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METRIZABILITY OF ORDERED ADDITIVE GROUPS

By

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Abstract. In terms of General Topology, we consider ordered additive groups having the order topology, including ordered fields. Namely, we investigate metrizability of these groups or fields, and topological properties of ordered fields in terms of Archimedes' axiom or the axiom of continuity. Also, we give a negative answer to a question in [9]. Finally, we revise the proof of [2, Theorem 2.6], and give some related results.

1. Introduction

As is well-known, an ordered field is a field which has a linear (total) order and the order topology by this order. Ordered fields have played important roles in the theory of real numbers in terms of Archimedes' axiom or the axiom of continuity.

In terms of General Topology, we consider ordered additive groups as a generalization of ordered fields, and we investigate metrizability of these groups. Then, we give characterizations for ordered fields to be metrizable, or satisfy the above axioms. Besides, we give a negative answer to a question in [9]. Finally, we revise the proof of D. E. Dobbs' result [2, Theorem 2.6], and simplify the proof of the result. Also, we give some related results.

Let **R**; Q; and N be respectively the usual real number field; rational number field; and the set of natural numbers.

We assume that all (topological) spaces are Hausdorff. We give main definitions used in this paper. Let X be a set which is linearly ordered (or, totally ordered) by $\leq$. For $a, b \in X$ with $a < b$, define the intervals (a, b), $[a, b]$ in X

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by the same way as in $\mathbf{R}$, and especially $(a, \infty) = \{x \in X : x > a\}, (-\infty, a) = \{x \in X : x < a\}$. $(X, \leq)$ is a *linearly ordered topological space or a LOTS* if X has the subbase $\{(a, \infty), (-\infty, a) : a \in X\}$. Such a topology on X is called the *order topology*. It is well-known that every LOTS is hereditarily (collectionwise) normal. For LOTS, see [3, 5], etc.

Let G (or, $(G, +)$) be an Abelian group (i.e., commutative group which is additive). Let us say that G is an *ordered additive group* (cf. [9]) if G has a linear order $\leq$ such that the order is preserving with respect to addition (i.e., for $a < b$, $a + x < b + x$), and G has the order topology by the order $\leq$ (hence $(G, \leq)$ is a LOTS). For $x \in G$, define $|x| \in G$ by $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$. Then, for $x, y \in G$, $|x + y| \leq |x| + |y|$ holds. For a commutative field $(K, +, \times)$ with a linear order $\leq$, K is an *ordered field* if $(K, +, \leq)$ is an ordered additive group, and the order $\leq$ is also preserving with respect to multiplication (i.e., for $a < b$ and $0 < x$, $a \times x < b \times x$). Any ordered field contains no isolated points. Ordered fields play important roles in the theory of the field $\mathbf{R}$.

Let $(G, \cdot)$ be a group. Then G is a *topological group* if it is a space, and the group operation of G is continuous; that is, for the map $(a, b) \to a \cdot b^{-1}$ from the product space $G \times G$ to G is continuous. As is well-known, every topological group is homogeneous.

2. Results

We investigate metrizability of ordered additive groups and ordered fields. Also, we consider topological properties of ordered fields in terms of Archimedes' axiom or the axiom of continuity.

Proposition 2.1. *Every ordered additive group $(G, +, \leq)$ is a topological group. When G is an ordered field $(G, +, \times, \leq)$, moreover the multiplication map $(a, b) \mapsto a \times b$ and the multiplication inverse map $a \mapsto a^{-1}$ $(a \neq 0)$ are continuous.*

Proof. This is folkloric or well-known, but let us give a proof. We can assume G is not discrete. Then for some $p \in G$, $p \in cl(G \backslash \{p\})$. Let $\Delta = \{|x - p| : x \in G \backslash \{p\}\}$. Let $a, b \in G$. For each $\varepsilon > 0$ $(\varepsilon \in G)$, take $\delta \in \Delta$ with $\delta + \delta < \varepsilon$. Then, for $|x - a|, |y - b| < \delta$, $|(x - y) - (a - b)| < \varepsilon$. This shows $(G, +, \leq)$ is a topological group. The latter part holds by the ε-δ method as in $\mathbf{R}$ (cf. [2, Lemma 2.1]). □

Every LOTS topological group is hereditarily paracompact by [6, Theorem 8] (which is valid with respect to addition). Thus, the following holds.

