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ASYMPTOTIC BEHAVIORS FOR MULTIDIMENSIONAL KIRCHHOFF EQUATIONS

By

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1. Introduction

In the previous paper [8] we discussed the Cauchy problem for Kirchhoff equation in multidimensional spaces and obtained the time global solutions to the Cauchy problem under the assumption that the initial data satisfy an integrable condition. In this paper under the same integrable conditions we shall investigate the asymptotic behaviors concerning $t \to \pm$ for the following Kirchhoff equation

$$u_{tt}(t,x) - (1 - \varepsilon(Au(t), u(t))_{L^2})Au(t,x) = 0, \quad t \in R,\ x \in R^n, \tag{1.1}$$

where $A = \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk}(x) \frac{\partial}{\partial x_k}$ and ε is a positive constant. We assume that $-A$ and $H = \sqrt{-A}$ are non negative definite selfadjoint operators in $L^2(R^n)$. Denote by $D(H) = \{u \in L^2(R^n); Hu \in L^2(R^n)\}$ the definition domain of H. For $(f,g) \in D(H^{(2+k)/2}) \times D(H^{k/2})$, $k \geq 0$ and j a non negative integer, we define

$$G_{k,j}(H,f,g,t) = |(e^{itH}H^{2+k}f, \langle t\rangle^j f)| + |(e^{itH}H^{1+k}f, \langle t\rangle^j g)| + |(e^{itH}H^k g, \langle t\rangle^j g)| \tag{1.2}$$

and

$$\|(f,g)\|_{Y_{k,j}(H)} = \int_{-\infty}^{\infty} G_{k,j}(H,f,g,t)\,dt$$

where $\langle t\rangle = \sqrt{1+t^2}$ and $(\cdot,\cdot)$ stands for an inner product of $L^2(R^n)$. Denote by $Y_{k,j}(H)$ the set of functions $(f,g) \in D(H^{(k+2)/2}) \times D(H^{k/2})$ satisfying $\|(f,g)\|_{Y_{k,j}(H)} < \infty$. For simplicity we denote $G_{k,0}(H,f,g,t)$ and $Y_{k,0}(H)$ by $G_k(H,f,g,t)$ and $Y_k(H)$ respectively.

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We shall investigate the the asymptotic behaviors concerning $t \to \pm$ among Kirchhoff equation (1.1) and the following linear equation,

$$u_{tt}^{\pm}(t,x) = (c_{\infty}^{\pm})^2 A u^{\pm}(t,x), \quad u^{\pm}(0,x) = f^{\pm}(x),$$
$$u_t^{\pm}(0,x) = g^{\pm}(x), \quad t \in R,\ x \in R^n. \tag{1.3}$$

We mention our main result.

Theorem 1.1. *Assume that* $-A$ *and* $H = \sqrt{-A}$ *are non negative definite selfadjoint operators and that the initial data* (f^-, g^-) *belong to* $Y_{1,1}(H)$. *Then there is* $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon \le \varepsilon_0$, *there are* $c_{\infty}^{\pm} > 0$ *and* $u \in \bigcap_{j=0}^{1} C^j(R^1; D(H^{3/2-j}))$ *a solution of* (1.1) *and* $(f^+, g^+) \in Y_{1,1}(H)$ *satisfying*

$$\|u_t(t) - u_t^{\pm}(c_{\infty}^{\pm -1} S(t))\|_{L^2} + \|H(u(t) - u^{\pm}(c_{\infty}^{\pm -1} S(t)))\|_{L^2} \to 0, \quad t \to \pm\infty, \tag{1.4}$$

and

$$(1 + \varepsilon \|Hu(t)\|_{L^2}^2)^{1/2} - c_{\infty}^{\pm} \to 0, \quad t \to \pm\infty, \tag{1.5}$$

where $S(t) = \int_0^t (1 + \varepsilon \|Hu(s)\|_{L^2}^2)^{1/2}\, ds$ *and* $u^{\pm}(t,x) \in \bigcap_{j=0}^{1} C^j(R; D(H^{3/2-j}))$ *denote the solutions of* (1.3). *Moreover assume* $G_0(H, f^-, g^-, t) \to 0$, $|t| \to \infty$ *holds. Then* $c_{\infty}^+ = c_{\infty}^-$ *(denote by* c_{∞}*) holds and* c^{∞} *solves the following equation*

$$c_{\infty}^2 = 1 + \frac{\varepsilon}{2c_{\infty}^2}(\|g^-\|^2 + c_{\infty}^2 \|Hf^-\|^2) \tag{1.6}$$

and (f^+, g^+) *satisfies*

$$\|g^+\|^2 + c_{\infty}^2 \|Hf^+\|^2 = \|g^-\|^2 + c_{\infty}^2 \|Hf^-\|^2 \tag{1.7}$$

and $\lim_{|t|\to\infty} G_0(H, f^+, g^+, t) = 0$.

In [8] we obtain the time global solutions to the Cauchy problem for (1.1) under the assumption that the initial data (f, g) belong to $D(H^{3/2}) \times D(H^{1/2})$ and satisfy $\|(f,g)\|_{Y_1(H)} < \infty$.

Remark 1.1. We note that $\|(f,g)\|_{Y_k(H)} < \infty$, $k = 0, 1$ imply the following condition

$$G_0(H, f, g, t) = |(e^{itH} H^2 f, f)| + |(e^{itH} Hf, g)| + |(e^{itH} g, g)| \to 0, \quad |t| \to \infty. \tag{1.8}$$

It should be remarked that in the case of $A = \Delta$ the asymptotic behavior of the solutions of (1.1) is showed by Greenberg and Hu in [3] ($n = 1$), by Ghisi [2]

($n \geq 1$). We note that we can not derive in general

$$\|u_t(t) - u_{0t}^{\pm}(t)\|_{L^2} + \|H(u(t) - u_0^{\pm}(t))\|_{L^2} \to 0, \quad t \to \pm\infty, \tag{1.9}$$

under the conditions $\|(f^-, g^-)\|_{Y_k(H)} < \infty$, $k = 0, 1$. In oder to get (1.9) we need a more strong condition. In fact, in the case of $A = \Delta$ and $H_0 = \sqrt{-\Delta}$, under the condition that there is $p > 2$ such that

$$\begin{aligned} G_k(H_0, f^-, g^-; t) &= |(e^{itH_0} H_0^{2+k} f^-, f^-)| + |(e^{itH_0} H_0^{1+k} f^-, g^-)| + |(e^{itH_0} H_0^k g^-, g^-)| \\ &= 0(|t|^{-p}), \end{aligned} \tag{1.10}$$

for $|t| \to \infty$, $k = 0, 1$, Yamazaki in [13] derived (1.9) (more precisely the decay order $0(|t|^{-p+1})$ of the right hand side). On the other hand Matsuyama in [9] gave initial data (f^-, g^-) such that (1.9) does not hold. We remark that (1.4) is equivalent to

$$\|u_t(T(c_\infty^{\pm}\tau)) - u_\tau^{\pm}(\tau)\|_{L^2} + \|H(u(T(c_\infty^{\pm}\tau) - u^{\pm}(\tau))\|_{L^2} \to 0, \quad \tau \to \pm\infty,$$

where T is the inverse function of $S(t) = \int_0^t (1 + \varepsilon\|Hu(s)\|_{L^2}^2)^{1/2}\, ds$.

Next we shall give an example of A different from Δ which satisfies the integrable condition $\|(f^-, g^-)\|_{Y_{1,1}(H)} < \infty$. Before mentioned our theorem introduce notations. Let $\mu \in R$ and $1 \leq p \leq \infty$ and $L^p = L^p(R^n)$ the set of integrable functions over R^n with integrable p th power. We denote by $W_\mu^{l,p}$ the set of functions $u(x)$ defined in R^n such that $(1 + |x|)^\mu \partial_x^\alpha u(x)$ is contained in L^p for $|\alpha| \leq l$. For brevity we denote $L_\mu^p = W_\mu^{0,p}$, $W^{l,p} = W_0^{l,p}$, $H_\mu^l = W_\mu^{l,2}$ and $H^l = W^{l,2}$. Denote $H = \sqrt{-A}$ and $H_0 = \sqrt{-\Delta}$, $\Delta = \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}\right)^2$.

Assume that A is elliptic and a purturbation of Δ, that is,

$$a(x, \xi) = \sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2, \quad (x, \xi) \in \mathbf{R}^n, \tag{1.11}$$

and

$$A(x, D) = \Delta, \quad |x| \geq R_0 > 0. \tag{1.12}$$

We assume also that the coefficients $a_{jk}(x)$ of A belong to $C_\infty(R^n)$. It is well known that under the assumptions (1.11) and (1.12) A is a non negative self adjoint operator in $L^2(R^n)$ and has no positive eigenvalues. For example see Mochizuki [11] (p. 46). Besides A has not zero eigenvalue and zero resonance. See Mizohata [10] (p. 386) for Δ and Kajitani [6] (p. 130–131) for A. Furthermore assume that A satisfies the non trapping condition, that is, there exists a real valued function $q \in C^\infty(\mathbf{R}^{2n})$ such that with $C_{\alpha\beta} > 0$, $C_1 > 0$, $C_2 \geq 0$

$$|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{-|\alpha|}(1+|x|)^{1-|\beta|}, \quad x,\xi \in \mathbf{R}^n, \tag{1.13}$$

for all α, β, and

$$\begin{aligned} H_a q &= \sum_{j=1}^n \{\partial_{\xi_j} a(x,\xi)\partial_{x_j} q(x,\xi) - \partial_{x_j} a(x,\xi)\partial_{\xi_j} q(x,\xi)\} \\ &\geq C_1|\xi| - C_2, \quad x,\xi \in \mathbf{R}^n. \end{aligned} \tag{1.14}$$

Then we can prove the following theorem.

Theorem 1.2. *Let $n \geq 3$. Assume that A satisfies* (1.11), (1.12), (1.13) *and* (1.14) *and moreover the initial data (f^-, g^-) belongs to $H_\mu^l \times H_\mu^{l-1}$, $l > \frac{3n+6}{2}$, $\mu > \frac{7}{2}$ if $n = 3,4,5$ and to $(D(H^{3/2}) \cap W^{l,1})) \times (D(H^{1/2}) \cap W^{l-1,1}))$, $l \geq 2n+6$ if $n \geq 6$. Then there is $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$ are valid, there are $c_\infty > 0$ and $u \in \bigcap_{j=0}^2 C^j(R^1; D(H^{3/2-j}))$ a unique solution of* (1.1) *for $t \in R^1$ and $(f^+, g^+) \in D(H_0^{3/2}) \times D(H_0^{1/2})$ such that*

$$\|u_t(t) - u_{0t}^\pm(c_\infty^{-1}S(t))\|_{L^2} + \|H(u(t) - u_0^\pm(c_\infty^{-1}S(t)))\|_{L^2} \to 0, \quad t \to \pm\infty, \tag{1.15}$$

and

$$(1 + \varepsilon\|Hu(t)\|_{L^2}^2)^2)^{1/2} - c_\infty \to 0, \quad t \to \pm\infty, \tag{1.16}$$

where $S(t) = \int_0^t (1+\varepsilon\|Hu(s)\|_{L^2}^2)^{1/2}\,ds$ and $u^\pm(t,x) \in \bigcap_{j=0}^2 C^j(R; D(H_0^{3/2-j})$ denote the solutions of the following equation

$$\begin{aligned} &u_{tt}^\pm(t,x) = c_\infty^2 \Delta u^\pm(t,x), \quad u^\pm(0,x) = f^\pm(x), \\ &u_t^\pm(0,x) = g^\pm(x), \quad t \in R,\ x \in R^n \end{aligned} \tag{1.17}$$

When $n = 1$ Kajitani proved Theorem 1.2 in [7].

2. Proof of Theorem 1.1

We let $A_1(t,x) = u_t + ic(t)Hu$ and $B_1(t,x) = u_t - ic(t)Hu$ where $c(t)^2 = 1 + \varepsilon\|Hu(t)\|^2$, and $A_1^-(t,x) = u_t^- + ic_\infty^- H_0 u^-$ and $B_1^-(t,x) = u_t^- - ic_\infty^- Hu^-$. Then the equation (1.1) and the equation (1.3) yield

$$A_{1t} - ic(t)HA_1 = \frac{c'(t)}{2c(t)}(A_1 - B_1), \quad B_{1t} + ic(t)HB_1 = -\frac{c'(t)}{2c(t)}(A_1 - B_1) \tag{2.1}$$

and

$$A_{1t}^{-} - ic_{\infty}^{-}HA_1^{-} = 0, \quad B_{1t}^{-} + ic_{\infty}^{-}HB_1^{-} = 0, \tag{2.2}$$

respectively. The initial data is given by

$$\begin{aligned} A_1^{-}(0,x) &= A_0^{-}(x); = g^{-} + ic_{\infty}Hf^{-}(x), \\ B_1^{-}(0,x) &= B_0^{-}(x); = g^{-}(x) - ic_{\infty}Hf^{-}(x) \end{aligned} \tag{2.3}$$

and (1.4) gives

$$\|A_1(t) - A^{-}((c_{\infty}^{-})^{-1}S(t))\| + \|B_1(t) - B^{-}((c_{\infty}^{-})^{-1}S(t))\| \to 0, \quad t \to -\infty, \tag{2.4}$$

where $S(t) = \int_0^t c(s)\,ds$. Let $T(\tau)$ be the inverse function of $S(t) = \tau$. Put $A(\tau,x) = A_1(T(\tau),x)$, $B(\tau,x) = B_1(T(\tau),x)$, $A^{-}(\tau,x) = A_1^{-}((c_{\infty}^{-})^{-1}\tau,x)$, $B^{-}(\tau,x) = B_1^{-}((c_{\infty}^{-})^{-1}\tau,x)$ and $\gamma(\tau) = c(T(\tau))$. Then (2.1) and (2.2) yield

$$A_\tau - iHA = \frac{\gamma'(\tau)}{2\gamma(\tau)}(A-B), \quad B_\tau + iHB = -\frac{\gamma'(\tau)}{2\gamma(\tau)}(A-B), \quad \tau \in R,\ x \in R^n \tag{2.5}$$

and

$$A_\tau^{-} - iHA^{-} = 0, \quad B_\tau^{-} + iHB^{-} = 0, \quad \tau \in R,\ x \in R^n, \tag{2.6}$$

respectively. Here we pose the condition below to solve (2.5)

$$\|A(\tau) - A^{-}(\tau)\| + \|B(\tau) - B^{-}(\tau)\| \to 0, \quad \tau \to -\infty, \tag{2.7}$$

which is equivalent to (2.4). γ satisfies

$$\gamma(\tau)^2 = 1 + \frac{\varepsilon}{4\gamma(\tau)^2}\|A(\tau) - B(\tau)\|^2. \tag{2.8}$$

Then we note that (1.5) with $-$sign is equivalent to

$$\gamma(\tau)^2 - (c_{\infty}^{-})^2 = 1 + \frac{\varepsilon}{4\gamma(\tau)^2}\|A(\tau) - B(\tau)\|^2 - (c_{\infty}^{-})^2 \to 0, \quad \tau \to -\infty. \tag{2.9}$$

For $k \geq 0$ and a non negative integer j we introduce

$$\begin{aligned} \tilde{G}_{k.j}(H,A,B,\tau) = {}& |(e^{i\tau H}H^kA, \langle\tau\rangle^jA)| + |(e^{i\tau H}H^kA, \langle\tau\rangle^jB)| \\ & + |(e^{i\tau H}H^kB, \langle\tau\rangle^jB)| \end{aligned} \tag{2.10}$$

and

$$\|(A,B)\|_{\tilde{Y}_{k,j}(H)} = \int_{-\infty}^{\infty} \tilde{G}_{k.j}(H,A,B,\tau)\,d\tau.$$

Denote by $\tilde{Y}_{k,j}(H)$ the set of $(A,B) \in D(H^{k/2}) \times D(H^{k/2})$ satisfying $\|(A,B)\|_{\tilde{Y}_{k,j}(H)} < \infty$. For simplicity we denote by $\tilde{G}_{k,0}(H,A,B,t)$ and $\tilde{Y}_{k,0}(H)$ by $\tilde{G}_k(H,A,B,t)$ and $\tilde{Y}_k(H)$ respectively. Taking account that $A^-(\tau) = e^{i\tau H}A_0^-$ and $B^-(\tau) = e^{-i\tau H}B_0^-$, we can see easily,

Lemma 2.1. *Let $k \geq 0$ and assume that $(A_0^-(x), B_0^-(x))$ belongs to $\tilde{Y}_k(H)$. Then the solution (A^-, B^-) of* (2.6) *with the initial condition* (2.3) *belongs to $\tilde{Y}_k(H)$ and satisfies*

$$\begin{aligned}\|(A^-(s_1), B^-(s_2))\|_{\tilde{Y}_k(H)} &= \int_{-\infty}^{\infty} |(e^{i\tau H}H^k A^-(s_1), A^-(s_2))|\,d\tau \\ &\quad + \int_{-\infty}^{\infty} |(e^{i\tau H}H^k A^-(s_1), B^-(s_2))|\,d\tau \\ &\quad + \int_{-\infty}^{\infty} |(e^{i\tau H}H^k B^-(s_1), B^-(s_2))|\,d\tau \\ &= \|(A_0^-, B_0^-)\|_{\tilde{Y}_k(H)}, \quad s_1, s_2 \in R \end{aligned} \tag{2.11}$$

and

$$\|H^{k/2}A^-(\tau)\|^2 + \|H^{k/2}B^-(\tau)\|^2 = \|H^{k/2}A_0^-\|^2 + \|H^{k/2}B_0^-\|^2, \quad \tau \in R. \tag{2.12}$$

