2}\operatorname{arccot}(\sqrt{\alpha}H)$. Then the corresponding rotation hypersurface with CMC-H is an extended Clifford torus*

$$S^1(r(\theta_0)) \times S^1(R(\theta_0))$$

in $(S^3, g_{\alpha,\beta})$, where

$$r(\theta_0) = \cos\theta_0\sqrt{\alpha + \beta\cos^2\theta_0}, \quad R(\theta_0) = \sin\theta_0\sqrt{\alpha + \beta\sin^2\theta_0}.$$

COROLLARY 5.4. *There exists an embedded minimal torus in $(S^3, g_{\alpha,\beta})$*

$$S^1\left(\frac{\sqrt{2\alpha+\beta}}{2}\right) \times S^1\left(\frac{\sqrt{2\alpha+\beta}}{2}\right).$$

Appendix

We consider the rotation surfaces in $\mathbf{R}^3$. Let Σ be a rotation surface and $\gamma(s) = (\phi(s), \theta(s))$ be its generating curve on (ϕ, θ)-plane, where s stands for the arc length and $\theta(s)$ is positive. Let $H = H(s)$ be the mean curvature of Σ. Then we have the following fundamental formula [8]:

$$\phi'\theta'' - \phi''\theta' - \frac{\phi'}{\theta} + 2H = 0. \tag{8}$$

Let $\theta^{\#}$ denote $d\theta/d\phi$. Then we have

$$\theta^{\#\#} = (\theta^{\#})^{\#} = \frac{\theta''\phi' - \theta'\phi''}{(\phi')^3}. \tag{9}$$

We consider the space $\Lambda(\theta, \theta^{\#})$ of motion with time ϕ and give a Lagrangian $\mathscr{L} = \mathscr{L}(\theta, \theta^{\#}) = \theta\sqrt{1 + (\theta^{\#})^2} + G(\theta)$, where $G(\theta)$ is a potential function. Then the corresponding Hamiltonian is $\mathscr{H} = -\left(\frac{\theta}{\sqrt{1+(\theta^{\#})^2}} + G(\theta)\right)$ and the Euler-

Lagrange equation is

$$\frac{dG}{d\theta} = \frac{\theta\theta^{\#\#}}{(1+(\theta^{\#})^2)\sqrt{1+(\theta^{\#})^2}} - \frac{1}{\sqrt{1+(\theta^{\#})^2}}. \tag{10}$$

Using (9) and (10), we have

$$\frac{dG}{d\theta} = \frac{\theta(\theta''\phi' - \theta'\phi'')}{((\phi')^2 + (\theta')^2)^{3/2}} - \frac{\phi'}{\sqrt{(\phi')^2 + (\theta')^2}} = \theta(\theta''\phi' - \theta'\phi'') - \phi'. \tag{11}$$

The formulas (8) and (11) yield the following.

Theorem. *The following Lagrangian $\mathscr{L}_H$ is corresponding to some CMC-H rotation surfaces immersed in* $\mathbf{R}^3$:

$$\mathscr{L}_H = \theta\sqrt{1+(\theta^{\#})^2} - H\theta^2.$$

$\mathscr{L}_0 = \theta\sqrt{1+(\theta^{\#})^2}$ is a Lagrangian corresponding to the catenoid and $\mathscr{L}_H = \theta\sqrt{1+(\theta^{\#})^2} - H\theta^2$ $(H \neq 0)$ corresponds to the unduloid and nodoid. Thus $\mathscr{L}_H$ is a Lagrangian which corresponds to the Delaunay surfaces of variable curvature [1], [8].

Acknowledgements

I am grateful to Prof. Yoshihiro Ohnita for his constant encouragement.

References

[1] Eells, J., The surfaces of Delaunay, Math. Intell. **9** (1987), 53–57.
[2] Eells, J. and Ratto, A., Harmonic maps and minimal immersions with symmetries, Ann. Math. Studies **130**, 1993.
[3] Ferapontov, E. V. and Fordy, A. P., Separable Hamiltonians and integrable systems of hydrodynamic type, J. Geometry and Physics **21** (1997), 169–182.
[4] Ferus, D. and Pedit, F., S^1-equivariant Minimal Tori in S^4 and S^1-equivariant Willmore Tori in S^3, Math. Z. **204** (1990), 269–282.
[5] Ferus, D. and Pinkall, U., Constant curvature 2-spheres in the 4-sphere, Math. Z. **200** (1989), 265–271.
[6] Hsiang, W.-Y., On generalization of theorems of A. D. Alexandrov and C. Delaunay on hypersurfaces of constant mean curvature, Duke Math. J. **49** (1982), 485–496.
[7] Hsiang, W.-Y. and Lawson, H. B., Minimal submanifolds of low cohomogeneity, J. Diff. Geo. **5** (1971), 1–38.
[8] Kenmotsu, K., Kyokumenron Kougi (Japanese), Baifukan, 2000.
[9] Muto, H., Ohnita, Y. and Urakawa, H., Homogeneous minimal hypersurfaces in the unit spheres and the first eigenvalues of their Laplacian, Tohoku Math. J. **36** (1984), 253–267.

[10] Otsuki, T., Minimal hypersurfaces in a Riemannian manifold of constant curvature, Amer. J. Math. **92** (1970), 145–173.
[11] Uhlenbeck, K., Equivariant Harmonic Maps into Spheres, preprint.
[12] Von Westenholz, C., Differential forms in Mathematical Physics, (Chap. 12, Hamiltonian mechanics and geometry), Series Studies in Mathematics and its Applications Vol. **3**, North-Holland, 1981.

Department of Mathematics
School of Science
Tokai University
2-3-23 Takanawa, Minatoku
Tokyo, Japan