Corollary 2.1. *Every ordered additive group* (*in particular*, *ordered field*) *is homogeneous*, *and hereditarily paracompact.*

A space X is a *P-space* if all G_δ-sets are open in X (equivalently, all F_σ-sets are closed in X). In terms of Proposition 2.1, the following lemma is shown by a (folkloric) metrization theorem for LOTS topological groups (see [6, Remark 10] (which is valid with respect to addition)).

Lemma 2.1. *Every ordered additive group G is metrizable or a P-space.*

Proof. Suppose G is not a P-space. Then there exists a G_δ-set $A = \bigcap_{n=1}^{\infty} U_n$ with U_n open such that for some $p \in A$, any neighborhood of p is not contained in A. Suppose $\chi(p,G)(= min\{|\mathscr{B}| : \mathscr{B}$ is a local base at $p\}) > \omega$. Since G is a LOTS, G has a decreasing local base $\{I_\alpha : \alpha < \kappa\}$ at p by open intervals in G. For each $n \in N$, pick $I_{\alpha_n} \subset U_n$. Since $\chi(p,G) > \omega$, there exists $\beta < \kappa$ which is larger than any α_n. Then $p \in I_\beta \subset A$. Hence A contains a neighborhood of p, a contradiction. Hence $\chi(p,G) = \omega$. Thus G is first countable by Corollary 2.1. Then G is metrizable by a classical theorem that every first countable topological group is metrizable (this is valid with respect to the addition in view of [5, VI.5]). □

Remark 2.1. It is well-known that every compact, LOTS, topological group is metrizable. But, as well known, every compact, connected, LOTS need not be metrizable ([3, 3.12.3(d)]), hence, not be a P-space. We have the same if we replace "LOTS" by "topological Abelian group". Indeed, let S be the circle in the plane. Then S is a topological Abelian group. Let G be the product S^{ω_1} of ω_1 many copies of S. Then G is a compact, connected, topological Abelian group with respect to coordinate addition, but G is not metrizable, nor a P-space since G contains a non-metrizable set $\{p,q\}^{\omega_1}$.

Also, not every ordered field is metrizable by the following: For a completely regular space X, let $C(X)$ be the (partially ordered) ring of all continuous functions from X into $\mathbf{R}$. For a maximal ideal M in $C(X)$, let $K = C(X)/M$ be the residue class ring. Then K is an ordered field (see [4, 5.4(c)]). In view of Theorems 5.5 and 13.8 in [4], K is (order-preserving) isomorphic (equivalently, homeomorphic) to $\mathbf{R}$, otherwise K is a P-space (hence not metrizable). Thus, if X is not pseudo-compact (i.e., $C(X)$ contains an unbounded function), there exists a non-metrizable ordered field $K = C(X)/M$ by [4, Theorem 5.8(b)]. (For $X = N$, such an ordered field K is directly shown by [9, Example 2]).

A space X is a *k-space* if $F \subset X$ is closed if and only if $F \cap C$ is closed in C for every compact subset C of X. Locally compact spaces or first-countable spaces are k-spaces.

PROPOSITION 2.2. *Let G be a non-discrete ordered additive group. Then* (a), (b), *and* (c) *below are equivalent* (*cf.* [9]). *When G is an ordered field,* (a)$\sim$(d) *below are equivalent* (*cf.* [2] *for* (a) $\Leftrightarrow$ (c)).

(a) *G is metrizable.*
(b) *G contains an infinite countably compact set* (*in particular, G is a k-space*).
(c) *G contains a countable set A having an accumulation point $p \in G$.*
(d) *G contains a countable set B having no upper* (*or no lower*) *bounds.*

PROOF. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is obvious. For the parenthetic part in (b), since G is not discrete, G contains some infinite compact subset. For (c) $\Rightarrow$ (a), the countable set $A - \{p\}$ is not closed in G, hence G is not a P-space. Thus, G is metrizable by Lemma 2.1. When G is an ordered field, for (c) $\Rightarrow$ (d), the countable set $\{1/|a-p| : a \in A, a \neq p\}$ has no upper bounds. For (d) $\Rightarrow$ (c), the countable set $\{1/b : b \in B, b \neq 0\}$ has an accumulation point $0 \in G$. □

PROPOSITION 2.3. *Let G be an ordered additive group. Then* (a)$\sim$(e) *below are equivalent. When G is an ordered field,* (a)$\sim$(f) *below are equivalent* (*cf.* [9] *for* (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (f)).