We continue to prove Theorem 1.1. First of all we note that the integrable condition $\|(f^-, g^-)\|_{Y_{k,j}(H)} < \infty$ is equivalent to $\|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)} < \infty$, if $(A_0^-, B_0^-) = (g^- + ic_\infty^- Hf^-, g^- - ic_\infty^- Hf^-)$ and that there are α_1^-, α_2^-, α_3^- and α_4^- such that if $\|(f^-, g^-)\|_{Y_1(H)} < \infty$, the terms $(e^{2i\tau H}Hf^-, Hf^-)$, $(e^{2i\tau H}Hf^-, g^-)$, $(e^{2i\tau H}Hg^-, f^-)$ and $(e^{2i\tau H}g^-, g^-)$ are convergent to α_1^-, α_2^-, α_3^- and α_4^- respectively, tending τ to $-\infty$. Because, their derivatives with respect to τ are in $L^1(R)$. Therefore we can see

$$(e^{i\tau H}A_0^-, e^{-i\tau H}B_0^-) = (e^{2i\tau H}(g^- + ic_\infty^- Hf^-), g^- - ic_\infty^- Hf^-)$$

is convergent to $\alpha_4^- + ic_\infty^-\alpha_2^- - ic_\infty^-\alpha_3^+(c_\infty^-)^2\alpha_1^-$ (denote by α^-), tending $\tau \to -\infty$. We begin to determine c_∞^-. It follows from (2.7) that we get

$$\|A(\tau) \pm B(\tau)\| - \|A^-(\tau) \pm B^-(\tau)\| \to 0, \quad \tau \to -\infty.$$

On the other hand, we have by use of (2.12) with $k = 0$

$$\begin{aligned}&\|A^-(\tau) \pm B^-(\tau)\|^2 - \|A_0^-\|^2 - \|B_0^-\|^2 \\ &\qquad = \pm 2\Re(A^-(\tau), B^-(\tau)) = \pm 2\Re(e^{i\tau H}A_0^-, e^{-i\tau H}B_0^-)\end{aligned}$$

$$
\begin{aligned}
&= \pm 2\Re(e^{i2\tau H}(g^- + ic_\infty^- Hf^-), g^- - ic_\infty^- Hf^-) \\
&\quad \to \pm 2\Re(\alpha_4^- + ic_\infty^-\alpha_2^- - ic_\infty^-\alpha_3^- + (c_\infty^-)^2\alpha_2^-) \\
&= \pm 2\Re\alpha^-, \quad \tau \to -\infty. \qquad (2.13)
\end{aligned}
$$

We define $c_\infty^- > 0$ as a positive root of the following equation which is solved uniquely by a positive root.

$$
\begin{aligned}
(c_\infty^-)^2 = 1 + \frac{\varepsilon}{4(c_\infty^-)^2}\{&\|g^-\|^2 + (c_\infty^-)^2\|Hf^-\|^2 \\
&- 2\Re(\alpha_4^- + ic_\infty^-\alpha_2^- - ic_\infty^-\alpha_3^- + (c_\infty^-)^2\alpha_2^-)\},
\end{aligned}
$$

which satisfies

$$
(c_\infty^-)^2 = 1 + \frac{\varepsilon}{4(c_\infty^-)^2}(\|A_0^-\|^2 + \|B_0^-\|^2 - 2\Re\alpha^-) \qquad (2.14)
$$

because of $\|A_0^-\|^2 + \|B_0^-\|^2 = \|g^-\|^2 + (c_\infty^-)^2\|Hf^-\|^2$. Moreover taking account that the relation (2.13) holds we get by use of (2.8) and (2.14)

$$
\begin{aligned}
|\gamma(\tau)^2 - (c_\infty^-)^2| &= \left|\frac{\sqrt{1+\varepsilon\|A(\tau)-B(\tau)\|^2} - \sqrt{1+\varepsilon\{\|A_0^-\|^2 + \|B_0^-\|^2) - 2\Re\alpha^-\}}}{2}\right| \\
&\le \frac{\varepsilon}{2}|\,\|A(\tau)-B(\tau)\|^2 - \|A_0^-\|^2 - \|B_0^-\|^2 + 2\Re\alpha^-| \\
&\le \varepsilon|\Re(A^-(\tau), B^-(\tau)) - \Re\alpha^-| \to 0, \quad \tau \to -\infty,
\end{aligned}
$$

which implies (2.9). Furthermore we assume that (f^-, g^-) satisfies $G_0(H, f^-, g^-; t) \to 0$, $t \to -\infty$ that is, $(e^{i\tau H}A_0^-, e^{-i\tau H}B_0^-) = (e^{2i\tau H}(g^- + ic_\infty^- Hf^-), g^- - ic_\infty^- Hf^-) \to 0$, $t \to -\infty$, then we have $\alpha_- = 0$ and consequently c_∞^- satisfies (1.6) from (2.14). Now we shall find the solution (A, B) and γ satisfying (2.5), (2.7) and (2.8). Let $\delta > 0$ and $M > 0$ and introduce

$$
X_{\delta,M} = \left\{\gamma(\tau) \in C^1((R); 1 \le \gamma(\tau) \le M, \tau \in R, \int_{-\infty}^{\infty} |\gamma'(\tau)|\, d\tau \le \delta\right\}.
$$

Denote by $|\gamma|_X = \sup_{\tau\in R}|\gamma(\tau)| + \int_{-\infty}^{\infty}|\gamma'(\tau)|\,d\tau$ a norm of $X_{\delta,M}$. For $\gamma \in X_{\delta,M}$ we consider the linear equation of (2.5) and (2.7). We change a unkown function (A, B) of (2.5) to (U, V) as $U = A - A^-$, $V = B - B^-$ which satisfies

$$U_\tau - iHU = \frac{\gamma'(\tau)}{2\gamma(\tau)}(U - V) + \frac{\gamma'(\tau)}{2\gamma(\tau)}W, \quad \tau \in R,\ x \in R^n, \tag{2.15}$$

$$V_\tau + iHV = -\frac{\gamma'(\tau)}{2\gamma(\tau)}(U - V) - \frac{\gamma'(\tau)}{2\gamma(\tau)}W, \quad \tau \in R,\ x \in R^n, \tag{2.16}$$

where $W = A^- - B^-$. Moreover (2.7) gives

$$\|U(\tau)\| + \|V(\tau)\| \to 0, \quad \tau \to -\infty. \tag{2.17}$$

In stead of (A, B) we shall find (U, V) satisfying (2.15), (2.16) and (2.17). Now we can prove the following proposition.

Proposition 2.1. *Let* γ *be in* $X_{\delta,M}$, $\int \frac{|\gamma'(s)|}{2}\,ds = \delta_1 < 1$ *and* $k \geq 0$. *Assume that* H *is a self adjoint operator.*

If (A_0^-, B_0^-) *belongs to* $D(H^k) \times D(H^k)$, *then* (2.15)–(2.17) *has a unique solution* (U, V) *satisfying*

$$\|H^k U(\tau)\| + \|H^k V(\tau)\| \leq \frac{2\sqrt{M}}{1 - \delta_1}(\|H^k A_0^-\| + \|H^k B_0^-\|), \tag{2.18}$$

for $\tau \in R$ *and there is* $(U^+, V^+) \in D(H^k) \times D(H^k)$ *such that*

$$\|U(\tau) - e^{i\tau H}U^+\| + \|V(\tau) - e^{-i\tau H}V^+\| \to 0, \quad \tau \to \infty. \tag{2.19}$$

Proof. Put $\alpha(\tau, x) = \gamma(\tau)^{-1/2}e^{-i\tau H}U(\tau, x)$ and $\beta(\tau, x) = \gamma(\tau)^{-1/2}e^{i\tau H}V(\tau, x)$. Then it follows from (2.15)–(2.17) that (α, β) satisfies

$$\frac{\partial\alpha}{\partial\tau}(\tau, x) = -\frac{\gamma'(\tau)}{2\gamma(\tau)}\{e^{-i2\tau H}\beta(\tau, x) - e^{-i\tau H}W_1(\tau, x)\}, \tag{2.20}$$

$$\frac{\partial\beta}{\partial\tau}(\tau, x) = -\frac{\gamma'(\tau)}{2\gamma(\tau)}\{e^{2i\tau H}\alpha(\tau, x) + e^{i\tau H}W_1(\tau, x)\}, \tag{2.21}$$

where $W_1(\tau, x) = \gamma(\tau)^{-1/2}W(\tau, x) = \gamma(\tau)^{-1/2}(e^{i\tau H}A_0^-(x) - e^{-i\tau H}B_0^-(x))$ and

$$\|\alpha(\tau)\| + \|\beta(\tau)\| \to 0, \quad \tau \to -\infty. \tag{2.22}$$

Therefore (α, β) solves the following integral equation

$$\alpha(\tau, x) = -\int_{-\infty}^{\tau} \frac{\gamma'(s)}{2\gamma(s)}\{e^{-2isH}\beta(s, x) - e^{-isH}W_1(s, x)\}\,ds, \tag{2.23}$$

and

$$\beta(\tau,x) = -\int_{-\infty}^{\tau} \frac{\gamma'(s)}{2\gamma(s)}\{e^{2isH}\alpha(s,x) + e^{isH}W_1(s,x)\}\,ds. \tag{2.24}$$

We shall show the existence of solutions of the integral equation (2.23)–(2.24). We seek a solution $(\alpha,\beta)(\tau,x)$ as

$$\alpha(\tau,x) = \sum_{n=0}^{\infty}\alpha_n(\tau,x), \quad \beta(\tau,x) = \sum_{n=0}^{\infty}\beta_n(\tau,x), \tag{2.25}$$

where

$$\alpha_0(\tau,x) = \int_{-\infty}^{\tau}\frac{\gamma'(s)}{2\gamma(s)}e^{-isH}W_1(s,x)\,ds,$$

$$\beta_0(\tau,y) = -\int_{-\infty}^{\tau}\frac{\gamma'(s)}{2\gamma(s)}e^{isH}W_1(s,x)\,ds, \tag{2.26}$$

and for $n \geq 1$

$$\alpha_n(\tau,x) = -\int_{-\infty}^{\tau}\frac{\gamma'(s)}{2\gamma(s)}e^{2isH}\beta_{n-1}(s,x)\,ds \tag{2.27}$$

and

$$\beta_n(\tau,x) = -\int_{-\infty}^{\tau}\frac{\gamma'(s)}{2\gamma(s)}e^{-2isH}\alpha_{n-1}(s,x)\,ds. \tag{2.28}$$

We can show easily by induction

$$\|H^k\alpha_n(\tau)\| + \|H^k\beta_n(\tau)\| \leq 2(\|H^kA_0^-\| + \|H^kB_0^-\|)\delta_1^{n+1}, \tag{2.29}$$

for $n = 0, 1, \ldots$ and consequently $(\alpha,\beta)(\tau,x)$ defined by (2.25) converges uniformly in τ, if $\delta_1 < 1$. Therefore $U(\tau,x) = \gamma^{1/2}e^{i\tau H}\alpha(\tau,x)$, $V(\tau,x) = \gamma^{1/2}e^{-i\tau H}\beta(\tau,x)$ solves (2.15)–(2.17) and satisfies (2.18).

It follows from (2.23), (2.24) and (2.18)

$$\|H^k(\alpha(s) - \alpha(s'))\| \leq C\left|\int_{s'}^{s}|\gamma'(t)|\,dt\right| \to 0, \quad s, s' \to \infty.$$

This means that $\{\alpha(s)\}_s$ is a Cauchy sequence in $D(H^k)$. Therefore there is $\alpha^+ \in D(H^k)$ such that

$$\|H^k(\alpha(s) - \alpha^+)\| \to 0, \quad s \to \infty. \tag{2.30}$$

Because $\gamma'(s)$ is in $L^1(R)$. Since γ' is in $L^1(R)$, we have $c_\infty^+ \geq 1$ such that $\gamma(s) - \gamma_\infty^+ \to 0$, $s \to \infty$. Put $U^+ = (\gamma_\infty^+)^{-1/2}\alpha^+$. The relation $e^{-isH}U(s) =$

$\gamma(s)^{-1/2}\alpha(s)$ together with (2.30) implies (2.19). Similarly we can show that $V(s)$ alos satisfies (2.19). □

Lemma 2.2. *Let γ be in $X_{\delta,M}$, $q(s)=\frac{\gamma'(s)}{2\gamma(s)}$, $\delta_1=\int\langle s\rangle^j|q(s)|\,ds$, $k\ge 0$ and j, p, q non negative integers and $c_j=2^j$. Assume that H is a self adjoint operator and (A_0^-,B_0^-) belongs to $\tilde{Y}_{k,j}(H)$. Let $(\alpha_p(\tau),\beta_q(\tau)$ be defined by* (2.26), (2.27) *and* (2.28). *Then (α_p,β_q) satisfies the following* 5 *properties.*

(i)

$$\sup_{s_1}\int_{-\infty}^{\infty}\{|(e^{i\tau H}H^k\alpha_p(s_1),\langle\tau\rangle^jC)|\,d\tau\le c_j(c_j\delta_1)^{p+1}\|(A_0^-,B_0^-)|\,|_{\tilde{Y}_{k,j}(H)},\quad(2.31)$$

$$\sup_{s_1}\int_{-\infty}^{\infty}|(e^{i\tau H}H^k\beta_p)(s_1),\langle\tau\rangle^jC)|\,d\tau\le c_j(c_j\delta_1)^{p+1}\|(A_0^-,B_0^-)\|_{\tilde{Y}_{k,j}(H)},\quad(2.32)$$

where $C=A_0^-$ *or* B_0^-.

(ii)

$$\sup_{s_1,s_2}\int_{-\infty}^{\infty}|(e^{i\tau H}H^k\alpha_p)(s_1),\langle\tau\rangle^j\alpha_q(s_2))|\,d\tau\le 3(c_j\delta_1)^{p+q+2}\|(A_0^-,B_0^-)\|_{\tilde{Y}_{k,j}(H)},\quad(2.33)$$

$$\sup_{s_1,s_2}\int_{-\infty}^{\infty}|(e^{i\tau H}H^k\alpha_p)(s_1),\langle\tau\rangle^j\beta_q(s_2))|\}\,d\tau\le 3(c_j\delta_1)^{p+q+2}\|(A_0^-,B_0^-)\|_{\tilde{Y}_{k,j}(H)},\quad(2.34)$$

$$\sup_{s_1,s_2}\int_{-\infty}^{\infty}|(e^{i\tau H}H^k\beta_p)(s_1),\langle\tau\rangle^j\beta_q(s_2))|\}\,d\tau\le 3(c_j\delta_1)^{p+q+2}\|(A_0^-,B_0^-)\|_{\tilde{Y}_{k,j}(H)}.\quad(2.35)$$

(iii)

$$\int_{-\infty}^{\infty}|(e^{i\tau H}H^k\alpha_p(\tau),\langle\tau\rangle^jC)|\,d\tau\le c_j(c_j\delta_1)^{p+1}\|(A_0^-,B_0^-)\|_{\tilde{Y}_{k,j}(H)},\quad(2.36)$$

$$\int_{-\infty}^{\infty}|(e^{i\tau H}H^k\beta_p(\tau),\langle\tau\rangle^jC)|\,d\tau\le c_j(c_j\delta_1)^{p+1}\|(A_0^-,B_0^-)\|_{\tilde{Y}_{k,j}(H)}\quad(2.37)$$

where $C=A_0^-$ *or* B_0^-.

(iv)

$$\sup_{s_1}\int_{-\infty}^{\infty}|(e^{i\tau H}H^k\alpha_p(\tau),\langle\tau\rangle^j\alpha_q(s_1))|\,d\tau\le 3(c_j\delta_1)^{p+q+2}\|(A_0^-,B_0^-)\|_{\tilde{Y}_{k,j}(H)},\quad(2.38)$$

$$\sup_{s_1}\int_{-\infty}^{\infty}|(e^{i\tau H}H^k\alpha_p(\tau),\langle\tau\rangle^j\beta_q(s_1))|\,d\tau\le 3(c_j\delta_1)^{p+q+2}\|(A_0^-,B_0^-)\|_{\tilde{Y}_{k,j}(H)},\quad(2.39)$$

$$\sup_{s_1}\int_{-\infty}^{\infty}|(e^{i\tau H}H^k\beta_p(\tau),\langle\tau\rangle^j\beta_q(s_1))|\,d\tau\le 3(c_j\delta_1)^{p+q+2}\|(A_0^-,B_0^-)\|_{\tilde{Y}_{k,j}(H)}.\quad(2.40)$$

(v)

$$\int_{-\infty}^{\infty} |(e^{i\tau H} H^k \alpha_p(\tau), \langle\tau\rangle^j \alpha_q(\tau))| \, d\tau \le 3(c_j\delta_1)^{p+q+2} \|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)}, \quad (2.41)$$

$$\int_{-\infty}^{\infty} |(e^{i\tau H} H^k \alpha_p(\tau), \langle\tau\rangle^j \beta_q(\tau))| \, d\tau \le 3(c_j\delta_1)^{p+q+2} \|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)}, \quad (2.42)$$

$$\int_{-\infty}^{\infty} |(e^{i\tau H} H^k \beta_p(\tau), \langle\tau\rangle^j \beta_q(\tau))| \, d\tau \le 3(c_j\delta_1)^{p+q+2} \|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)}. \quad (2.43)$$

Proof. (i) We shall prove (2.31) and (2.32) by induction of p. For $p = 0$ we have from (2.26)

$$\begin{aligned}
&\int_{-\infty}^{\infty} |(e^{i\tau H} H^k \alpha_0(s_1), \langle\tau\rangle^j C)| \, d\tau \\
&\quad = \int_{-\infty}^{\infty} \left| \left(e^{i\tau H} H^k \int_{-\infty}^{s_1} q(s)(A_0^- - e^{-2isH} B_0^-) \, ds, \langle\tau\rangle^j C \right) \right| d\tau \\
&\quad \le \int_{-\infty}^{\infty} \int |q(s)| \, ds |(e^{\tau H} H^k A_0^-, \langle\tau\rangle^j C)| + |(e^{i\tau H} H^k B_0^-, \langle 2s+\tau\rangle^j C)| \, d\tau \\
&\quad \le c_j^2 \delta_1 |(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)}.
\end{aligned}$$

Here we used $\langle 2s+\tau\rangle \le 3\langle s\rangle\langle\tau\rangle$, $c_j = 2^j$ and $3^j + 1 \le 4^j = c_j^2$. Similarly (2.32) is proved for $p = 0$. Assume that (2.31) and (2.32) are valid for $p - 1$. Then we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} |(e^{i\tau H} H^k \alpha_p(s_1), \langle\tau\rangle^j C)| \, d\tau \\
&\quad = \int_{-\infty}^{\infty} \left| \left(e^{i\tau H} H^k \int_{-\infty}^{s_1} q(s) e^{-isH} \beta_{p-1}(s) \, ds, \langle\tau\rangle^j C \right) \right| d\tau \\
&\quad \le \int_{-\infty}^{s_1} |q(s)| \, ds \int_{-\infty}^{\infty} |(e^{i\tau H} H^k \beta_{p-1}(s), \langle s+\tau\rangle^j C)| \, d\tau \\
&\quad \le c_j \int_{-\infty}^{s_1} |\langle s\rangle^j q(s)| \, ds \int_{-\infty}^{\infty} |(e^{i\tau H} H^k \beta_{p-1}(s), \langle\tau\rangle^j C)| \, d\tau \\
&\quad \le c_j \delta_1 \sup_s \int_{-\infty}^{\infty} |(e^{i\tau H} H^k \beta_{p-1}(s), \langle\tau\rangle^j C)| \, d\tau
\end{aligned}$$

which implies (2.31) for p together with the assumption of induction. (2.32) is proved by the same way.