(a) *G is separable and metrizable.*
(b) *G is separable.*
(c) *$G - \{0\}$ is Lindelöf.*
(d) *G is a Lindelöf space with $\chi(0, G) = \omega$.*
(e) *G is a Lindelöf space with $\chi(0, G) \neq \omega_1$.*
(f) *G is Lindelöf.*

PROOF. The equivalences among (a)$\sim$(d) are shown by means of Lemma 2.1 with Corollary 2.1, because (b), (c), or (d) implies that G is countable discrete, or not a P-space (for (c), G is a Lindelöf space in which $G - \{0\}$ is an F_σ-set). (a) $\Rightarrow$ (e) is obvious. For (e) $\Rightarrow$ (a), if $\chi(0, G) = \omega$, (a) holds, so let $\chi(0, G) \geq \omega_2$. Since G is not metrizable, G is a P-space by Lemma 2.1. Thus, G has a base by open-and-closed sets ([4, 4K]). While, G has a decreasing local base at 0 by open intervals. Thus G has a decreasing local base $\{B_\tau : \tau < \alpha\}$ ($\alpha \geq \omega_2$) at 0 by open-and-closed sets (or, see [6, Theorem 6]). For each $\tau < \alpha$, let $C_\tau = B_\tau - B_{\tau+1}$, here we assume that $C_\tau \neq \varnothing$. Then, G has a disjoint open cover $\{G - B_1\} \cup$

$\{C_\tau : 1 \le \tau < \omega_1\} \cup \{B_{\omega_1}\}$ of cardinality ω_1. Thus G is not Lindelöf, a contradiction. For (f) $\Rightarrow$ (a) in the latter part, since G is Lindelöf, G has a countable open cover $\{(-\infty, a_n) : n \in N\}$. Since the set $\{a_n : n \in N\}$ has no upper bounds, G is metrizable by Proposition 2.2. $\square$

Let us show that every Lindelöf, ordered additive group need not be metrizable in Proposition 2.3. This gives a negative answer to a question in [9].

Let $X = \prod_{a \in A} X_a$ with X_a spaces. For $p = (p_a) \in X$, the *$\sigma(p)$-product* of X is the set of $x = (x_a) \in X$ with $x_a \neq p_a$ for at most a finite number of a. The *ω-box topology* on X has a base by the sets of the form $\prod_{a \in A} B_a$ such that each B_a is open in X_a, but $B_a \neq X_a$ for at most a countable number of a.

The following is similarly shown as in the proof of [1, Proposition 3].

Lemma 2.2. *Let $X = \prod_{a \in A} X_a$ have the ω-box topology. If $\prod_{b \in B} X_b$ is Lindelöf for any finite $B \subset A$, then each $\sigma(p)$-product of X is Lindelöf.*

Example 2.1. A Lindelöf, ordered additive group which is not metrizable.

Proof. Let Z be the usual ordered additive group of integers, and let $X = Z^{\omega_1}$. For $\mathbf{0} = (0, 0, \dots) \in X$, let G be the $\sigma(\mathbf{0})$-product of X. Then G is an additive group with respect to coordinatewise addition. Endow X with the ω-box topology, and let G be a subspace of X. Then G is Lindelöf by Lemma 2.2. But, G is not first countable (hence, not metrizable), because the local base $\{V(\mathbf{0}; \alpha) : \alpha < \omega_1\}$ at $\mathbf{0}$ does not have any countable subfamily which becomes a local base at $\mathbf{0}$, where $V(\mathbf{0}; \alpha) = \{x = (x_\alpha) \in G : x_\beta = 0 \text{ for any } \beta < \alpha\}$. The topology on the additive group G is equivalent to the order topology by the lexicographic order $\le$ on G (i.e., for $x = (x_\alpha)$, $y = (y_\alpha)$, $x < y$ if and only if for some $\alpha < \omega_1$, $x_\beta = y_\beta$ for any $\beta < \alpha$, but $x_\alpha < y_\alpha$). For $a, b \in G$, $a < b$ if and only if $\mathbf{0} < b - a$. Thus, $(G, \le)$ is a desirable ordered additive group. $\square$

A space X is *totally disconnected* if any component in X is a singleton. A space is X is *zero-dimensional* ([3]) if X has a base by open-and-closed sets (namely, $\operatorname{ind} X = 0$). Every completely regular P-space is zero-dimensional, and every zero-dimensional space is totally disconnected. For a LOTS X, X is totally disconnected; $\operatorname{ind} X = 0$; $\operatorname{Ind} X = 0$; and $\dim X = 0$ are all equivalent by [3, 6.3.2(e), 7.1.10].

Concerning topological embeddings for ordered additive groups, Proposition 2.4 below holds. For the *Baire* (*zero-dimensional*) *space* $B(m) = D^\omega$, D is a discrete space of cardinality $m \geq \aleph_0$, see [3, 4.2.12]. The space $B(m)$ is considered as an ordered additive group (by the lexicographic order on D^ω). For the *hedgehog* $J(m)$ of spininess $m \geq \aleph_0$, see [3, 4.1.5]. The spaces $B(m)$ and $J(m)$ are complete metrizable.

Proposition 2.4. *For a non-discrete ordered additive group G, the following hold.*

(1) *G is not totally disconnected if and only if G is homeomorphic to $\mathbf{R} \times D$, where D is a discrete space of cardinality $|G/H|$ for some open subgroup H.*

(2) *For G being totally disconnected, G is not a P-space if and only if it is topologically embedded in the space $B(m)$, $m = w(G)$ (i.e., the weight of G).*

(3) *G is Čech-complete if and only if it is homeomorphic to a closed subset of the countable product $J(m)^\omega$, $m = w(G)$. When G is totally disconnected, we can replace "$J(m)^\omega$" by "$B(m)$".*

Proof. The "if" parts of (1), (2), and (3) are obvious. Then, let us show their "only if" parts. For (1), in view of [11, Theorem 2.4] the result holds for G being a LOTS topological group (this is valid with respect to addition). Thus (1) holds by Proposition 2.1. For (2), G is a totally disconnected LOTS, then $\operatorname{Ind} G = 0$. While, since G is not a P-space, G is metrizable by Lemma 2.1. Hence, the "only if" part holds by [3, Theorem 7.3.15]. For (3), let G be Čech-complete. Thus, by [3, 3.9.5], G is a k-space. Thus G is metrizable by Proposition 2.2. Then G is completely metrizable. Hence the "only if" part holds by [3, 4.4.B]. The latter part holds by [3, 7.3.H] since $\operatorname{Ind} G = 0$. □

Theorem 2.1. *Let G be an ordered additive group. Then* (a), (b), *or* (c) *below holds. When G is an ordered field,* (a), (b)*, *or* (c) *holds.*

(a) *G is a P-space.*

(b) *G is homeomorphic to a topological sum of $\mathbf{R}$.*

(b)* *G is homeomorphic to $\mathbf{R}$.*

(c) *G is topologically embedded in the space $B(m)$, $m = w(G)$.*

Proof. This holds in view of Propositions 2.4. For the latter part, it suffices to show that every ordered field G is connected or totally disconnected. Indeed, suppose G is not totally disconnected. Then G has a component L at 0, containing $a \neq 0$. For any $p \in G$, $pa^{-1}L$ contains 0, and it is connected by the continuity of the multiplication (in Proposition 2.1). Then $p \in pa^{-1}L = L$, thus $p \in L$. Hence, $G = L$ is connected. □

Remark 2.2. Every ordered additive group need not satisfy (a), (b)*, or (c) in Theorem 2.1. Indeed, let $G = Z \times \mathbf{R}$ (Z is the ordered additive group of integers) be the product space, and let $\leq$ be the lexicographic order on G. Then $(G, \leq)$ is an ordered additive group with respect to coordinatewise addition, but G satisfies none of (a), (b)*, (c).

Now, let $(K, \leq)$ be an ordered field. A pair $(A|B)$ of non-empty subsets A and B in K is a (Dedekind) *cut* if $K = A \cup B$, $A \cap B = \emptyset$, and for any $x \in A$ and any $y \in B$, $x < y$. Let us recall the following *Archimedes' axiom*, and the *axiom of continuity* which is stronger than Archimedes' axiom.

Archimedes' axiom: For each $\alpha, \beta \in K$ with $0 < \alpha < \beta$, there exists $n \in N$ such that $\beta < n\alpha$.