(ii) We shall prove (2.33)–(2.35) by induction of $p+q$. For $p+q=0$ we can see from (2.26)

$$\int_{-\infty}^{\infty} |((e^{i\tau H}H^k\alpha_0)(s_1), \langle\tau\rangle^j\alpha_0(s_2))|\,d\tau$$

$$= \int_{-\infty}^{\infty} \left| \left(e^{i\tau H}H^k \int_{-\infty}^{s_1} q(s)(A_0^- - e^{-2isH}B_0^-)\,ds, \right.\right.$$

$$\left.\left. \langle\tau\rangle^j \int_{-\infty}^{s_2} q(t)(e^{2itH}A_0^- - B_0^-)\,dt \right) \right| d\tau$$

$$\le \left(\int_{-\infty}^{\infty} |\langle s\rangle^j q(s)|\,ds \right)^2 \int_{-\infty}^{\infty} \{4^j|(e^{i\tau H}H^kA_0^-, \langle\tau\rangle^jA_0^-)|$$

$$+ 2^{j+1}|(e^{i\tau H}H^kA_0^-, \langle\tau\rangle^jB_0^-)| + 4^j|(e^{i\tau H}H^kB_0^-, \langle\tau\rangle^jB_0^-)|\}\,d\tau$$

$$\le 3(c_j\delta_1)^2\|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)}. \tag{2.44}$$

where we used $\langle\tau - 2(s-t)\rangle \le 4\langle\tau\rangle\langle s\rangle\langle t\rangle$ and $\langle 2s+\tau\rangle \le 3\langle s\rangle\langle\tau\rangle$. Assume (2.33)–(2.35) for $p+q-1$. We shall prove (2.33)–(2.35) for $p+q$. We may assume $p \ge 1$. Then

$$\int_{-\infty}^{\infty} |(e^{i\tau H}H^k\alpha_p)(s_1), \langle\tau\rangle^j\alpha_q(s_2))|\,d\tau$$

$$= \int_{-\infty}^{\infty} \left| \left(e^{i\tau H}H^k \int_{-\infty}^{s_1} q(s)e^{-isH}\beta_{p-1}(s)\,ds, \langle\tau\rangle^j\alpha_q(s_2) \right| d\tau \right.$$

$$\le c_j \int_{-\infty}^{\infty} |\langle s\rangle^j q(s)|\,ds \sup_{s,s_2} \int_{-\infty}^{\infty} |(e^{i\tau H}H^k\beta_{p-1}(s), \langle\tau\rangle^j\alpha_q(s_2))|\,d\tau,$$

which implies (2.33) from the assumption of induction. We can show (2.34) and (2.35) similarly.

(iii) Put $\alpha_{-1} = \gamma(\tau)^{-1/2}(e^{i\tau H}A_0^- - e^{-i\tau H}B_0^-)$ and $\beta_{-1} = -\gamma(\tau)^{-1/2}(e^{i\tau H}A_0^- - e^{-i\tau H}B_0^-)$. Then we have from (2.26) for $p \ge 0$

$$\int_{-\infty}^{\infty} |(e^{i\tau H}H^k\alpha_p(\tau), \langle\tau\rangle^jC)|\,d\tau$$

$$= \int_{-\infty}^{\infty} \left| \left(e^{i\tau H}H^k \int_{-\infty}^{\tau} q(s)e^{-isH}\beta_{p-1}(s)\,ds, \langle\tau\rangle^jC \right) \right| d\tau$$

$$\le \int_{-\infty}^{\infty} |q(s)|\,ds \int_{-\infty}^{\infty} |(e^{i\tau H}H^k\beta_{p-1}(s), \langle s+\tau\rangle^jC)|\,d\tau$$

$$\le c_j \int_{-\infty}^{s_1} |\langle s\rangle^j q(s)|\, ds \int_{-\infty}^{\infty} |(e^{i\tau H} H^k \beta_{p-1}(s), \langle\tau\rangle^j C)|\, d\tau$$

$$\le c_j\delta_1 \sup_s \int_{-\infty}^{\infty} |(e^{i\tau H} H^k \beta_{p-1}(s), \langle\tau\rangle^j C)|\, d\tau,$$

which implies (2.36) for p together with (2.32) with $p-1$. We can prove (2.37) similarly.

(iv) We get

$$\int_{-\infty}^{\infty} |(e^{i\tau H} H^k \alpha_p)(\tau), \langle\tau\rangle^j \alpha_q(s_2))|\, d\tau$$

$$= \int_{-\infty}^{\infty} \left| \left(e^{i\tau H} H^k \int_{-\infty}^{\tau} q(s) e^{-isH} \beta_{p-1}(s)\, ds, \langle\tau\rangle^j \alpha_q(s_2) \right| d\tau\right.$$

$$\le c_j \int_{-\infty}^{\infty} |\langle s\rangle^j q(s)|\, ds \sup_{s, s_2} \int_{-\infty}^{\infty} |(e^{i\tau H} H^k \beta_{p-1}(s), \langle\tau\rangle^j \alpha_q(s_2)|\, d\tau,$$

which implies (2.38) with (2.34). We can show (2.39)–(2.40) by the same argument.

(v) For any (p,q), $p,q \ge 0$ we have

$$\int_{-\infty}^{\infty} |(e^{i\tau H} H^k \alpha_p(\tau), \langle\tau\rangle^j \alpha_q(\tau))|\, d\tau$$

$$= \int_{-\infty}^{\infty} \left| \left(e^{i\tau H} H^k \alpha_p(\tau), \tau^j \int_{-\infty}^{\tau} q(s) e^{-isH} \beta_{q-1}(s)\, ds \right) \right| d\tau$$

$$\le c_j \int |\langle s\rangle^j q(s)|\, ds \sup_s \int_{-\infty}^{\infty} |(e^{i\tau H} H^k \alpha_p(\tau), \langle\tau\rangle^j \beta_{q-1}(s))|\, d\tau,$$

which yields (2.41) together with (2.39) for $(p, q-1)$. (2.42) and (2.43) can be poved by the same way. □

We remark that we can replace $e^{i\tau H}$ in the integrands of left hand sides of all cases in Lemma 2.2 to $e^{i\mu\tau H}$ and then the constant M in the right hand sides is changed to $\frac{M}{|\mu|^{1+j}}$, $(\mu \ne 0)$.

Proposition 2.2. *Let γ be in $X_{\delta,M}$, $q(\tau) = \frac{\gamma'(\tau)}{2\gamma(\tau)}$ and $k \ge 0, j$ a non negative integer. Assume that H is a self adjoint operator.*

(i) *If (A_0, B_0) belongs to $\tilde{Y}_{k,j}(H)$ and $\delta_1 = \int \langle s\rangle^j |q(s)|\, ds < 2^{-j} = c_j^{-1}$, then for any $s, t \in R$, $(U(s), V(t))$ belongs to $\tilde{Y}_{k,j}(H)$ and satisfies*

$$\sup_{s_1, s_2 \in R} \|(U(s_1), V(s_2))\|_{\tilde{Y}_{k,j}(H)} \le \frac{6(c_j\delta_1)^2 M}{(1-c_j\delta_1)^2} \|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)} \tag{2.45}$$

and

$$\int_{-\infty}^{\infty} \tilde{G}_{k,j}(H, U(\tau), V(\tau), \mu\tau))\, d\tau \le \frac{6(c_j\delta_1)^2 M}{|\mu|^{j+1}(1-c_j\delta_1)^2} \|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)}, \quad (2.46)$$

for real $\mu \neq 0$. *Moreover*

$$\int_{-\infty}^{\infty} |(H^k U(\tau), \langle\tau\rangle^j V(\tau))|\, d\tau \le \frac{2(c_j\delta_1)^2 M}{(1-c_j\delta_1)^2} \|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)} \quad (2.47)$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} &\{|(e^{i\mu\tau H} H^k U(\tau), \langle\tau\rangle^j B_0^-)| + |(e^{i\mu\tau H} H^k V(\tau), \tau^j A_0^-)| \\ &+ |(e^{i\mu\tau H} H^k U(\tau), \tau^j A_0^-)| + |(e^{i\mu\tau H} H^k V(\tau), \langle\tau\rangle^j B_0^-)|\}\, d\tau \\ &\le \frac{c_j\delta_1 M}{|\mu|^{j+1}(1-c_j\delta_1)} \|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)} \quad (2.48)\end{aligned}$$

hold for $\mu \neq 0$, *where* $c_j = 2^j$.

(ii) *If* (A_0, B_0) *belongs to* $\tilde{Y}_k(H)$ *and* $\delta_1 = \int |q(\tau)|\, d\tau < 1$, *then* (U^+, V^+) *given in Proposition* 2.1 *belongs to* $\tilde{Y}_k(H)$ *and satisfies*

$$\begin{aligned}\lim_{s\to\infty} \int_{-\infty}^{\infty} &|(e^{i\tau H} H^k U(s), U(s))| + |(e^{i\tau H} H^k U(s), V(s))| + |(e^{i\tau H} H^k V(s), V(s))|\, d\tau \\ = \int_{-\infty}^{\infty} &|(e^{i\tau H} H^k U^+, U^+)| + |(e^{i\tau H} H^k U^+, V^+)| + |(e^{i\tau H} H^k V^+, V^+)|\, d\tau \quad (2.49)\end{aligned}$$

and

$$\begin{aligned}\lim_{s\to\infty} \int_{-\infty}^{\infty} &|(e^{i\tau H} U(s), B_0^-)| + |(e^{i\tau} V(s), A_0^-)|\, d\tau \\ &= \int_{-\infty}^{\infty} |(e^{i\tau H} U^+, B_0^-)| + |(e^{i\tau H} V^+, A_0^-)|\, d\tau. \quad (2.50)\end{aligned}$$

(iii) *If* $(A_0, B_0) \in L^2(R^n)^2$, $\int |q(\tau)|\, d\tau < 1$ *and*

$$\tilde{G}_0(H, A_0^-, B_0^-, \tau) \to 0, \quad \tau \to \infty, \quad (2.51)$$

then for any $s_1, s_2 \in R$

$$\tilde{G}_0(H, U(s_1), V(s_2), \tau) \to 0, \quad \tau \to \infty \quad (2.52)$$

$$\tilde{G}_0(H, U(s_1), B^-(s_2), \tau) \to 0, \quad \tau \to \infty \quad (2.53)$$

and

$$\tilde{G}_0(H, V(s_1), A^-(s_2), \tau) \to 0, \quad \tau \to \infty \tag{2.54}$$

hold.

Proof. (i) Taking the summation of p and q in (2.33), we get

$$\sup_{s,t\in R}\int_{-\infty}^{\infty} |(e^{i\tau H}H^k\alpha(s), \langle\tau\rangle^j\alpha(t))|\, d\tau \le \sup_{s,t\in R}\sum_{p,q}\int_{-\infty}^{\infty} |(e^{i\tau H}H^k\alpha_p(s), \langle\tau\rangle^j\alpha_q(t))|\, d\tau$$

$$\le \frac{3(c_j\delta_1)^2}{(1-c_i\delta_1)^2}\|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)}, \tag{2.55}$$

if $0 < c_j\delta_1 < 1$. Similarly we can estimate from (2.34) and (2.35)

$$\sup_{s,t\in R}\int_{-\infty}^{\infty} |(e^{i\tau H}H^k\alpha(s), \langle\tau\rangle^j\beta(t))|\, d\tau \le \frac{3(c_j\delta_1)^2}{(1-c_j\delta_1)^2}\|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)} \tag{2.56}$$

and

$$\sup_{s,t\in R}\int_{-\infty}^{\infty} |(e^{i\tau H}H^k\beta(s), \langle\tau\rangle^j\beta(t))|\, d\tau \le \frac{3(c_j\delta_1)^2}{(1-c_j\delta_1)^2}\|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)}. \tag{2.57}$$

Taking account of the relations below

$$U(\tau, x) = \gamma^{1/2}e^{i\tau H}\alpha(\tau, x), \quad V(\tau, x) = \gamma^{1/2}e^{-i\tau H}\beta(\tau, x) \tag{2.58}$$

and $\gamma(\tau) \le M$ for any $\tau \in R$, we obtain (2.78) from (2.55), (2.56) and (2.57). Similarly we can prove (2.46) from (v) of Lemma 2.1. We have from the relation (2.58) and from (2.42)

$$\int_{-\infty}^{\infty} |(H^kU(\tau), \langle\tau\rangle^jV(\tau)|\, d\tau \le M\int_{-\infty}^{\infty} |(e^{2i\tau H}H^k\alpha(\tau), \langle\tau\rangle^j\beta(\tau)|\, d\tau$$

$$\le M\sum_{p,q}\int_{-\infty}^{\infty} |(e^{2\tau H}H^k\alpha_p(\tau), \langle\tau\rangle^j\beta_q(\tau)|\, d\tau$$

$$\le \frac{3(c_j\delta_1)^2M}{(1-c_j\delta_1)^2}\|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)},$$

which means (2.47). We can prove (2.48) easily from (iii) of Lemma 2.2.

(ii) In order to prove (2.49) it suffices to show that (α^+, β^+) belongs to $\tilde{Y}_k(H)$ and satisfies

$$\lim_{s\to\infty}\int_{-\infty}^{\infty}\{|(e^{i\tau H}H^k\alpha(s),\alpha(s))|+|(e^{i\tau H}H^k\alpha(s),\beta(s))|+|(e^{i\tau H}H^k\beta(s),\beta(s))|\}\,d\tau$$

$$=\int_{-\infty}^{\infty}\{|(e^{i\tau H}H^k\alpha^+,\alpha^+)|+|(e^{i\tau H}H^k\alpha^+,\beta(s))|+|(e^{i\tau H}H^k\beta^+,\beta^+)|\}\,d\tau. \quad (2.59)$$

Because $U^+=(\gamma_\infty^+)^{1/2}\alpha^+$, $V^+=(\gamma_\infty^+)^{1/2}\beta^+$ and the relation (2.58) imply (2.49) together with (2.59) and with the fact that $\gamma(s)\to\gamma^+$, $s\to\infty$. We shall show that for each (p,q) there is (α_p^+,β_q^+) such that $\|H^k(\alpha_p(s)-\alpha_p^+)\|+\|H^k(\beta_q(s)-\beta_q^+)\|\to 0$, $s\to\infty$ and

$$\begin{aligned}\int_{-\infty}^{\infty}\{&|(e^{i\tau H}H^k\alpha_p(s),\alpha_q(s))-(e^{i\tau H}H^k\alpha_p^+,\alpha_q^+)|\\&+|(e^{i\tau H}H^k\alpha_p(s),\beta_q(s))-(e^{i\tau H}H^k\alpha_p^+,\beta_q^+)|\\&+|(e^{i\tau H}H^k\beta_p(s),\beta_q(s))-(e^{i\tau H}H^k\beta_p^+,\beta_q^+)|\}\,d\tau\\ \le 18\int_s^{\infty}&|q(t)|\,dt\delta_1^{p+q+1}\|(A_0^-,B_0^-)\|_{\tilde{Y}_{k,j}(H)}.\end{aligned} \quad (2.60)$$

In fact, we can give from (2.27) (denote by $q(s)=\frac{\gamma'(s)}{2\gamma(s)}$),

$$\alpha_p^+=\int_{-\infty}^{\infty}q(\eta)e^{2i\eta H}\beta_{p-1}(\eta)\,d\eta,$$

which satisfies

$$\alpha_p^+-\alpha_p(s)=\int_s^{\infty}q(\eta)e^{2i\eta H}\beta_{p-1}(\eta)\,d\eta.$$

Therefore we get by use of (2.34) with $j=0$

$$\int_{-\infty}^{\infty}|(e^{i\tau H}(\alpha_p^+-\alpha_p(s)),\alpha_q(s))|\,d\tau\le 3\int_s^{\infty}|q(\eta)|\,d\eta\delta_1^{p+q+1}\|(A_0^-,B_0^-)\|_{\tilde{Y}_k(H)}.$$

Similarly we can show that $(e^{i\tau H}(\alpha_p^+-\alpha_p(s)),\beta_q)$, $(e^{i\tau H}(\beta_p^+-\beta_p(s)),\beta_q)$ and $(e^{i\tau H}(\beta_p^+-\beta_p(s)),\alpha_q)$ satisfy the same inequality as above. Thus we get (2.60). Taking the summation of (2.60) and tending s to ∞, we obtain (2.59), because $q(t)$ is in $L^1(R)$ and $\delta_1<1$. We can show also (2.50) analogously.