Axiom of continuity: For each cut $(A|B)$ in K, K contains $max\, A$ or $min\, B$.

An ordered field K is *Archimedean*; *Dedekind-complete* if K respectively satisfies Archimedes' axiom; the axiom of continuity. For $S \subset K$, S is *Dedekind-complete* if we replace "K" by "S". Then K is Dedekind-complete if and only if so is any $[a, b] \subset K$, here we can replace "any $[a, b]$" by "some $[a, b]$ (or $[0, 1]$)".

We can assume that any ordered field K contains Q as a subfield. The field Q is Archimedean, but not Dedekind-complete.

Let us recall the following characterizations for an ordered field to be Archimedean or Dedekind-complete (many of these are well-known); see [7, 8, 9], for example.

Proposition 2.5. *For an ordered field K*, (1) *and* (2) *below hold.*

(1) *The following are equivalent.*
   (a) *K is Archimedean.*
   (b) *The sequence $\{1/n : n \in N\}$ has a limit point 0 in K.*
   (c) *The set $\{1/n : n \in N\} \cup \{0\}$ is compact in K.*
   (d) *Q is a dense subset of K.*
   (e) *Q has an accumulation point in K.*
(2) *The following are equivalent.*
   (a) *K is Dedekind-complete.*
   (b) *Every lower bounded decreasing sequence* (*in Q*) *has a limit point in K.*
   (c) *Every lower bounded subset* (*of Q*) *has an infimum in K.*
   (d) *Every bounded infinite subset* (*of Q*) *has an accumulation point in K.*
   (e) *Some* (*or any*) *interval $[a, b]$ is compact in K.*
   (f) *K is connected* (*we can replace "K" by "Some* (*or any*) *interval $[a, b]$ in K"*).

REMARK 2.3. (1) Let K be an ordered field. As is well-known, K is Archimedean if and only if (*) K is order-preserving isomorphic to a subfield F of $\mathbf{R}$ which has the usual order $\leq$ in $\mathbf{R}$, in particular, K is Dedekind-complete if and only if $F = \mathbf{R}$. In (*), K is homeomorphic to $(F, \leq)$ (indeed, F has the order topology by $\leq$ in view of [3, 2.7.5(a)], because the field F contains Q which is dense in $\mathbf{R}$). If K is Dedekind-complete, then K is homeomorphic to $\mathbf{R}$ (the converse holds in view of (f) in Proposition 2.5(2)).

(2) Every Archimedean ordered field K is separable metrizable by (1) (this is also shown by Proposition 2.2 and 2.5(1)). But, every separable metrizable ordered field need not be Archimedean ([9]).

Let $(G, \leq)$ be an ordered additive group. A sequence $\{a_n : n \in N\}$ in G is *Cauchy* if for each $\varepsilon > 0$ ($\varepsilon \in G$), there exists $n_0 \in N$ such that $|a_m - a_n| < \varepsilon$ if $m, n > n_0$. Let $\mathscr{C} = \{[a_n, b_n] : n \in N\}$ be a decreasing sequence of closed intervals in G. Let us call $\mathscr{C}$ *shrinking* if $(b_n - a_n) \to 0$. When $\mathscr{C}$ has a non-empty intersection, $\mathscr{C}$ is shrinking if and only if $\mathscr{C}$ has only one common point.

PROPOSITION 2.6. *For an ordered additive group G, the following are equivalent.*

(a) (*Cauchy's theorem*) *Every Cauchy sequence in G has a limit point.*

(b) (*Principle of successive division*) *Every shrinking sequence $\{[a_n, b_n] : n \in N\}$ in G has a non-empty intersection.*