(iii) We shall prove by induction of $p+q$

$$\begin{aligned}&|(e^{i\tau H}\alpha_p(s_1),\alpha_q(s_2)|+|(e^{i\tau H}\alpha_p(s_1),\beta_q(s_2)|\\&\quad+|(e^{i\tau H}\beta_p(s_1),\beta_q(s_2)|\to 0,\quad \tau\to\infty\end{aligned} \quad (2.61)$$

uniformly in $s_1, s_2 \in R$. When $p = q = 0$ we have

$$
\begin{aligned}
&(e^{i\tau H}\alpha_0(s_1), \alpha_0(s_2)) \\
&\quad = \int_{-\infty}^{s_1}\int_{-\infty}^{s_2} q(s)q(t)(\gamma(s)\gamma(t))^{-1/2}(e^{i\tau H - isH}W_1(s,\cdot), e^{-itH}W_1(t,\cdot)\ dsdt \\
&\quad = \int_{-\infty}^{s_1}\int_{-\infty}^{s_2} q(s)q(t)(\gamma(s)\gamma(t))^{-1/2}\{(e^{i\tau H}A_0, A_0) - (e^{i(\tau+2t)H}A_0^-, B_0^-) \\
&\qquad\qquad - (e^{i(\tau-2s)H}B_0^-, A_0^-) + (e^{i(\tau-2s+2t)H}B_0^-, B_0^-)\}\ dsdt, \\
&\quad =; \int_{-\infty}^{s_1}\int_{-\infty}^{s_2} q(s)q(t)f(\tau,s,t)\ dsdt,
\end{aligned}
$$

which converges to 0, $\tau \to \infty$. In fact, since $q(s)q(t)$ is in $L^1(R^2)$, $f(\tau,s,t)$ is bounded from the assumption $(A_0^-, B_0^-) \in L^2(R^n) \times L^2(R^n)$ and the assumption of (iii) implies $f(\tau,s,t) \to 0$, $\tau \to \infty$, we can apply to the right hand side of the above terms Lebesgue convergence theorem. Similarly we can show that (α_0, β_0) and (β_0, β_0) satisfy (2.61) for $p = q = 0$. Next assume (2.61) is valid for $p + q - 1$. We shall show (2.61) for $p + q \geq 1$. We may assume $p \geq 1$. Then we see from (2.27)

$$
\begin{aligned}
(e^{i\tau H}\alpha_p(s_1), \alpha_q(s_2)) &= -\int_{-\infty}^{s_1} q(s)\gamma(s)^{1/2}(e^{i(\tau-2s)H}\beta_{p-1}(s), \alpha_q(s_2))\ ds \\
&=; \int_{-\infty}^{s_1} q(s)f_{p-1,q}(\tau,s)\ ds. \qquad (2.62)
\end{aligned}
$$

which tends to zero uniformly in s_1, $\tau \to \infty$. Because $q(s)$ is in $L^1(R)$ and $f_{p-1,q}(\tau,s)$ is bounded and converges to zero, $\tau \to \infty$. Similarly we can see that $(e^{i\tau H}\alpha_p(s_1), \beta_q(s_2))$, and $(e^{i\tau H}\beta_p(s_1), \beta_q(s_2))$ satisfy the property (2.62). It follows from (2.29) that for any $\varepsilon > 0$ there is an integer N independent of (τ, s_1, s_2) such that

$$
\sum_{p+q \geq N}\{|(e^{i\tau H}\alpha_p(s_1), \alpha_q(s_2))| + |(e^{i\tau H}\alpha_p(s_1), \beta_q(s_2))| + |(e^{i\tau H}\beta_p(s_1), \beta_q(s_2))|\} \leq \varepsilon.
$$

Hence we see

$$
\begin{aligned}
&|(e^{i\tau H}\alpha(s_1), \alpha(s_2))| + |(e^{i\tau H}\alpha(s_1), \beta(s_2))| + |(e^{i\tau H}\beta(s_1), \beta(s_2))| \\
&\quad \leq \sum_{p,q}\{|(e^{i\tau H}\alpha_p(s_1), \alpha_q(s_2))| + |(e^{i\tau H}\alpha_p(s_1), \beta_q(s_2))| + |(e^{i\tau H}\beta_p(s_1), \beta_q(s_2))|\}
\end{aligned}
$$

$$\le \sum_{p+q\le N} \{|(e^{i\tau H}\alpha_p(s_1), \alpha_q(s_2))| + |(e^{i\tau H}\alpha_p(s_1), \beta_q(s_2))|$$
$$+ |(e^{i\tau H}\beta_p(s_1), \beta_q(s_2))|\} + \varepsilon.$$

On the other hand from (2.61)

$$\sum_{p+q\le N} \{|(e^{i\tau H}\alpha_p(s_1), \alpha_q(s_2))| + |(e^{i\tau H}\alpha_p(s_1), \beta_q(s_2))|$$
$$+ |(e^{i\tau H}\beta_p(s_1), \beta_q(s_2))|\} \to 0, \quad \tau \to \infty.$$

Therefore we get

$$\limsup_{\tau\to\infty}\{|(e^{i\tau H}\alpha(s_1), \alpha(s_2))| + |(e^{i\tau H}\alpha(s_1), \beta(s_2))| + |(e^{i\tau H}\beta(s_1), \beta(s_2))|\} \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\lim_{\tau\to\infty}\{|(e^{i\tau H}\alpha(s_1), \alpha(s_2))| + |(e^{i\tau H}\alpha(s_1), \beta(s_2))| + |(e^{i\tau H}\beta(s_1), \beta(s_2))|\} = 0$$

which implies (2.52). Noting that

$$(e^{i\tau H}\alpha_p(s_1), B^-(s_2)) = -\int_{-\infty}^{s_1} q(s)\gamma(s)^{1/2}(e^{i(\tau-2s)H}\beta_{p-1}(s), e^{-is_2 H}B_0^-)\,ds$$

and

$$(e^{i\tau H}\beta_p(s_1), A^-(s_2)) = -\int_{-\infty}^{s_1} q(s)\gamma(s)^{1/2}(e^{i(\tau-2s)H}\alpha_{p-1}(s), e^{-is_2 H}A_0^-)\,ds$$

we can prove (2.53) and (2.54) analogously to (2.52). □

The solution (U, V) of (2.15)–(2.17) depends on $\gamma \in X_{\delta,M}$. So we denote it by (U_γ, V_γ).

Proposition 2.3. *Let γ_1, γ_2 be in $X_{\delta,M}$, $k \ge 0$ and j a non negative integer. Assume that H is a selfadjoint operator.*

(i) *Assume that (A_0^-, B_0^-) belongs to $D(H^k)\times D(H^k)$. Then $(U_{\gamma_1}, V_{\gamma_1})$ and $(U_{\gamma_2}, V_{\gamma_2})$ satisfy*

$$\|H^k(U_{\gamma_1}(\tau,\cdot) - U_{\gamma_2}(\tau,\cdot))\| + \|H^k(V_{\gamma_1}(\tau,\cdot) - V_{\gamma_2}(\tau,\cdot))\|$$
$$\le C(\|H^k A_0^-\| + \|H^k B_0^-\|)|\gamma_1 - \gamma_2|_X. \tag{2.63}$$

(ii) *Assume that* (A_0^-, B_0^-) *belongs to* $\tilde{Y}_{k,j}(H)$ *and* $\delta_1 = \int \langle s\rangle^j |q(s)|\, ds < c_j^{-1}$, $(c_j = 2^j)$. *Then*

$$\int_{-\infty}^{\infty} \{|(H^k(U_{\gamma_1} - U_{\gamma_2}(\tau)), \langle\tau\rangle^j V_{\gamma_1}(\tau))| + |(H^k(V_{\gamma_1} - V_{\gamma_2}(\tau)), \langle\tau\rangle^j U_{\gamma_2}(\tau))|\}\, d\tau$$

$$\leq \frac{3e^{c_j\delta_1}M}{1 - c_j\delta_1}|\gamma_1 - \gamma_2|_X \|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)} \tag{2.64}$$

and

$$\int_{-\infty}^{\infty} \{|(H^k(U_{\gamma_1} - U_{\gamma_2}(\tau)), \langle\tau\rangle^j e^{-i\tau H} B_0^-)| + |(H^k(V_{\gamma_1} - V_{\gamma_2}(\tau)), \langle\tau\rangle^j e^{i\tau H} A_0^-)|\}\, d\tau$$

$$\leq C|\gamma_1 - \gamma_2|_X \|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)} \tag{2.65}$$

hold, where $C = C(M, \delta_1, j)$.

Proof. (i) Put

$$\alpha(\tau, x) = H^k(U_{\gamma_1} - U_{\gamma_2})(\tau, x), \quad \beta(\tau, x) = H^k(V_{\gamma_1} - V_{\gamma_2})(\tau, x).$$

Then (α, β) satisfies from (2.15) and (2.16)

$$\alpha(\tau, x) = e^{i\tau H}\int_{-\infty}^{\tau} (F_{\gamma_1} - F_{\gamma_2})(s, x)\, ds,$$

$$\beta(\tau, x) = e^{-i\tau H}\int_{-\infty}^{\tau} (G_{\gamma_1} - G_{\gamma_2})(s, x)\, ds, \tag{2.66}$$

where

$$F_\gamma(\tau, x) = \frac{\gamma'(\tau)}{2\gamma(\tau)} e^{-isH} H^k\{(U_\gamma - V_\gamma)(\tau, x) + \gamma(\tau)^{-1/2}(A^- - B^-)(\tau, x)\} \tag{2.67}$$

and

$$G_\gamma(\tau, x) = -\frac{\gamma'(\tau)}{2\gamma(\tau)} e^{i\tau H} H^k\{(U_\gamma - V_\gamma)(\tau, x) - \gamma(\tau)^{-1/2}(A^- - B^-)(\tau, x)\} \tag{2.68}$$

Hence we see

$$(F_{\gamma_1} - F_{\gamma_2})(s, x) = \frac{1}{2}\left(\frac{\gamma_1'(s)}{\gamma_1(s)} - \frac{\gamma_2'(s)}{\gamma_2(s)}\right) e^{-isH} H^k(U_{\gamma_1} - V_{\gamma_1} + \gamma_1(s)^{-1/2}(A^- - B^-))(s, x)$$
$$+ \frac{\gamma_2'(s)}{2\gamma_2(s)} e^{-isH} H^k\{(U_{\gamma_1} - U_{\gamma_2} - V_{\gamma_1} + V_{\gamma_2})(s, x)$$
$$+ (\gamma_1(\tau)^{-1/2} - \gamma_2(s)^{-1/2})(A^- - B^-)\}(s, x)$$

and

$$
\begin{aligned}
&(G_{\gamma_1} - G_{\gamma_2})(s,x)) \\
&\quad = -\frac{1}{2}\left(\frac{\gamma_1'(s)}{\gamma_1(s)} - \frac{\gamma_2'(s)}{\gamma_2(s)}\right) e^{i\tau H} H^k (U_{\gamma_1} - V_{\gamma_1} - \gamma_1(s)^{-1/2}(A^- - B^-))(s,x) \\
&\qquad - \frac{1}{2}\frac{\gamma_2'(s)}{\gamma_2(s)} e^{i\tau H} H^k \{(U_{\gamma_1} - U_{\gamma_2} - V_{\gamma_1} + V_{\gamma_2})(s,x) \\
&\qquad\qquad - (\gamma_1(s)^{-1/2} - \gamma_2(s)^{-1/2})(A^- - B^-)\}(s,x).
\end{aligned}
$$

Define

$$
e(\tau) = \sup_{s\le\tau}(\|\alpha(s)\| + \|\beta(s)\|).
$$

Then the above equations yield

$$
\begin{aligned}
e(\tau) \le \int_{-\infty}^{\tau} &\left\{ \frac{1}{2}|\gamma_2'(s)| e(s) + \left|\frac{\gamma_1'(s)}{2\gamma_1(s)} - \frac{\gamma_2'(s)}{2\gamma_2(s)}\right| \right. \\
&\times \|H^k(U_{\gamma_1} - V_{\gamma_1} - \gamma_1(s)^{-1/2}(A^- - B^-))(s))\| \\
&\left. + \left|\frac{\gamma_2'(s)}{2\gamma_2(s)}(\gamma_1(s)^{-1/2} - \gamma_2(s)^{-1/2})\right| \|H^k(A^- - B^-)\| \right\} ds
\end{aligned}
$$

which implies (2.63) together with Gronwall inequality.

(ii) Put

$$
\begin{aligned}
\zeta(s) &= \gamma_2(s)^{-1/2} e^{-isH}(U_{\gamma_1}(s) - U_{\gamma_2}(s)), \\
\eta(s) &= \gamma_2(s)^{-1/2} e^{isH}(V_{\gamma_1}(s) - V_{\gamma_2}(s)).
\end{aligned} \tag{2.69}
$$

Then (ζ, η) satisfies from (2.15)–(2.17)

$$
\zeta(\tau) = -\int_{-\infty}^{\tau} (q_2(s)(e^{-2isH}\eta(s) + f_1(s))\, ds \tag{2.70}
$$

and

$$
\eta(\tau) = -\int_{-\infty}^{\tau} (q_2(s)(e^{2isH}\zeta(s) + f_2(s))\, ds, \tag{2.71}
$$

where $q_j(s) = \frac{\gamma_j'(s)}{2}$, $j = 1, 2$,

$$
\begin{aligned}
f_1(s) = \gamma_2(s)^{-1/2} e^{-isH}\{&(q_1(s) - q_2(s))(U_{\gamma_1}(s) - V_{\gamma_1}(s) + W_{\gamma_1}(s)) \\
&+ q_2(s)(W_{\gamma_1}(s) - W_{\gamma_2}(s))\},
\end{aligned}
$$

$$f_2(s) = -\gamma_2(s)^{-1/2} e^{isH}\{(q_1(s) - q_2(s))(U_{\gamma_1}(s) - V_{\gamma_1}(s) + W_{\gamma_1}(s)) - q_2(s)(W_{\gamma_1}(s) - W_{\gamma_2}(s))\}$$

and $W_\gamma(s) = \gamma(s)^{-1/2}(e^{isH}A_0^- - e^{-isH}B_0^-)$. It follows from Proposition 2.2 that we can see

$$\int_{-\infty}^{\infty} |(H^k f_m(\tau), \langle\tau\rangle^j U_{\gamma_l}(\tau))| + |(H^k f_m(\tau), \langle\tau\rangle^j V_{\gamma_l}(\tau))|\, d\tau \le C|\gamma_1 - \gamma_2|_X \|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)}$$

and

$$\int_{-\infty}^{\infty} |(H^k f_m(\tau), \langle\tau\rangle^j e^{i\tau H} A_0^-)| + |(H^k f_m(\tau), \langle\tau\rangle^j e^{-i\tau H} B_0^-)|\, d\tau \le C|\gamma_1 - \gamma_2|_X \|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)}$$

for $m, l = 1, 2$, where $C = C(M, \delta)$. Therefore we can prove (2.64) and (2.65) analogously to (2.47) and (2.48). In fact, we seek the solution (ζ, η) of (2.70) and (2.71) as $\zeta(\tau) = \sum_{p=0}^{\infty} \zeta_p(\tau)$, $\eta(\tau) = \sum_{p=0}^{\infty} \eta_p(\tau)$, where

$$\zeta_0(\tau) = -\int_{-\infty}^{\tau} q_2(s) f_1(s)\, ds, \quad \eta_0(\tau) = -\int_{-\infty}^{\tau} q_2(s) f_2(s)\, ds$$

and for $p \ge 1$

$$\zeta_p(\tau) = -\int_{-\infty}^{\tau} q_2(s) e^{-2isH} \eta_{p-1}(s)\, ds, \quad \eta_p(\tau) = -\int_{-\infty}^{\tau} q_2(s) e^{2isH} \zeta_{p-1}(s)\, ds.$$

Then we can show similarly to the proof of (v) in Lemma 2.2

$$\int_{-\infty}^{\infty} |(e^{i\tau H} H^k \zeta_p(\tau), \langle\tau\rangle^j \beta_q(\tau))| + |(e^{i\tau H} H^k \eta_p(\tau), \langle\tau\rangle^j \alpha_q(\tau))|\, d\tau \le C(c_j\delta_1)^{p+q+2} |\gamma_1 - \gamma_1|_X \|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)},$$

where (α_p, β_p) is defined by (2.27) and (2.28), and

$$\int_{-\infty}^{\infty} |(e^{i\tau H} H^k \zeta_p(\tau), \langle\tau\rangle^j B_0^-)| + |(e^{-i\tau H} H^k \eta_p(\tau), \langle\tau\rangle^j A_0^-)|\, d\tau \le C(c_j\delta_1)^{p+1} |\gamma_1 - \gamma_1|_X \|(A_0^-, B_0^-)\|_{\tilde{Y}_{k,j}(H)}.$$

Then we can show that (ζ_p, η_p) satisfies the above estimates similarly to (iii) and (v) of Lemma 2.1. □

We continue again to prove Theorem 1.1. For $\gamma \in X_{\delta,M}$ we denote by (A_γ, B_γ) the solution of (2.5)–(2.7) and define

$$\Phi(\gamma)^2(\tau) = 1 + \frac{\varepsilon}{4\gamma(\tau)^2}\|A_\gamma(\tau) - B_\gamma(\tau)\|^2$$

$$= 1 + \frac{\varepsilon}{4\gamma(\tau)^2}\|U_\gamma(\tau) - V_\gamma(\tau) + W(\tau)\|^2,$$

where $(U_\gamma, V_\gamma) = (A_\gamma, B_\gamma) - (A^-, B^-)$ denotes the solution of (2.15)–(2.17) and $W(\tau, x) = (A^- - B^-)(\tau, x)$. We shall prove that $\Phi(\gamma)$ is a contraction mapping in $X_{\delta,M}$. It is trivial that $1 \le \Phi(\gamma)(\tau)^2 \le 1 + C(\delta)\varepsilon \le M^2$, if ε and M are chosen suitablly, because U_γ, V_γ, and W are bounded in $L^2(R^n)$ from Proposition 2.1. Next we shall prove that $\Phi(\gamma)'(\tau)$ belongs to $L^1(R)$. Differentiating $\Phi^2(\gamma)(\tau)$ with respect to τ we have

$$2\Phi(\gamma)(\tau)\Phi(\gamma)'(\tau) = \frac{-\varepsilon\gamma'(\tau)}{2\gamma(\tau)^3}\|A_\gamma(\tau) - B_\gamma(\tau)\|^2$$

$$+ \frac{\varepsilon}{2\gamma(\tau)^2}\Re((A_\gamma(\tau) - B_\gamma)'_\tau(\tau), A_\gamma(\tau) - B_\gamma(\tau))$$