PROOF. For (a) $\Rightarrow$ (b), let $L = \{a_1, b_1, a_2, b_2, \ldots\}$ be a sequence of endpoints of the closed intervals $[a_n, b_n]$ in (b). Since L is Cauchy, L has a limit point p. Then $p \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. For (b) $\Rightarrow$ (a), let $L = \{x_n : n \in N\}$ be an infinite Cauchy sequence. For each $n \in N$, let $\varepsilon_n = |x_n - x_{n+1}|$. Since L is Cauchy, we can assume that $\varepsilon_n \to 0$ with $0 < \varepsilon_{n+1} < \varepsilon_n$. Since any subsequence of L is Cauchy, by induction, we can choose a decreasing sequence $\mathscr{C} = \{[a_n, b_n] : n \in N\}$ such that $(b_n - a_n) \leq \varepsilon_n$, $[a_n, b_n] \cap L$ is a subsequence of L, but for some subsequence $S = \{s_n : n \in N\}$ of L, each s_n is one of the endpoints in $[a_n, b_n]$, and the rest of the endpoints is s_i or $s_j \pm \varepsilon_j$ for some $i < n$ and $j \leq n$. Since $\mathscr{C}$ is shrinking, it has only one common point $a \in G$. Then, the sequence S converges to the point a. To see L converges to the point a, let $\varepsilon > 0$. Since $\varepsilon_n \to 0$, $2\varepsilon_i (= \varepsilon_n + \varepsilon_n) \to 0$ by Proposition 2.1, so take ε_i with $2\varepsilon_i < \varepsilon$. Since L is a Cauchy sequence, and S is a subsequence of L converging to the point a, there exists $m \in N$ such that for $n \geq m$, $|x_n - s_n| < \varepsilon_i$, and $|s_n - a| < \varepsilon_i$. Then $|x_n - a| < \varepsilon$. Thus L converges to the point a. $\square$

In Propositions 2.7 and 2.8 below, the equivalences of (a) and (b) were shown in [9].

PROPOSITION 2.7. *For a non-discrete ordered additive group G, the following are equivalent.*

(a) *G is metrizable.*
(b) *G contains an infinite Cauchy sequence.*
(c) *G contains a shrinking sequence $\{[a_n, b_n] : n \in N\}$.*

PROOF. (a) $\Rightarrow$ (b) holds, because every convergent sequence is Cauchy in G. (b) $\Rightarrow$ (c) holds, putting $a_n = 0$ and $b_n = \varepsilon_n$ in the proof of Proposition 2.6. For (c) $\Rightarrow$ (a), G has the convergent sequence $\{b_n - a_n : n \in N\}$. Hence G is metrizable by Proposition 2.2. □

PROPOSITION 2.8. *For an ordered field K, (1) and (2) below hold.*

(1) *The following are equivalent.*
 (a) *K is Archimedean.*
 (b) *K contains an infinite Cauchy sequence in Q.*
 (c) *K contains a shrinking sequence of closed intervals with endpoints in Q.*
(2) *The following are equivalent.*
 (a) *K is Dedekind-complete.*
 (b) *K contains an infinite Cauchy sequence in Q, and any of these Cauchy sequences has a limit point in K.*
 (c) *K contains a shrinking sequence of closed intervals with endpoints in Q, and any of these shrinking sequences has a non-empty intersection in K.*

PROOF. For (1), (a) $\Rightarrow$ (b) is obvious, for K contains a convergent sequence $\{1/n : n \in N\}$ by Proposition 2.5(1). For (b) or (c) $\Rightarrow$ (a), let $\varepsilon > 0$. Then there exist $p, q \in Q$ such that $0 < |p - q| < \varepsilon$ by (b) or (c). This shows that Q has an accumulation point 0 in K. Hence, K is Archimedean by Proposition 2.5(1). For (b) $\Rightarrow$ (c), it is shown as in the proof of Proposition 2.7 by replacing "ε_n" by "$1/n$". For (2), (a) $\Rightarrow$ (b) holds by *Cauchy's theorem* in $\mathbf{R}$. (b) $\Rightarrow$ (c) holds by (1) and the proof of Proposition 2.6. For (c) $\Rightarrow$ (a), K is Archimedean by (1), so Q is dense in K by Proposition 2.5(1). Then K satisfies *Principle of successive division*. Since K is Archimedean, as is well-known, (a) holds (see [7], etc.). □

Let $f : [a, b] \to K$ with K an ordered field, and $A \subset [a, b]$. Without loss of generalities, let us consider "$[0, 1]$" instead of "$[a, b]$", and consider "maxima (or upper bounds)" instead of "minima (or lower bounds)" of $f(A)$.

Theorem 2.2. *For an ordered field* K, *the following are equivalent.*

(a) K *is metrizable.*

(b) *There exists an infinite* (*countable closed*) *set* $A \subset [0,1]$ *such that for any continuous function* $f : [0,1] \to K$, $f(A)$ *has a maximum.*

Proof. For (a) $\Rightarrow$ (b), $0 \in K$ is not isolated in $[0,1]$, then there exists an infinite sequence $L \subset [0,1]$ converging to the point 0. Thus for any continuous function $f : [0,1] \to K$, $M = f(L \cup \{0\})$ has a maximum in K (indeed, for some $p \in M$, if $p > f(0)$, then $\{q \in M : q > p\}$ is finite since any subsequence of M converges to the point $f(0)$). For (b) $\Rightarrow$ (a), suppose that K is not metrizable. Let A be any infinite subset of $[0,1]$, and let $D = \{d_n : n \in N\}$ be an infinite countable subset of A. Then, by Proposition 2.2, D is closed discrete in $[0,1]$. Thus, since $[0,1]$ is normal, as is well-known, there exists a closed discrete collection $\mathscr{D} = \{[a_n, b_n] : n \in N\}$ in $[0,1]$ with $a_n < d_n < b_n$. Define a function $f : [0,1] \to K$ as follows:

$$f(x) = \begin{cases} (n/(d_n - a_n))(x - a_n) & \text{if } x \in (a_n, d_n] \ \ (n \in N) \\ (n/(b_n - d_n))(b_n - x) & \text{if } x \in (d_n, b_n) \ \ (n \in N) \\ 0 & \text{if } x \in [0,1] - \bigcup_{n=1}^{\infty} (a_n, b_n) \end{cases}$$

Then f is continuous, because any $f^{-1}((\alpha, \beta))$ is open in $[0,1]$ since the collection $\mathscr{D}$ is closed discrete in $[0,1]$. But, $f(A)$ has no maxima since $f(d_n) = n$ $(n \in N)$. This is a contradiction. Hence K is metrizable. $\square$

Corollary 2.2. *For an ordered field* K, (1), (2), *and* (3) *below hold.*

(1) *The following are equivalent.*
   (a) K *is metrizable, but not Archimedean.*
   (b) *There exists a continuous function* $f : [0,1] \to K$ *such that* $f(S_0)$ *has no upper bounds in* K, *where* $S_0 = \{1/n : n \in N\} \cup \{0\} \subset K$.

(2) *The following are equivalent.*
   (a) K *is metrizable, but not Dedekind-complete.*
   (b) *There exists a continuous function* $f : [0,1] \to K$ *such that for some countable* (*closed*) *set* L *in* $[0,1]$, $f(L)$ *has no upper bounds in* K.

(3) *The following are equivalent.*
   (a) K *is metrizable.*
   (b) *Same as* (b) *in* (2), *but replace* "$[0,1]$" *by an open interval* "$(0,1)$" *twice.*

Proof. For (1), (2) and (3), to see (a) $\Rightarrow$ (b) holds, assume (a) holds. Since K is metrizable, K has a countable set $\{\alpha_n : n \in N\}$ having no upper bounds by

Proposition 2.2. While, for (1), the sequence S_0 is closed discrete in $[0,1]$ by Proposition 2.5(1). For (2), K has a decreasing sequence L in $[0,1]$ having no limit points, hence L is closed discrete in $[0,1]$. For (3), the same holds in a normal space $(0,1)$. Let us denote these discrete countable sets by $\{d_n : n \in N\}$. Then, we obtain a desirable continuous function f in (b) such that $f(d_n) = \alpha_n$ $(n \in N)$ by the same way as in the proof of Theorem 2.2. To see (b) $\Rightarrow$ (a), assume (b) holds. Then, K is metrizable by Proposition 2.2, for K contains a countable set having no upper bounds. While, for (1), $f(S_0)$ has no upper bounds in (b), then it is not compact in K, thus neither is S_0. Hence, K is not Archimedean by Proposition 2.5(1). Similarly, for (2), $[0,1]$ is not compact. Thus, K is not Dedekind-complete by Proposition 2.5(2). Hence (a) holds. $\square$

The following holds in view of the proofs of Theorems 2.2 and Corollary 2.2.

Corollary 2.3. *For an ordered field* K, (1) *and* (2) *below hold* (see [10]).