It follows from (2.5)

$$\Re((A_\gamma(\tau) - B_\gamma(\tau))'_\tau, A_\gamma(\tau) - B_\gamma(\tau))$$

$$= \Re(iH(A_\gamma(\tau) + B_\gamma(\tau)) + \frac{\gamma'}{\gamma}(A_\gamma(\tau) - B_\gamma(\tau), A_\gamma(\tau) - B_\gamma(\tau))$$

$$= 2\Im(HA_\gamma(\tau), B_\gamma(\tau)) + \frac{\gamma'}{\gamma}\|(A_\gamma(\tau) - B_\gamma(\tau))\|^2.$$

Hence we obtain

$$2\Phi(\gamma)(\tau)\Phi(\gamma)'(\tau) = \frac{\varepsilon}{\gamma(\tau)^2}\Im(HA_\gamma(\tau), B_\gamma(\tau)). \tag{2.72}$$

Moreover taking account that $A_\gamma = e^{i\tau H}A_0^- + U_\gamma$, $B_\gamma = e^{-i\tau H}B_0^- + V_\gamma$ we have from (2.72)

$$2\Phi(\gamma)(\tau)\Phi(\gamma)'(\tau) = \frac{\varepsilon}{2\gamma(\tau)^2}\Im\{(HU_\gamma(\tau), V_\gamma(\tau)) + (e^{i\tau H}HU_\gamma(\tau), B_0^-)$$

$$+ (e^{i\tau H}HA_0^-, V_\gamma(\tau)) + (e^{2i\tau H}HA_0^-, B_0^-)\}, \tag{2.73}$$

which belongs to $L^1(R)$ from (i) of Proposition 2.2 with $k = 1$, $j = 0$. Now we can show that Φ is a contraction mapping in $X_{\delta,M}$. It follows from Proposition

2.1 and Proposition 2.3 that we can show

$$|\Phi(\gamma_1) - \Phi(\gamma_2)|_X \le C\varepsilon|\gamma_1 - \gamma_2|_X \|(A_0^-, B_0^-)\|_{\tilde{Y}_1(H)} \tag{2.74}$$

for any $\gamma_1, \gamma_1 \in X_{\delta,M}$, which implies that Φ is a contraction mapping in $X_{\delta,M}$, if $\varepsilon > 0$ is small. In fact we have from (2.73) and from (i) of Proposition 2.2 with $k = 1$, $j = 0$

$$\begin{aligned}\int_{-\infty}^{\infty} |\Phi(\gamma)'(\tau)|\, d\tau &\le \int_{-\infty}^{\infty} \frac{1}{2\Phi(\gamma)(\tau)} \frac{\varepsilon}{2\gamma(\tau)^2} |\Im\{(HU_\gamma(\tau), V_\gamma(\tau)) + (e^{i\tau H} HU_\gamma(\tau), B_0^-) \\ &\quad + (e^{i\tau H} HA_0^-, V(\tau)) + (e^{2i\tau H} HA_0^-, B_0^-)\}|\, d\tau \\ &\le C\varepsilon\|(A_0^-.B_0^-)\|_{\tilde{Y}_1(H)} \le \delta\end{aligned}$$

if $\varepsilon > 0$ is small. Besides from (i) of Proposition 2.3 we have

$$\begin{aligned}|\Phi(\gamma_1)^2 - \Phi(\gamma_2)^2| &= \left|\frac{\varepsilon}{4\gamma_1(\tau)^2}\|H(A_{\gamma_1} - B_{\gamma_1})(\tau)\|^2 - \frac{\varepsilon}{4\gamma_2(\tau)}\|H(A_{\gamma_2} - B_{\gamma_2})(\tau)\|^2\right| \\ &\le C\varepsilon|\gamma_1 - \gamma_2|_X\end{aligned}$$

and it follows from (2.72) that we see

$$\begin{aligned}\Phi'(\gamma_1)\Phi(\gamma_1) - \Phi'(\gamma_2)\Phi(\gamma_2) &= \varepsilon\left(\frac{1}{4\gamma_1(\tau)^2} - \frac{1}{4\gamma_2(\tau)^2}\right)\Im(HA_1, B_1) \\ &\quad + \frac{\varepsilon}{4\gamma_2(\tau)^2}\Im\{(H(A_1 - A_2), B_1) + (HA_2, B_1 - B_2)\}.\end{aligned}$$

where $A_j = e^{i\tau H}A_0^- + U_{\gamma_j}(\tau)$ and $B_j = e^{-i\tau H}B_0^- + V_{\gamma_j}(\tau)$, $j = 1, 2$. Besides

$$A_1 - A_2 = U_{\gamma_1} - U_{\gamma_2}, \quad B_1 - B_2 = V_{\gamma_1} - V_{\gamma_2}$$

hold. Hence we can get applying Proposition 2.2 and Proposition 2.3

$$\int_{-\infty}^{\infty} |\Phi'(\gamma_1)\Phi(\gamma_1) - \Phi'(\gamma_2)\Phi(\gamma_2)|\, d\tau \le C\varepsilon|\gamma_1 - \gamma_2|_X.$$

Moreover taking account of the following relation,

$$\Phi'(\gamma_1) - \Phi'(\gamma_2) = \frac{\Phi'(\gamma_1)\Phi(\gamma_1) - \Phi'(\gamma_2)\Phi(\gamma_2)}{\Phi(\gamma_1)} + \frac{\Phi'(\gamma_2)(\Phi(\gamma_1) - \Phi(\gamma_2))}{\Phi(\gamma_2)}$$

we can obtain (2.74). Therefore Φ is a contraction mapping in $X_{\delta,M}$ and we have the fixed point $\gamma \in X_{\delta,M}$ of Φ. Consequently we obtain $(A(\tau), B(\tau), \gamma(\tau))$ a solution of (2.5) and (2.7) satisfying (2.8). Next we shall investigate the behavior of $(A(\tau), B(\tau))$ when $\tau \to \infty$.

PROPOSITION 2.4. *Assume that* (A_0^-, B_0^-) *satisfies* $\|(A_0^-, B_0^-)\|_{\tilde{Y}_1(H)} < \infty$ *and let* $(A(\tau), B(\tau), \gamma(\tau))$ *be the solution of* (2.5), (2.7) *and* (2.8). *Then there is* $(A_0^+, B_0^+) \in D(H^{1/2}) \times D(H^{1/2})$ *satisfying* $\|(A_0^+, B_0^+)\|_{\tilde{Y}_1(H)} < \infty$ *and*

$$\|e^{-i\tau H}A(\tau) - A_0^+\| + \|e^{i\tau H}B(\tau) - B_0^+\| \to 0, \quad \tau \to \infty. \tag{2.75}$$

and there is $c_\infty^+ > 0$ *such that*

$$\gamma(\tau) - c_\infty^+ \to 0, \quad \tau \to \infty, \tag{2.76}$$

and c_∞^+ *solves the following equation*

$$c_\infty^{+2} = 1 + \frac{\varepsilon}{4c_\infty^{+2}}\{\|A_0^+\|^2 + \|B_0^+\|^2 - 2\Re\alpha^+\}, \tag{2.77}$$

where $\alpha^+ = \lim_{\tau\to\infty}(e^{2i\tau H}A_0^+, B_0^+)$. *Furthermore if* (A_0^-, B_0^-) *satisfies*

$$\tilde{G}(H, A_0^-, B_0^-, \tau) \to 0, \quad \tau \to \pm\infty, \tag{2.78}$$

we have $\alpha^\pm = \lim_{\tau\to\pm\infty}(e^{i\tau H}A_0^\pm, B_0^\pm) = 0$, $c_\infty^+ = c_\infty^-$ *and* $\|A_0^+\|^2 + \|B_0^+\|^2 = \|A_0^-\|^2 + \|B_0^-\|^2$, *and* (A_0^+, B_0^+) *also satisfies* (2.78).

Moreover assume that (A_0^-, B_0^-) *satisfies* $\|(A_0^-, B_0^-)\|_{\tilde{Y}_{1,1}(H)} < \infty$. *Then* (A^+, B^+) *also belongs to* $\tilde{Y}_{1,1}(H)$ *and there is* $C(\tau) \in C^0(R^1; D(H))$ *such that*

$$A(\tau) - B(\tau) = HC(\tau) \tag{2.79}$$

and

$$A_0^+ - B_0^+ = H\left\{C(0) + \int_0^\infty \frac{\gamma'(s)}{\gamma(s)}(e^{isH} + e^{-isH})C(s)\,ds\right\}. \tag{2.80}$$

PROOF. It follows from (2.5)

$$(e^{-i\tau H}A(\tau))' = e^{-i\tau H}\frac{\gamma'}{2\gamma}(A - B)(\tau), \quad (e^{i\tau H}B(\tau))' = -e^{i\tau H}\frac{\gamma'}{2\gamma}(A - B)(\tau). \tag{2.81}$$

Therefore $e^{-i\tau H}A(\tau)$ and $e^{i\tau H}B(\tau)$ are Cauchy sequeneces in L^2 tending $\tau \to \infty$ of which limit (A_0^+, B_0^+) satisfies (2.75), because, $\gamma'(\tau) \in L^1(R)$ and $(A(\tau), B(\tau))$ is bounded in L^2 from Proposition 2.1. We can prove that c_∞^+ satisfies (2.76) and (2.77) by the similar way as in the proof of the fact that c_∞^- satisfies (2.9). Assume that (A^-, B^-) belongs to $\tilde{Y}_{1,1}(H)$, Since $e^{-isH}A(s) = e^{-isH}U(s) + A_0^-$, $e^{isH}B(s) =$

$e^{isH}V(s) + B_0^-$, we get $A^+ = U^+ + A_0^-$, $B^+ = V^+ + B_0^-$. It follows from (i) of Proposition 2.2 that (U^+, V^+), (A_0^-, B_0^-) and $(U^+, V^+) + (A_0^-, B_0^-)$ belong to $\tilde{Y}_{1,1}(H)$, that is, (A_0^+, B_0^+) is in $\tilde{Y}_{1,1}(H)$. Assume (A_0^-, B_0^-) satisfies (2.78) with $-$. Then evidently $\alpha^- = 0$ holds, because $U(s), V(s) \to 0$, $\tau \to -\infty$ in L^2. If (A_0^-, B_0^-) satisfies (2.78) with $+$, then it follows from (iii) of Proposition 2.2 that $\lim_{\tau\to\infty}(e^{i\tau H}A_0^+, B_0^+) = \alpha^+ = 0$ also holds. In fact, it follows from (2.75) that $(e^{i(\tau-2s)H}A(s), B(s)) \to (e^{i\tau H}A^+, B^+)$ uniformly in τ, tending $s \to \infty$. Therefore for any $\varepsilon > 0$ there is s_1 such that for any $\tau \in R$

$$|(e^{i(\tau-2s_1)H}A(s_1), B(s_1)) - (e^{i\tau H}A^+, B^+)| < \varepsilon.$$

On the other hand it follows from (iii) of Proposition 2.2 that $(e^{i(\tau-2s_1)H}A(s_1), B(s_1)) = (e^{i(\tau-2s_1)H}(U(s_1) + A^-(s_1)), V(s_1) + B^-(s_1)) \to 0$, $\tau \to \infty$. Hence we get $\limsup_{\tau\to\infty}|(e^{i\tau H}A^+, B^+)| \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain $\alpha^+ = 0$. Next we shall prove $c_\infty^+ = c_\infty^-$. In fact, we get from (2.7) and (2.75) by use of $\alpha^\pm = 0$

$$\begin{aligned}
\lim_{\tau\to\pm\infty} \|A(\tau) + B(\tau)\|^2 &= \lim_{\tau\to\pm\infty} \|e^{i\tau H}A_0^\pm + e^{-i\tau H}B_0^\pm\|^2 \\
&= \|A_0^\pm\|^2 + \|B_0^\pm\|^2 + \lim_{\tau\to\pm\infty} 2\Re(e^{i\tau H}A_0^\pm, e^{-i\tau H}B_0^\pm) \\
&= \|A_0^\pm\|^2 + \|B_0^\pm\|^2
\end{aligned}$$

and similarly

$$\lim_{\tau\to\pm\infty}\left\|\frac{A(\tau) - B(\tau)}{\gamma(\tau)}\right\|^2 = \frac{\|A_0^\pm\|^2 + \|B_0^\pm\|^2}{c_\infty^{\pm 2}}.$$

Moreover we get from (2.8)

$$c_\infty^{\pm 2} = 1 + \frac{\varepsilon(\|A_0^\pm\|^2 + \|B_0^\pm\|^2)}{4c_\infty^{\pm 2}}. \tag{2.82}$$

Introduce

$$E(\tau) = \frac{1}{2}\left\{\|A(\tau) + B(\tau)\|^2 + F\left(\left\|\frac{A(\tau) - B(\tau)}{\gamma(\tau)}\right\|^2\right)\right\}$$

which is a constant E_0, where $F(\eta) = \int_0^\eta (1 + \varepsilon s)\, ds$. Because $\frac{dE(\tau)}{d\tau} = 0$ holds from (2.5) and (2.8). Therefore tending τ to $\pm\infty$ we obtain

$$\frac{1}{2}\left\{\|A_0^\pm\|^2 + \|B_0^\pm\|^2 + F\left(\frac{\|A_0^\pm\|^2 + \|B_0^\pm\|^2}{c_\infty^{\pm 2}}\right)\right\} = E_0.$$

Since it follows from (2.82) that $\|A_0^{\pm}\|^2 + \|B_0^{\pm}\|^2 = \frac{4(c_\infty^{\pm 2}-1)c_\infty^{\pm 2}}{\varepsilon}$, we obtain the following equation

$$\frac{(c_\infty^{\pm 2}-1)c_\infty^{\pm 2}}{\varepsilon} + F\left(\frac{4(c_\infty^{\pm 2}-1)}{\varepsilon}\right) = 2E_0,$$

which implies $c_\infty^+ = c_\infty^-$ and consequently $\|A_0^-\|^2 + \|B_0^-\|^2 = \|A_0^+\|^2 + \|B_0^+\|^2$ follows from (2.82), because the equation $\frac{(x-1)x}{2}F\left(\frac{x-1}{2}\right) = 2E_0$ has a unique positive solution.

Next we shall prove (2.79). It follows from (2.5)

$$(A-B)_\tau - \frac{\gamma'}{\gamma}(A-B) = iH(A+B),$$

that is, $(\gamma^{-1}(A-B)' = i\gamma^{-1}H(A+B)$. Hence we obtain

$$\gamma(\tau)^{-1}(A(\tau)-B(\tau)) - \gamma(0)^{-1}(A(0)-B(0)) = i\int_0^\tau \gamma(s)^{-1}H(A(s)+B(s))\,ds. \quad (2.83)$$

Moreover since $A = \gamma^{1/2}\alpha + A^-$, $B = \gamma^{1/2}\beta + B^-$ we have

$$A(0) - B(0) = \gamma(0)^{1/2}(\alpha(0)-\beta(0)) + 2ic_\infty Hf^-. \quad (2.84)$$

We shall prove that $\alpha(0)-\beta(0)$ is contained in the image of H. It follows from (2.22), (2.23) and (2.24)

$$\begin{aligned}\alpha(\tau)-\beta(\tau) &= \int_{-\infty}^\tau \frac{\gamma'(s)}{2\gamma(s)}e^{-2isH}(\alpha(s)-\beta(s))\,ds \\ &\quad + \int_{-\infty}^\tau \frac{\gamma'(s)}{2\gamma(s)}\{i\sin(2sH)\alpha(s) \\ &\qquad + e^{isH}\gamma^{-1/2}(2i\sin(sH)g^- + ic_\infty H\cos(sH)f^-)\}\,ds \\ &=: F(\tau) + G(\tau), \end{aligned} \quad (2.85)$$

where we use $e^{isH} = \cos(sH) + i\sin(sH)$. Then noting that $F'(\tau) = \frac{\gamma'}{2\gamma}e^{-2i\tau H}(\alpha(\tau)-\beta(\tau))$ we can see that F satisfies $(e^{-Q}F)'(\tau) = e^{-Q(\tau)}\frac{\gamma'(\tau)}{2\gamma(\tau)}e^{-2i\tau H}G$, where $Q = \int_{-\infty}^\tau \frac{\gamma'}{2\gamma}e^{-2isH}\,ds$. Hence we get from (2.85)

$$F(\tau) = \int_{-\infty}^\tau e^{Q(\tau-s)}\frac{\gamma'(s)}{2\gamma(s)}e^{-2isH}G(s)\,ds. \quad (2.86)$$

On the other hand, we can see

$$
\begin{aligned}
G(\tau) &= \int_{-\infty}^{\tau} \frac{\gamma'(s)}{2\gamma(s)} \{i \sin(2sH)\alpha(s) \\
&\qquad + 2e^{isH}\gamma^{-1/2}(2i \sin(sH)g^- + 2ic_\infty H \cos(sH)f^-)\}\, ds \\
&= H \int_{-\infty}^{\tau} \frac{\gamma'(s)s}{2\gamma(s)} \{i(sH)^{-1} \sin(2sH)\alpha(s) \\
&\qquad + e^{isH}\gamma^{-1/2}(2i(sH)^{-1} \sin(sH)g^- + ic_\infty \cos(sH)f^-)\}\, ds \\
&=; H\tilde{G}(\tau)
\end{aligned}
$$

is in the image of H. In fact, $(sH)^{-1} \sin(sH) = \int_0^\infty \frac{\sin(s\lambda)}{s\lambda}\, dE(\lambda)$ is a bounded operator in $L^2(R^n)$, where $E(\lambda)$ is the spectral family of H, and it follows from the assumption $(A_0^-, B_0^-) \in \tilde{Y}_{1,1}(H)$ that $s\gamma'(s) \in L^1(R)$. Because from (2.73) and Proposition 2.2 we can see

$$
\tau\gamma'(\tau) = \tau\Phi(\gamma)'(\tau) = \frac{\varepsilon}{2\gamma(\tau)^2} \Im\{(HU_\gamma(\tau), \tau V_\gamma(\tau)) + (e^{i\tau H} HU_\gamma(\tau), \tau B_0^-)\}
$$

which belongs to $L^1(R)$. Therefore we get from (2.83)–(2.86),

$$
\begin{aligned}
C(\tau) &= \frac{\gamma(\tau)}{\gamma(0)^{1/2}} \left\{ \int_{-\infty}^{0} e^{Q(-s)} \frac{\gamma'(s)}{2\gamma(s)} e^{-2i\tau H} \tilde{G}(s)\, ds + \tilde{G}(0) \right\} \\
&\qquad + \frac{2i\gamma(\tau)c_\infty}{\gamma(0)} f^- + \int_{-\infty}^{\tau} \frac{\gamma(\tau)}{\gamma(s)} (A(s) + B(s))\, ds, \qquad (2.87)
\end{aligned}
$$

which satisfies (2.79).

Finally we shall prove (2.80). Integrating (2.81) from 0 to ∞ we get by use of (2.79)

$$
\begin{aligned}
A_0^+ - B_0^+ &= A(0) - B(0) + \int_0^\infty \frac{\gamma'(s)}{\gamma(s)} (e^{isH} + e^{-isH})(A(s) - B(s))\, ds \\
&= H\left\{ C(0) + \int_0^\infty \frac{\gamma'(s)}{\gamma(s)} (e^{isH} + e^{-isH}) C(s)\, ds \right\},
\end{aligned}
$$

which means (2.80). □

Now we can prove Theorem 1.1. Define $T(\tau) = \int_0^\tau \gamma(s)^{-1}\, ds$ and denote by $S(t)$ the inverse of $T(\tau) = t$. Put $c(t) = \gamma(S(t))$ and $(A_1(t,x), B_1(t,x)) = (A(S(t),x), B(S(t),x))$, $(A_1^-(t,x), B_1^-(t,x)) = (A^-(c_\infty^- t, x), B^-(c_\infty^- t, x))$ which solve

(2.1)–(2.4). Moreover we can define by use of (2.79)

$$u(t,x)=\frac{C(S(t))}{2ic(t)},$$

which belongs to $D(H)$ and satisfies $Hu(t)=\frac{A_1(t)-B_1(t)}{2ic(t)}$ and $u_t(t)=\frac{A_1(t)+B_1(t)}{2}$, and consequently u solves (1.1) from (2.1). On the other hand, define $(f^+,g^+)=\left(H^{-1}\frac{(A_0^+-B_0^+)}{2ic_\infty^+}=\frac{1}{2ic_\infty^+}\left(C(0)+\int_0^\infty\frac{\gamma'(s)}{\gamma(s)}(e^{isH}+e^{-isH})C(s)\,ds\right),\frac{(A_0^++B_0^+)}{2}\right)$ which belongs to $Y_1(H)$ because of $(A_0^+,B_0^+)\in\tilde{Y}_1(H)$ and let $u^+(t)$ be a solution of (1.3) with $+$ and put $A_1^+(t)=A(c_\infty^+t)$, $B_1^+(t)=B^+(c_\infty^+t)$. Then it follows from (2.7), (2.9) and (2.75), (2.76) that $u(t)$ and $u^+(t)$ satisfy (1.3), (1.4), (1.5) and (1.7). Besides (1.6) and $\lim_{t\to\infty}G_0(H,f^+,g^+,t)=0$ follow from Proposition 2.4. Thus we completed the proof of Theorem 1.1.

3. Sufficient Conditions for $\|(f,g)\|_{Y_{k,j}(H)}<\infty$

In this section we shall investigate the condition $\|(f,g)\|_{Y_{k,j}(H)}<\infty$ and prove Theorem 1.2. To do so we use the wave operator among A and Δ defined by

$$W_\pm=s-\lim_{t\to\pm\infty}e^{itA}e^{-it\Delta}$$

which exists in $L^2(\mathbf{R}^n)$ if we assume that A is a perturbation of Δ, that is, A is elliptic and there is $R>0$ such that

$$a_{jk}(x)=\delta_{jk}, \tag{3.1}$$

for $|x|\geq R$. The wave operators $W_\pm$ are unitary in $L^2(\mathbf{R}^n)$ and satisfies $AW_\pm=W_\pm\Delta$, so we can see easily that if $(f_0,g_0)=(W_\pm^*f,W_\pm^*g)$, where $W_\pm^*$ is the ajoint operator of $W_\pm$, then (f,g) satisfies $\|(f,g)\|_{Y_2(H)}<\infty$ if and only if (f_0,g_0) satisfies

$$\|(f_0,g_0)\|_{Y_{k,j}(H_0)}=\int_{-\infty}^{\infty}(1+|t|^2)^{j/2}\{|(e^{itH_0}H_0^{2+k}f_0,f_0)|\,|(e^{itH_0}H_0^kg_0,g_0)| + |(e^{itH_0}H_0^{1+k}f_0,g_0)|\}\,dt<\infty, \tag{3.2}$$

where $H_0=\sqrt{-\Delta}$ and $H=\sqrt{-A}$. Because that $\|(f_0,g_0)\|_{Y_{k,j}(H_0)}=\|(f,g)\|_{Y_{k,j}(H)}$ holds, if $(f_0,g_0)=(W_\pm^*f,W_\pm^*g)$. The following proposition is well known. For example see Mochizuki [11].

PROPOSITION 3.1. *Assume that A is elliptic and the coefficients of A satisfies (3.1). Then there is the wave operator $W_\pm$ which is unitary in $L^2(\mathbf{R}^n)$, has the* *relation* $H = W_{\pm} H_0 W_{\pm}^*$ *and satisfies*

$$C_l^{-1} \|f\|_{H^l} \le \|W_{\pm} f\|_{H^l} \le C_l \|f\|_{H^l}, \tag{3.3}$$

for any $l \in \mathbf{R}^1$ *and* $f \in H^l$.

We continue to explain what functions (f, g) satisfy the condition $\|(f, g)\|_{Y_{k,j}(H)} < \infty$. We need the following lemma which is proved by Greenberg and Hu [3], D'Ancona and Spagnolo [1] and Yamazaki [13].

Lemma 3.1. *Let* μ_1 *and* μ_2 *nonnegative numbers and put* $\mu = \mu_1 + \mu_2$. *When* $n \ge 2$ *we take* $H_0 = \sqrt{-\Delta}$ *and* k *be a nonnegative number such that* $k \le n + \mu$ *and when* $n = 1$ *we take* $H_0 = i\frac{d}{dx}$ *and* k, μ *arbitrary non negative integers. Then there is a positive constant* $C_{n,\mu}$ *such that the following inequality holds for every* $f_j \in H_k^{\mu_j}$, $(j = 1, 2)$,

$$(1 + |t|)^k |(e^{itH_0} H_0^{\mu} f_1, f_2)_{L^2}| \le C_{n,k,\mu} \|f_1\|_{H_k^{\mu_1}} \|f_2\|_{H_k^{\mu_2}}. \tag{3.4}$$

Here we shall give an outline of the proof of this lemma in a simple case following D'Ancona and Spagnolo [1]. When $n \ge 2$, $\mu = 1$ and an integer $k = n$, we can see easily (3.4) holds. In fact, taking account of $\sum \frac{\xi_j}{it|\xi|} \partial_{\xi_j} e^{it|\xi|} = e^{it|\xi|}$, we see

$$\begin{aligned} t^n (e^{itH_0} H_0 f_1, f_2)_{L^2} &= \int_{R^n} \left(\sum \frac{\xi_j}{i|\xi|} \partial_{\xi_j} \right)^n e^{it|\xi|}) \hat{f}_1(\xi) \bar{\hat{f}}_2(\xi) |\xi| \, d\xi \\ &= \int_{R^n} e^{it|\xi|} \left(\sum \partial_{\xi_j} \frac{i\xi_j}{|\xi|} \right)^n (\hat{f}_1(\xi) |\xi|^{\mu_1} \bar{\hat{f}}_2(\xi) |\xi|^{\mu_2}) \, d\xi \end{aligned}$$

which implies

$$|t^n (e^{itH_0} H_0 f_1, f_2)_{L^2}| \le \int_{R^n} \left| \left(\sum \partial_{\xi_j} \frac{\xi_j}{|\xi|} \right)^n (|\xi|^{\mu_1} \hat{f}_1 |\xi|^{\mu_2} \bar{\hat{f}}_2(\xi)) \right| d\xi \le C \|f_1\|_{H_n^{\mu_1}} \|f_2\|_{H_n^{\mu_2}},$$

where $\mu_1 + \mu_2 = 1$ and we used the inequalities

$$\left\| \frac{\xi_j}{|\xi|} \partial_{\xi_j} \hat{f} \right\|_{L^2} = \|R_j x_j f\|_{L^2} \le C \|f\|_{L_1^2}$$

and

$$\left\| \partial_{\xi_j} \frac{\xi_j}{|\xi|} \hat{f} \right\|_{L^2} = \|x_j R_j f\|_{L^2} \le C \|f\|_{L_1^2},$$

here R_j are Riez operator of which symbol is $\frac{\xi_j}{|\xi|}$. When k is not an integer, we can derive (3.4) by use of the interpolation theorem. See Lemma 2.1 in Yamazaki [13] for detail.

It follows from Lemma 3.1 that we can see that if $(f_0, g_0) = (W_\pm^* f, W_\pm^* g)$ belongs to $H_\mu^{(2+k)/2} \times H_\mu^{k/2}$ for $\mu > 1$, then (f, g) satisfies $\|(f, g)\|_{Y_{k,j}(H)} < \infty$. We investigate the conditions under that $(f_0, g_0) = (W_\pm^* f, W_\pm^* g)$ belongs to $H_\mu^{(2+k)/2} \times H_\mu^{k/2}$. In Proposition 3.1 we indicated the boundedness in Sobolev space H^l of the wave operators $W_\pm$, $W_\pm^*$. To use Lemma 3.1 we need the boundedness of the wave operators in weighted Sobolev space $W_\mu^{l,2}$. To get such boundedness, we need moreover some condition. Namely we assume that there is a real valued function $q \in C^\infty(\mathbf{R}^{2n})$ satisfying (1.13) and (1.14). This condition is equivalent to the non trapping condition. See [4] and [5]. Then the following theorem holds, of which proof is given partially in Kajitani [6].

Theorem 3.1. *Assume that $n \geq 2$ and the coefficients of A satisfy* (1.11) *and* (3.1). *Moreover we assume that there is a function $q \in C^\infty(\mathbf{R}^{2n})$ satisfying* (1.13) *and* (1.14). *Let $l, \mu \in R$ and m an integer. Then there is $C_{ln} > 0$ such that*

$$\|W_\pm \varphi\|_{H^l_{m-\mu_0}} \leq C_{ln} \|\varphi\|_{H^{l+l_0+2}_{m+l_0}}, \tag{3.5}$$

for $0 \leq m < \frac{n+1}{2}$, $\mu_0 > \frac{1}{2}$, $l_0 > \frac{n}{2}$ and for $\varphi \in H_m^{l+l_0+2}$ and

$$\|W_\pm^* \psi\|_{H^l_{m+\mu}} \leq C_{ln} \|\psi\|_{H^{l+l_0}_{m+1+\mu_0}}, \tag{3.6}$$

for $0 \leq m + \mu < \frac{n+2}{2}$, $0 \leq \mu < 1$, $\mu_0 > \frac{1}{2}$, $l_0 > \frac{3n}{2} + 5$ and $\psi \in W^{l+l_0,2}_{m+1+\mu_0}$.

The proof of Theorem 3.1 will be given in the section 4.

We can get the following proposition by applying (3.6).

Proposition 3.2. *Let $n \geq 2$, $k \geq 0$ and $j \geq 0$ an integer. Assume that* (1.11) *and* (3.1) *are valid and that there is a function $q \in C^\infty(\mathbf{R}^{2n})$ satisfying* (1.13) *and* (1.14). *If $(f, g) \in D(H^{(2+k)/2}) \times D(H^{k/2})$ belongs to $H^{(2+k)/2+l_0}_{\mu_0+j+2} \times H^{k/2+l_0}_{\mu_0+j+2}$, $l_0 > \frac{3n}{2} + 3$, $\mu_0 > \frac{1}{2}$ and $j < \frac{n}{2}$, then $\|(f, g)\|_{Y_{k,j}(H)} < \infty$ holds.*

Proof. We put $(f_0, g_0) = (W_\pm^* f, W_\pm^* g)$. By use of Lemma 3.1 and Theorem 3.1 we can estimate

$$\begin{aligned}
\|(f, g)\|_{Y_{k,j}(H)} &= \|(f_0, g_0)\|_{Y_{k,j}(H_0)} \\
&= \int_{-\infty}^{\infty} (1 + |t|^2)^{j/2} \{|(e^{itH_0} H_0^{2+k} f_0, f_0)| + |(e^{itH_0} H_0^k g_0, g_0)| \\
&\qquad + |(e^{itH_0} H_0^{1+k} f_0, g_0)|\} \, dt
\end{aligned}$$

$$\leq C\int_{-\infty}^{\infty}(1+|t|)^{j-\mu}\,dt\{\|f_0\|^2_{H_\mu^{k/2+1}}+\|g_0\|^2_{H_\mu^{k/2}}\}\,dt$$

$$\leq C\{\|W_\pm^* f\|^2_{H_\mu^{(k+2)/2}}+\|W_\pm^* g\|^2_{H_\mu^{k/2}}\}$$

$$\leq C\{\|f\|^2_{H^{(k+2)/2+l_0}_{[\mu]+1+\mu_0}}+\|g\|^2_{H^{k/2+l_0}_{[\mu]+1+\mu_0}}\}<\infty,$$

where we take $\mu > 0$ such that $j-\mu < -1$, $\mu < \frac{n+2}{2}$, $[\mu] = j+1$ and $l_0 > \frac{3n}{2}+3$, $\mu_0 > \frac{1}{2}$. □

Next we mention a sufficient condition for (3.2) without decay weight with respect to the space variables. We need the following proposition of which proof is given for example in [8].

Lemma 3.2. *Let k a non negative integer. Then there is $C > 0$ such that if $n \geq 2$ we take $H_0 = \sqrt{-\Delta}$ and we have*

$$|e^{itH_0}H_0^k f(x)| \leq C\int_{\mathbf{R}^n}\{(1+|t|)^{-n-k}+(1+|t|)^{-(n-1)/2}((1+||x-y|+t|)^{-(n-1)/2-k}$$
$$+(1+||x-y|-t|)^{-(n-1)/2-k})\}|(1-\Delta_y)^{l/2}f(y)|\,dy, \quad (3.7)$$

for $f \in W^{l,1}$ and for $l > k+n$.

Using the above Lemma 3.2 we can prove the following proposition.

Proposition 3.3. *Let (f_0, g_0) be in $(D(H_0^{(2+k)/2}) \cap W^{(l+2+k)/2,1}) \times (D(H_0^{k/2}) \cap W^{(l+k)/2,1})$, $l > n$, $k \geq 0, j$ non negative integer. Assume that $j - \frac{n-1}{2} \leq 0$ and $k + \frac{n-1}{2} > 1$ or that $j - \frac{n-1}{2} < -1$ hold. Then (f_0, g_0) satisfies $\|(f_0, g_0)\|_{Y_{k,j}(H_0)} < \infty$. Moreover $G_0(H_0, f, g, t) \to 0$, $|t| \to \infty$ for $n \geq 2$.*

Proof. Applying Lemma 3.2 we can see that $(f_0, g_0) \in (D(H_0^{(2+k)/2}) \cap W^{(l+2+k)/2,1}) \times (D(H_0^{k/2}) \cap W^{(l+k)/2,1})$, $l > n$ satiasfies (3.2), that is, we calculate

$$\|(f_0, g_0)\|_{Y_{k,j}(H_0)}$$
$$=\int_{-\infty}^{\infty}(1+|t|^2)^{j/2}\{|(e^{itH_0}H_0^{2+k}f_0, f_0)|+|(e^{itH_0}H_0^k g_0, g_0)|$$
$$+|(e^{itH_0}H_0^{1+k}f_0, g_0)|\}\,dt$$

$$\le \int_{-\infty}^{\infty}\int_{\mathbf{R}^n}(1+|t|^2)^{j/2}\{|e^{itH_0}H_0^{2+k}(1-\Delta)^{-l_1}f_0(x)|\,|(1-\Delta)^{l_1}f_0(x)|$$

$$+|e^{itH_0}H_0^{k}(1-\Delta)^{-l_1}g_0(x)|\,|(1-\Delta)^{l_1}g_0(x)|$$

$$+|e^{itH_0}H_0^{1+k}(1-\Delta)^{-l_1}f_0(x)|\,|(1-\Delta)^{l_1}g_0(x)|\}\,dxdt$$

$$<\infty,$$

if $n+k>j+2$. Because for example the second term in the right hand side can be estimated

$$\int_{-\infty}^{\infty}\int_{\mathbf{R}^n}(1+|t|^2)^{j/2}|e^{itH_0}H_0^{k}(1-\Delta)^{-l_1}g_0(x)|\,|(1-\Delta)^{l_1}g_0(x)|\,dxdt$$

$$\le \int_{-\infty}^{\infty}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(1+|t|^2)^{j/2}$$

$$\times\{(1+|t|)^{-n-k}+(1+|t|)^{-(n-1)/2}((1+|\,|x-y|+|t|\,|)^{-(n-1)/2-k}$$

$$+(1+|\,|x-y|-|t|\,|)^{-(n-1)/2-k})\}|(1-\Delta_y)^{l/2-l_1}g_0(y)|\,dy$$

$$\times|(1-\Delta)^{l_1}g_0(x)|\,dxdt$$

$$\le C\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|(1-\Delta_y)^{l/2-l_1}g_0(y)|\,|(1-\Delta_x)^{l_1}g_0(x)|\,dydx$$

$$\le C\|(1-\Delta)^{l/2-l_1}g_0\|_{L^1}\|(1-\Delta_y)^{l_1}g_0\|_{L^1}\le C\|g_0\|_{W^{l/4,1}},$$

if we choose $l_1=\frac{l}{4}$, $l>n+k$. Here we used

$$\sup_{x,y}\int_{-\infty}^{\infty}(1+|t|^2)^{j/2}\{(1+|t|)^{-(n-1)/2}((1+|\,|x-y|+t|)^{-(n-1)/2-k}$$

$$+(1+|\,|x-y|-t|)^{-(n-1)/2-k})\}\,dt\le C,$$

if $j-\frac{n-1}{2}\le 0$ and $k+\frac{n-1}{2}>1$ or if $j-\frac{n-1}{2}<-1$. We can estimate the other terms by the same way. Thus we get $\|(f_0,g_0)\|_{Y_{k,j}(H_0)}<\infty$, if $j-\frac{n-1}{2}\le 0$ and $k+\frac{n-1}{2}>1$ or if $j-\frac{n-1}{2}<-1$. Moweover we get by Lebesgue's convergence theorem $(e^{itH_0}g_0,g_0)\to 0$, $|t|\to\infty$ for $n\ge 2$ which implies $G_0(H_0,f_0,g_0,t)\to 0$, $|t|\to\infty$ for $n\ge 2$. □

Proposition 3.3 with $k=j=1$ implies the following theorem.

Theorem 3.2. *Let* $n \geq 3$. *Assume that* $A = \Delta$ *and the initial data* (f^-, g^-) *belongs to* $(D(H^{3/2}) \cap W^{(l+3)/2,1}) \times (D(H^{1/2}) \cap W^{(l+1)/2,1})$, $l > n$. *Then the conclusion of Theorem* 1.1 *is valid.*

Proposition 3.4. *Assume that* $n \geq 2$ *and the coefficients of* A *satisfy* (1.11) *and* (3.1). *Moreover we assume that there is a function* $q \in C^\infty(\mathbf{R}^{2n})$ *satisfying* (1.13) *and* (1.14). *Let* (f, g) *be in* $D(H^{(2+k)/2}) \cap W^{l,1} \times D(H^{k/2}) \cap W^{l-1,1}$, $l \geq 2n+6$. *Then if* $\frac{n-3}{2} > j$, $k \geq 0$, $\|(f,g)\|_{Y_{k,j}(H)} < \infty$ *holds.*

Proof. It follows from Theorem 1.2 in Kajitani [6] that the uniform decay estimates of solutions of the wave equation associated to A holds, that is, the solution w of the equation $w_{tt} = Aw$ satisfies

$$\|w_t(t)\|_{L^\infty} + \|Hw(t)\|_{L^\infty} \leq C(1+|t|)^{-(n-1)/2}(\|w(0)\|_{W^{l,1}} + \|w_t(0)\|_{W^{l-1,1}})$$

for $l \geq 2n+6$. Using the above decay estimates we get

$$\|e^{itH}H^k f\|_{L^\infty} \leq C(1+|t|)^{-(n-1)/2}\|f\|_{W^{l,1}}, \quad k = 0, 1$$

for $n \geq 2$, $l \geq 2n+6$. In fact, let $w(t)$ be satisfied with $w_{tt} = Aw$, $w(0) = 0$, $w_t(0) = H^k f$ and put $A(t) = (\partial_t + iH)w(t)$. Then $A(t)$ satisfies $(\partial_t - iH)A(t) = 0$, $A(0) = H^k f$. Hence we can write $A(t) = e^{itH}H^k f$ which satisfies a desired estimate evidently. When $n \geq 6$ the above estimate implies that $(f, g) \in W^{l,1} \times W^{l+1,1}$, $l \geq 2n+6$ satisfies the condition $\|(f,g)\|_{Y_{k,j}(H)} < \infty$. □

Proof of Theorem 1.2. It follows from Proposition 3.2 with $j = 1$ for $n = 3, 4, 5$ and Proposition 3.4 with $k = j = 1$ for $n \geq 6$ that Theorem 1.1 with $k = j = 1$ holds for a perturbed operator A. Furthermore we can obtain Theorem 1.2 using Theorem 1.1 for an perturbed operator A and the scattering operator among A and Δ which existence is assured in Proposition 3.1.

4. Estimates of Wave Operators

In this section we shall prove Theoren 3.1. First we mention a well known result which can be found for example in the textbook (p. 75, Theorem 15.3 and its proof) of Mochizuki [11].

Proposition 4.1. *Let* $n \geq 2$. *Assume that* A *satisfies* (1.11) *and* $A = \Delta$ *for* $|x| \geq R_0$. *Then the wave operators* $W_\pm$ *has a following integral representation*

$$W_\pm \varphi(x) = (2\pi)^{-n} \int w_\pm(x, \xi)\hat{\varphi}(\xi)\, d\xi, \tag{4.1}$$

where

$$w_{\pm}(x,\xi) = e^{ix\xi} + w_{\pm}^{1}(x,\xi) \tag{4.2}$$

and

$$w_{\pm}^{1}(x,\xi) = (-A - (|\xi|^2 \pm i0))^{-1} e^{ix\xi} \sum_{l,k=1}^{n} \{(a_{lk}(x) - \delta_{lk})\xi_l\xi_k + \partial_{x_l} a_{lk}(x)\xi_k\}. \tag{4.3}$$

Denote the resolvent of $-A$ by $R(z) = (-A - z)^{-1}$ and put $u(\lambda) = R(\lambda^2)f$. Modifying the proofs of Theorem 4.8 and of Theorem 4.10 in Kajitani [6] we can prove the following proposition.

Proposition 4.2. *Let $n \geq 2$. Assume that A satisfies* (1.11) *and $A = \Delta$ for $|x| \geq R_0$ and that there is a function $q(x,\xi) \in C^\infty(\mathbf{R}^{2n})$ satisfying* (1.13) *and* (1.14). *Let $\lambda = \sqrt{\sigma^2 \pm i\varepsilon}$, $\sigma \in \mathbf{R}$, $\varepsilon \geq 0$, $l \geq \left[\frac{n}{2}\right] + 1$ and $f \in C_0^\infty(\mathbf{R}^n)$ with* $\operatorname{supp} f \subset K(K:$ *a compact set in $\mathbf{R}^n$), $\mu > \frac{1}{2}$ and u be a solution of the equation $(-A - \lambda^2)u = f$. Denote $u^k = \partial_\lambda^k u$. Then there are $\lambda_0 > 0$, $C_{kl}(K) > 0$ and $C_{lk} > 0$ such that for $|\lambda| \leq \lambda_0$*

$$\|u^k(\lambda)\|_{W^{l,2}_{-\mu-k}} \leq \frac{C_{kl}(K)}{|\lambda|^k} \|f\|_{H^l} \tag{4.4}$$

and for $|\lambda| \geq \lambda_0$

$$\|u^k(\lambda)\|_{W^{l,2}_{-1/2-\mu-k}} \leq C_{lk}|\lambda|^{-1-k} \|f\|_{W^{l+1,2}_{1/2+\mu+k}} \tag{4.5}$$

for $k = 0, 1, \ldots$.

Let decompose $w_0(x,\xi) = R(\lambda^2)V(\cdot,\xi)\rho(\xi)$ and $w_\infty(x,\xi) = R(\lambda^2)V(\cdot,\xi) \cdot (1 - \rho(\xi))$, where $\lambda^2 = |\xi|^2 \pm i0$, $V(x,\xi) = e^{ix\xi}\sum_{l,k=1}^{n}\{(a_{lk}(x) - \delta_{lk})\xi_l\xi_k + \partial_{x_l}a_{lk}(x)\xi_k\}$ and $\rho(\in C_0^\infty) = 1$ for $|\xi| \leq \lambda_0$ and $\rho = 0$ for $|\xi| \geq \lambda_0 + 1$. Denote

$$W_0 f(x) = \int w_0(x,\xi)\hat{f}(\xi)\,d\xi, \quad W_\infty f(x) = \int w_\infty(x,\xi)\hat{f}(\xi)\,d\xi.$$

Then we have $W_\pm f(x) = W_0 f(x) + W_\infty f(x)$. We shall estimate the terms $W_0 f(x)$ and $W_\infty f(x)$. It follows from Proposition 4.2 that we can see for any $\beta \in \mathbf{N}^n$

$$\|\partial_\xi^\beta w_0(x,\xi)\|_{L^2_{-|\beta|-\mu_0}} \leq C_\beta |\xi|^{1-|\beta|}, \quad \left(\mu_0 > \tfrac{1}{2}\right), \tag{4.6}$$

and

$$\|\partial_\xi^\beta w_\infty(x,\xi)\|_{L^2_{-\mu_1-|\beta|}} \leq C_\beta (1 + |\xi|), \quad \left(\mu_1 > \tfrac{3}{2}\right). \tag{4.7}$$

LEMMA 4.1. *Let w be satisfied with $(-A-z)w = V \in C_0^\infty(\mathbf{R}^n)$, where $z = \sigma \pm i0$, $\sigma \in R$ and $\alpha, \beta \in \mathbf{N}^n$. Then w satisfies*

$$x^\beta D_x^\alpha w = \sum_{k=1}^{|\beta|+1} R(z)^k \{b_{\alpha\beta k} R(z) + d_{\alpha\beta k}\} V, \tag{4.8}$$

where $b_{\alpha\beta k}$ is a differential operator of order $|\alpha| + k$ with compact support coefficients and $d_{\alpha\beta k}$ is a differential operator of order $|\alpha| + k - 1$.

PROOF. We shall prove (4.8) by induction of β. For $\beta = 0$ we have

$$(-A-z)D_x^\alpha w = [-A, D_x^\alpha]w + D_x^\alpha V,$$

which gives $b_{\alpha 01} = [-A, D_x^\alpha]$ and $d_{\alpha 01} = D_x^\alpha$. Assume that (4.8) is valid for $|\beta| = l-1$ and for any $\alpha \in \mathbf{Z}_+^n$. We have

$$(-A-z)(x^\beta D_x^\alpha w) = x^\beta D_x^\alpha V + [x^\beta D_x^\alpha, -A + \Delta]w - [x^\beta D_x^\alpha, \Delta]w \tag{4.9}$$

On the other hand, the assumption of induction yields

$$\begin{aligned}
[x^\beta D_x^\alpha, \Delta]w &= -\sum_{j=1}^{n} \{\beta_j(\beta_j - 1)x^{\beta-2e_j} D_x^\alpha + 2\beta_j x^{\beta-e_j} D^{\alpha+e_j}\} w \\
&= -\sum_{j=1}^{n} \Bigg\{ \beta_j(\beta_j - 1) \sum_{k=1}^{|\beta|-1} R(z)^k (b_{\alpha\beta-2e_j k} R(z) + d_{\alpha\beta-2e_j k}) V \\
&\qquad + 2\beta_j \sum_{k=1}^{|\beta|} R(z)^k (b_{\alpha\beta-e_j k} R(z) + d_{\alpha\beta-e_j k}) V \Bigg\},
\end{aligned}$$

here we denote $e_j = (0, \dots, 1, 0, \dots, 0)$ of which jth component equals to 1. Hence taking account that the support of the coefficients of $[x^\beta D_x^\alpha, A - \Delta]$ are compact, we obtain from (4.9) by use of the assumption of induction,

$$\begin{aligned}
x^\beta D_x^\alpha w = R(z) \Bigg\{ & x^\beta D_x^\alpha V + [x^\beta D_x^\alpha, -A + \Delta] R(z) V \\
& + \sum_{k=1}^{|\beta|} R(z)^k (b_{\alpha\beta-e_j k} R(z) + d_{\alpha\beta-e_j k}) V \Bigg\},
\end{aligned}$$

which implies (4.8). □

Noting that it follows from Racke [12] that $R(\lambda^2)^k = \frac{1}{(k-1)!}\left(\frac{\xi\cdot\nabla_\xi}{|\xi|^2}\right)^{k-1} R(\lambda^2)$ for $k \geq 1$ if $\lambda^2 = |\xi| \pm i0$ we get from Lemma 4.1

$$x^\beta D_x^\gamma w_\pm^1(x,\xi) = \sum_{k=1}^{|\beta|+1}\left(\frac{\xi\cdot\nabla_\xi}{|\xi|^2}\right)^{k-1} R(\lambda^2)\{b_{\gamma\beta k}R(\lambda^2) + d_{\gamma\beta k}\}V. \tag{4.10}$$

Proof of Theorem 3.1. First we shall prove (3.5). Since $W_\pm = I - W_\pm^1$ and $W_\pm^1 = W_0 + W_\infty$, it suffices to prove that (3.5) ia valid for W_0 and W_∞. From (4.3) we have

$$W_0 f(x) = \int R(\lambda^2)V(\cdot,\xi)\rho(\xi)\hat{f}(\xi)\,d\xi,$$

where $\hat{f}$ means the Fourier transform of f and denote $\lambda = \sqrt{|\xi|^2 \pm i0}$. Hence taking account of $V = 0(|\xi|)$, $|\xi| \to 0$ we get by use of (4.10) and by integration by part

$$\begin{aligned}
&x^\beta D_x^\gamma W_0 f(x)\\
&\quad= \int x^\beta D_x^\gamma R(\lambda^2)V(\cdot,\xi)\rho(\xi)\hat{f}(\xi)\,d\xi\\
&\quad= \sum\nolimits_{k=1}^{|\beta|+1}\int R(\lambda^2)^k\{b_{\gamma\beta k}R(\lambda^2) + d_{\gamma\beta k}\}\rho(\xi)V(\cdot,\xi)\hat{f}(\xi)\,d\xi\\
&\quad= \sum\int\frac{1}{(k-1)!}\left(\frac{\xi\cdot\nabla_\xi}{2|\xi|^2}\right)^{k-1} R(\lambda^2)\{b_{\gamma\beta k}R(\lambda^2) + d_{\gamma\beta k}\}V(\cdot,\xi)\rho(\xi)\hat{f}(\xi)\,d\xi\\
&\quad= \sum\int\frac{1}{(k-1)!}R(\lambda^2)\left(\frac{\xi\cdot\nabla_\xi}{2|\xi|^2}\right)^{*(k-1)}\{b_{\gamma\beta k}R(\lambda^2) + d_{\gamma\beta k}\}V(\cdot,\xi)\rho(\xi)\hat{f}(\xi)\,d\xi
\end{aligned} \tag{4.11}$$

if $|\beta| \leq n$, where $\left(\frac{\xi\cdot\nabla_\xi}{|\xi|^2}\right)^*$ means an ajoint operator of $\left(\frac{\xi\cdot\nabla_\xi}{|\xi|^2}\right)$. We can see

$$\frac{1}{(n-1)!}\left(\frac{\xi\cdot\nabla_\xi}{2|\xi|^2}\right)^{*k} = \sum_{|\alpha|\leq k} a_\alpha^{(k)}(\xi)\partial_\xi^\alpha,$$

where $a_\alpha^{(k)}(\xi)$ is a homogeous function of ξ of order $-2k + |\alpha|$. Therefore we can calculate

$$\frac{1}{(k-1)!}\left(\frac{\xi\cdot\nabla_\xi}{2|\xi|^2}\right)^{*(k-1)}\{(b_{\gamma\beta k}R(\lambda^2)+d_{\gamma\beta k})V(\cdot,\xi)\rho(\xi)\hat{f}(\xi)\}$$

$$=\sum_{|\alpha|\le k-1}a_\alpha^{(k-1)}(\xi)\left\{b_{\gamma\beta k}\sum_{|\alpha'|\le|\alpha|}C_{\alpha,\alpha'}\partial_\xi^{\alpha'}(R(\lambda^2)\rho(\xi))\partial_\xi^{\alpha-\alpha'}(V(\cdot,\xi)\hat{f}(\xi))\right.$$

$$\left.+d_{\gamma\beta k}\partial_\xi^\alpha(V(\cdot,\xi)\rho(\xi)\hat{f}(\xi))\right\},$$

where $C_{\alpha,\alpha'}=\frac{\alpha!}{(\alpha-\alpha')!\alpha'!}$. It follows from (4.11) that using (4.4) of Proposition 4.2 and noting that the supports of the coefficients of $b_{\gamma\beta k}$ and $V(x,\xi)$ are compact with respect to x, we get for $l\in R$ and for $\mu_0>\frac{1}{2}$

$$\|x^\beta D_x^\gamma W_0 f(x)\|_{L^2_{-\mu_0}}$$

$$=\left\|\sum\nolimits_k\int R(\lambda^2)\left\{\sum\nolimits_{|\alpha|\le k-1}a_\alpha^{(k-1)}(\xi)b_{\gamma\beta k}\sum\nolimits_{|\alpha'|\le|\alpha|}C_{\alpha,\alpha'}\partial_\xi^{\alpha'}(R(\lambda^2))\right.\right.$$

$$\left.\left.\times\,\partial_\xi^{\alpha-\alpha'}(V(\cdot,\xi)\rho(\xi)\hat{f}(\xi))+d_{\gamma\beta k}\partial_\xi^\alpha(V(\cdot,\xi)\rho(\xi)\hat{f}(\xi))\right\}d\xi\right\|_{L^2_{-\mu_0}(R_x^n)}$$

$$\le\int_{|\xi|\le\lambda_0+1}\left\|R(\lambda^2)\left\{\sum\nolimits_k\sum\nolimits_{|\alpha|\le k-1}a_\alpha^{(k-1)}(\xi)b_{\gamma\beta k}\sum\nolimits_{|\alpha'|\le|\alpha|}C_{\alpha,\alpha'}\partial_\xi^{\alpha'}(R(\lambda^2))\right.\right.$$

$$\left.\left.\times\,\partial_\xi^{\alpha-\alpha'}(V(\cdot,\xi)\rho(\xi)\hat{f}(\xi))+d_{\gamma\beta k}\partial_\xi^\alpha(V(\cdot,\xi)\rho(\xi)\hat{f}(\xi))\right\}\right\|_{L^2_{-\mu_0}(R_x^n)}d\xi$$

$$\le C\int\sum|\xi|^{(-2(k-1)+|\alpha|)}\left\{\|\partial_\xi^{\alpha'}(R(\lambda^2))\partial_\xi^{\alpha-\alpha'}(V(\cdot,\xi)\rho(\xi)\hat{f}(\xi)\|_{H^{|\gamma|+k}_{-\mu_0-|\beta|}(R_x^n)}\right.$$

$$\left.+\|\partial_\xi^\alpha(V(\cdot,\xi)\rho(\xi))\hat{f}(\xi))\|_{H^{|\gamma|+k-1}_{-\mu_0}(R_x^n)}\right\}d\xi$$

$$\le C\sum\nolimits_{k\le|\beta|+1}\int_{|\xi|\le\lambda_0+1}|\xi|^{-2(k-1)+1}\,d\xi\,\sup\nolimits_{|\xi|\le\lambda_0+1}|\nabla_\xi^{k-1}\hat{f}(\xi)|^2$$

$$\le C\int_{|\xi|\le\lambda_0+1}|\xi|^{-2|\beta|+1}\,d\xi\|f\|_{L^2_{|\beta|+l_0}}\le C\|f\|_{L^2_{|\beta|+l_0}},\tag{4.12}$$

if $|\beta|<\frac{n+1}{2}$ and $l_0>\frac{n}{2}$.

Next we shall esticmate $W_\infty f$. Similarly using (4.11) we get

$$x^\beta D_x^\gamma W_\infty f(x) = \int x^\beta D_x^\gamma R(\lambda^2) V(\cdot,\xi)(1-\rho(\xi))\hat{f}(\xi)\,d\xi$$

$$= \sum \int R(\lambda^2) \frac{1}{(n-1)!} \left(\frac{\xi \cdot \nabla_\xi}{|\xi|^2} \right)^{*(k-1)}$$

$$\times (b_{\gamma\beta k} R(\lambda^2) + d_{\gamma\beta k}) V(\cdot,\xi)(1-\rho(\xi))\hat{f}(\xi)\,d\xi \qquad (4.13)$$

for any $\beta, \gamma \in \mathbf{N}^n$. Taking account of $D_x^\gamma \partial_\xi^\alpha V = 0(|\xi|^{2+|\gamma|})$, $|\xi| \to \infty$, similarly to (4.12) we get from (4.5) for $\sigma > \frac{n}{2}$ and any β

$$\|x^\beta D_x^\gamma W_\infty f(x)\|_{L^2_{-\mu_0}}$$

$$\le C \sum \int |\xi|^{-2(k-1)+|\alpha|} \{ \|\partial_\xi^\alpha (R(\lambda^2) V(\cdot,\xi)(1-\rho(\xi))\hat{f}(\xi)\|_{H^{|\gamma|+k}_{-\mu_0-|\beta|}(R_x^n)}$$

$$+ \|\partial_\xi^\alpha (V(\cdot,\xi)(1-\rho(\xi))\hat{f}(\xi))\|^2_{H^{|\gamma|+k-1}_{-\mu_0}(R_x^n)} \}\,d\xi$$

$$\le C \int_{|\xi| \ge \lambda_0} |\xi|^{|\gamma|+4} |\nabla_\xi^{k-1} \hat{f}(\xi)|^2\,d\xi \le C\|f\|_{H^{l_1+l}_{|\beta|+\mu_0}}, \qquad (4.14)$$

if $l_1 > \frac{n}{2} + 4$ and $|\gamma| \le l$. Thus we get (3.5) for $0 \le |\beta| < \frac{n}{2}$ from (4.11), (4.12) and (4.14).

Next we shall estimate $W_\pm^* \psi = W_0^* \psi + W_\infty^* \psi$, that is, $W_0^* \psi$ and $W_\infty^* \psi$. Since we can write $W_0^* \psi$ as

$$W_0^* \psi(x) = \iint e^{ix\xi} w_0^*(y,\xi) \psi(y)\,dy d\xi,$$

the derivative of Fourier image of $W_0^* \psi$ is given by

$$\partial_\xi^\gamma \widehat{W_0^* \psi}(\xi) = \int \partial_\xi^\gamma w_0^*(y,\xi) \psi(y)\,dy, \qquad (4.15)$$

where $w_0^*(y,\xi) = \overline{w_\pm(y,\xi)}\rho(\xi)$ which satisfies (4.6). Hence we have for $|\gamma| = m$ by use of (4.15) and (4.6),

$$\|\partial_x^\beta x^\gamma W_0^* \psi\|_{L^2} = \|\xi^\beta \partial_\xi^\gamma \widehat{W_0^* \psi}(\xi)\|_{L^2} = \left\| \xi^\beta \partial_\xi^\gamma \int w_0^*(y,\xi) \psi(y)\,dy \right\|_{L^2}$$

$$\le \left\| \sqrt{\int |\langle y \rangle^{-(|\gamma|+\mu_0)} \partial_\xi^\gamma w_0^*(y,\xi)|^2\,dy \int |\langle y \rangle^{(|\gamma|+\mu_0)} \psi(y)|^2\,dy} \right\|_{L^2}$$

$$\le C \sqrt{\int |\xi|^{2(1-|\gamma|)}\,d\xi} \|\psi\|_{L^2_{|\gamma|+\mu_0}} \le C\|\psi\|_{L^2_{m+\mu_0}}, \qquad (4.16)$$

if we choose $m < \frac{n+2}{2}$ and $\mu_0 > \frac{1}{2}$. Let $m > 0$ be an integer and $1 > \mu > 0$ such that $m + \mu < \frac{n+2}{2}$. We shall use the following well known estimate

$$\|\langle x\rangle^{\mu} f\|_{L^2} \le C\left\{ \sup_{0<|h|\le 1, h\in R^n} |h|^{-\mu} \|\hat{f}(\xi+h) - \hat{f}(\xi)\|_{L^2} + \|f\|_{L^2} \right\}, \quad (4.17)$$

for any $f \in L^2_{\mu}$ and $\hat{f}$ means the Fourier transform of f. Applying the inequalty (4.17) to $x^{\beta}\partial_x^{\gamma} W_0^* \psi$ we can see for $|\beta| = m$,

$$\begin{aligned}
&\|\langle x\rangle^{\mu} x^{\beta}\partial_x^{\gamma} W_0^* \psi\|_{L^2} \\
&\le C\Bigg(\sup_{0<|h|\le 1, h\in R^n} |h|^{-\mu} \sqrt{\int\!\!\int |(\partial_{\xi}^{\beta}\partial_y^{\gamma} w_0^*(y,\xi+h) - \partial_{\xi}^{\beta}\partial_y^{\gamma} w_0^*(y,\xi))\psi(y)\, dy|^2 \, d\xi} \\
&\qquad + \|x^{\beta}\partial_x^{\gamma} W_{\pm}^* \psi\|_{L^2} \Bigg) \\
&\le C\Bigg(\sup_{|h|\le 1} |h|^{-\mu} \sqrt{\int\!\!\int |\langle y\rangle^{-m-1-\mu_0} (\partial_{\xi}^{\beta}\partial_y^{\gamma} w_0^*(y,\xi+h) - \partial_{\xi}^{\beta}\partial_y^{\gamma} w_0^*(y,\xi))|^2 \, dy d\xi} \|\psi\|_{L^2_{m+1+\mu_0}} \\
&\qquad + \|x^{\beta}\partial_x^{\gamma} W_0^* \psi\|_{L^2} \Bigg).
\end{aligned}$$

By use of (4.6) we can calculate

$$\begin{aligned}
&\int_{|\xi|\le 2|h|} \int |\langle y\rangle^{-m-1-\mu_0} (\partial_{\xi}^{\beta}\partial_y^{\gamma} w_0^*(y,\xi+h) - \partial_{\xi}^{\beta}\partial_y^{\gamma} w_0^*(y,\xi))|^2 \, dy d\xi \\
&\quad \le 2\int_{|\xi|\le 2|h|} |\langle y\rangle^{-m-1-\mu_0} \partial_{\xi}^{\beta}\partial_y^{\gamma} w_0^*(y,\xi+h)|^2 + |\langle x\rangle^{-m-1-\mu_0} \partial_{\xi}^{\beta}\partial_y^{\gamma} w_0^*(y,\xi)|^2 \, dy d\xi \\
&\quad \le C\int_{|\xi|\le 2|h|} (|\xi+h|^{-2(m-1)} + |\xi|^{-2(m-1)}) \, d\xi \le C|h|^{-2(m-1)+n},
\end{aligned}$$

if $|\beta| = m \le \frac{n}{2} + 1$ and $\mu_0 > \frac{1}{2}$. On the other hand, using (4.6) again

$$\begin{aligned}
&\int_{\lambda_0+1\ge|\xi|\ge 2|h|} \int |\langle y\rangle^{-m-1-\mu_0} (\partial_{\xi}^{\beta}\partial_y^{\gamma} w_0^*(y,\xi+h) - \partial_{\xi}^{\gamma}\partial_y^{\gamma} w_0^*(y,\xi))|^2 \, dy d\xi \\
&\qquad \le C\int_{\lambda_0+1\ge|\xi|\ge 2|h|} |h|^2 \int_0^1 |\xi+\theta h|^{-2m} \, d\theta d\xi \le C|h|^{-2(m-1)+n}.
\end{aligned}$$

Thus we get

$$\|\langle x\rangle^{\mu}x^{\beta}\partial_x^{\gamma}W_0^*\psi\|^2 \le C\left(\sup_{0<|h|\le 1}|h|^{-2(m+\mu-1)+n}\|\psi\|^2_{L^2_{m+1+\mu_0}} + \|x^{\beta}\partial_x^{\gamma}W_0^*\psi\|_{L^2}\right)$$
$$\le C\|\psi\|^2_{H^l_{m+1+\mu_0}}, \tag{4.18}$$

if $m+\mu \le \frac{n+2}{2}$ and $|\gamma|\le l$. Next we shall estimate $W_\infty^*\psi$. Since we can write

$$\partial_\xi^{\beta}\xi^{\gamma}(\widehat{W_\infty^*\psi})(\xi)) = \int \partial_\xi^{\beta}\xi^{\gamma}w_\infty^*(y,\xi)\psi(y)\,dy$$
$$= \int(\eta(y,D)^{-N}\partial_\xi^{\beta}\xi^{\gamma}w_\infty^*(y,\xi))(\eta(y,D)^{*N}\psi(y))\,dy, \tag{4.19}$$

for any integer N, where $\eta(x,D)=(-A+ih)^{1/2}$, $h\gg 1$. Hence we get

$$|\partial_\xi^{\beta}\xi^{\gamma}\widehat{W_\infty^*}\psi(\xi))| \le \|\eta(\cdot,D)^{-N}\partial_\xi^{\beta}\xi^{\gamma}w_\infty^*(\cdot,\xi)\|_{L^2_{-|\beta|-\mu_0}}\|\eta(y,D)^{*N}\psi(y)\|_{L^2_{|\beta|+\mu_0}}. \tag{4.20}$$

Denote $V(x,\xi)=e^{ix\xi}\tilde{V}(x,\xi)$. Since $\eta(x,D)=(-A+ih)^{1/2}$ commutes with $R(\lambda^2)=(-A-\lambda^2)^{-1}$,

$$(\eta(y,D)^{-N}\partial_\xi^{\beta}\xi^{\gamma}w_\infty^*(y,\xi)) = \eta(y,D)^{-N}\partial_\xi^{\beta}\xi^{\gamma}\overline{R(\lambda^2)e^{iy\xi}\tilde{V}(y,\xi)(1-\rho(\xi))}$$
$$= \partial_\xi^{\beta}\xi^{\gamma}\overline{R(\lambda^2)\overline{\eta(y,D)^{-N}}e^{iy\xi}\tilde{V}(y,\xi)(1-\rho(\xi))}$$

holds. On the other hand, taking account that $V(y,\xi)$ has a compact support in y and $|D_x^{\alpha}\partial_\xi^{\beta}\tilde{V}(y,\xi)(1-\rho(\xi)|\le C(1+|\xi|)^{2-|\beta|}$, by use of (4.5) we can estimate

$$\|\partial_\xi^{\beta}(\xi^{\gamma}R(\lambda^2)\overline{\eta(\cdot,D))^{-N}}e^{i\cdot\xi}\tilde{V}(\cdot,\xi)(1-\rho(\xi))\|_{L^2_{-|\beta|-\mu_0}(R_x^n)}$$
$$\le C(1+|\xi|)^{n+5+|\gamma|-2N}, \tag{4.21}$$

for any integer $N\ge 0$. In fact, since

$$\partial_\xi^{\beta}\{\xi^{\gamma}R(\lambda^2)\overline{\eta(\cdot,D)^{-N}}e^{i\cdot\xi}\tilde{V}(\cdot,\xi)(1-\rho(\xi))\}$$
$$= \sum C_{\beta,\beta'}(\partial_\xi^{\beta'}(R(\lambda^2))\overline{\eta(\cdot,D)^{-N}}\partial_\xi^{\beta-\beta'}\xi^{\gamma}\{e^{i\cdot\xi}\tilde{V}(\cdot,\xi)(1-\rho(\xi))\}$$

holds, we can see from (4.5)

$$\|\partial_\xi^{\beta}\{\xi^{\gamma}R(\lambda^2)\eta(\cdot,D)^{-N}e^{i\cdot\xi}\tilde{V}(\cdot,\xi)(1-\rho(\xi))\}\|_{L^2_{-|\beta|-\mu_0}}$$
$$\le \sum C_{\beta,\beta'}(1+|\xi|)^{-1-|\beta'|}$$
$$\times\|(\eta(\cdot,D))^{-N}\partial_\xi^{\beta-\beta'}\xi^{\gamma}e^{i\cdot\xi}\overline{\tilde{V}(\cdot,\xi)}(1-\rho(\xi))\|_{H^1_{\mu_0+|\beta|}}. \tag{4.22}$$

Put

$$e^{ix\xi}\partial_\xi^{\beta-\beta'}(\xi^\gamma e^{-ix\xi}\overline{\tilde{V}(x,\xi)}(1-\rho(\xi)) = U_{\beta-\beta'}(x,\xi).$$

Since V has a compact support in x and satisfies $|\partial_\xi^\delta\partial_x^\alpha\tilde{V}(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{2-|\delta|}$, we have

$$|\partial_\xi^\delta\partial_x^\alpha U_{\beta-\beta'}(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{2+|\gamma|-|\delta|}.$$

Hence the Fourier image of $e^{-ix\xi}U_{\beta-\beta'}(x,\xi)$ is given by

$$F(e^{-ix\xi}U_{\beta-\beta'}(x,\xi))(\zeta) = \int e^{-i(\zeta+\xi)y}U_{\beta-\beta'}(y,\xi)\,dy$$

which satisfies

$$|\partial_\zeta^\tau(F(e^{-ix\xi}U_{\beta-\beta'}(x,\xi))(\zeta))| \le C_{M\tau}(1+|\xi|)^{2+|\gamma|}(1+|\xi+\zeta|)^{-M} \tag{4.23}$$

for any non negative integer M. Denote the symbol of $\eta(x,D)^{-N}$ by $\eta_N(x,\zeta)$, which satisfies

$$|\partial_x^\alpha\partial_\zeta^\beta\eta_N(x,\zeta)| \le C_{\alpha\beta N}(1+|\zeta|)^{-N-|\beta|}, \tag{4.24}$$

if we choose $h > 0$ sufficiently large. Since we have for any α

$$\begin{aligned}
&x^\alpha\partial_x^\rho\eta(x,D)^{-N}e^{ix\xi}\tilde{V}(x,\xi)\\
&\quad= \sum_{\alpha',\rho'} C_{\alpha,\alpha'}C_{\rho,\rho'}\int e^{ix\zeta}(i\partial_\zeta)^{\alpha-\alpha'}(i\zeta)^{\rho-\rho'}\{\partial_x^{\rho'}\eta_N(x,\zeta)\partial_\zeta^{\alpha'}F(e^{-ix\xi}U_{\beta-\beta'}(x,\xi))(\zeta)\}\,d\zeta\\
&\quad= \sum_{\alpha',\rho'} C_{\alpha,\alpha'}\int e^{ix\zeta}\sum C_{\rho,\rho'}(i\partial_\zeta)^{\alpha''}(i\zeta)^{\rho-\rho'}\\
&\qquad\times\{\partial_x^{\rho'}\eta_N(x,\zeta)(i\partial_\zeta)^{\alpha-\alpha'-\alpha''}\{\partial_x^{\alpha'}F(e^{-ix\xi}U_{\beta-\beta'}(x,\xi))(\zeta)\}\,d\zeta,
\end{aligned}$$

we get by use of (4.23) and (4.24)

$$\begin{aligned}
&|\partial_x^\rho\eta(x,D)^{-N}e^{-ix\xi}U_{\beta-\beta'}(x,\xi)|\\
&\qquad\le C_{\alpha\rho NM}(1+|x|)^{-|\alpha|}(1+|\xi|)^{2+|\gamma|}\int(1+|\zeta|)^{-N+|\rho|}(1+|\zeta+\xi|)^{-M}\,d\zeta\\
&\qquad\le C_{\alpha\rho NM}(1+|x|)^{-|\alpha|}(1+|\xi|)^{2+|\gamma|-M}\int(1+|\zeta|)^{-N+M+|\rho|}\,d\zeta\\
&\qquad\le C_{\alpha\rho NM}(1+|x|)^{-|\alpha|}(1+|\xi|)^{4+|\gamma|-N+n+|\rho|}
\end{aligned}$$

if we choose $-N+M+|\rho| < -(n+1)$, that is, $M = N-n-2-|\rho|$, where we used $(1+|\xi+\zeta|)^{-1} \le (1+|\zeta|)(1+|\xi|)^{-1}$. Therefore if we take $|\alpha| > \frac{n+1}{2}+\mu_0+$

$|\beta|$, $|\rho| = 1$ and $|\gamma| \leq l$ we obtain

$$\|\eta(x,D)^{-N} e^{-ix\xi} U_{\beta-\beta'}(x,\xi)\|_{H^1_{\mu_0+|\beta|}} \leq C_N(1+|\xi|)^{5-N+n+l}. \tag{4.25}$$

which implies (4.21) together with (4.22). On the other hand, if we take $N > \frac{3n}{2} + l + 5$ we get from (4.20) and (4.25)

$$\|W^*_\infty \psi\|_{W^{l,2}_m} = \sum_{|\beta| \leq m, |\gamma| \leq l} \|\partial^\beta_\xi \xi^\gamma \widehat{W^*_\infty \psi}(\xi))\|_{L^2(\mathbf{R}^n)} \leq C\|\psi\|_{W^{l+l_0,2}_{m+\mu_0}},$$

for $l_0 > \frac{3n}{2} + 5$, $\mu_0 > \frac{1}{2}$ and for any integer m, which implies (3.6) together with (4.18). Thus we have completed the proof of Theorem 3.1.

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