(1) *The following are equivalent.*
   (a) *K is Archimedean.*
   (b) *For any continuous function $f : [0,1] \to K$, $f(S_0)$ has a maximum, where $S_0 = \{1/n : n \in N\} \cup \{0\} \subset K$.*

(2) *The following are equivalent.*
   (a) *K is Dedekind-complete.*
   (b) *For any continuous function $f : [0,1] \to K$, and for any decreasing sequence L in $[0,1]$, $f(cl\, L)$ has a maximum.*
   (c) *K is Archimedean, and same as* (b), *but $f(L)$ has an upper bound.*
   (d) *K is Archimedean, and for any continuous function $f : [0,1] \to K$, $f([0,1])$ has an upper bound.*

Remark 2.4. (1) We can not replace "maximum" by "upper bound" in Theorem 2.2. (Indeed, let K be a non-metrizable ordered field in Remark 2.1. Then, since K is not metrizable, by Corollary 2.2(2) (or (3)), K satisfies (b) with the substitution "upper bound" in Theorem 2.2.

(2) In Corollary 2.2, we can not replace "no upper bounds" with "no maxima". (Indeed, by Theorem 2.2, for a non-metrizable ordered field K, K satisfies (b) with the substitution "no maxima" in Corollary 2.2).

(3) For continuous functions from $[0,1] (\subset K)$ into $\mathbf{R}$, we can replace "maximum" by "upper bound" in Theorem 2.2 and Corollary 2.3 (hence, we can omit "Archimedean" in Corollary 2.3(2)) in view of their proofs, using a classical *Tietze's extension theorem* (*Tietze-Urysohn theorem*).

3. Revision of Dobbs' Paper [2]

Dobbs [2, Theorem 2.6] shows that for each uncountable cardinal number $\aleph$, there exist ordered fields F_1 and F_2 of cardinality $\aleph$, such that F_1 is metizable and F_2 is not metrizable. For his (long) proof of this result, let us revise and simplify the proof. Then, we give some related results.

THEOREM 3.1. *For each infinite cardinal number* $\aleph$, *there exist* (*non-Archimedean*) *ordered fields* F_1 *and* F_2 *of cardinality* $\aleph$ *satisfying*:

(a) F_1 *is metrizable*; *and*
(b) F_2 *is not metrizable*, *but* $\aleph$ *is uncountable.*

PROOF. Let K be an ordered field. For a set I, let $\mathbf{X} = \{x_i : i \in I\}$ be the set of algebraically independent indeterminates. Let $F = K(\mathbf{X})$ be the field of all rational functions in the variables $x_i \in \mathbf{X}$ with coefficients in K. Then $|F| = max\{|K|, |I|\} \geq \aleph_0$. We define a linear order $\leq$ on F by the steps (i), (ii), and (iii) below. (The order $\leq$ on F is denoted by $\leq_1$ in the proof of [2, Theorem 2.6], but we do not use the order $\leq_2$ on F defined there).

(i) Let ω_i $(i = 0, 1)$ be the smallest ordinal of cardinality $\aleph_i$. For the index-set I being infinite countable, define $I = [0, \omega_0)$ (or N) which has the usual order, so let I be uncountable. We will define a well-order $\leq_*$ in I to satisfy (*): for each countable subset C in I, there exists $i_0 \in I$ such that $i <_* i_0$ for all $i \in C$. Indeed, since I is uncountable, for some $A_1 \subset I$, we can consider A_1 as $[0, \omega_1)$ having the usual order $\preceq_1$. Let $A_0 = I - A_1$, and give a well-order $\preceq_0$ in A_0. Define the lexicographic order $\leq_*$ in $I = A_0 + A_1$ (i.e., for $a_0, a_1 \in I$, define $a_0 <_* a_1$ if $a_i \in A_i$; otherwise, if $a_i \in A_0$ with $a_0 \prec_0 a_1$, or $a_i \in A_1$ with $a_0 \prec_1 a_1$). Then $(I, \leq_*)$ is a well-ordered set satisfying (*). (Not every uncountable ordered set satisfy (*) (by the usual ordered field $\mathbf{R}$, etc.)).

(ii) Any monomial $x_{i_1}^{m_1} \cdots x_{i_n}^{m_n}$ in F $(m_i \in (0, \omega_0))$ is arranged by $i_n <_* i_{n-1} <_* \cdots <_* i_2 <_* i_1$. Among the monomials in F, define the lexicographic order $\preceq$ in terms of $I \times (0, \omega_0)$; that is, for distinct monomials $u = x_{i_1}^{m_1} \cdots x_{i_n}^{m_n}$ and $v = x_{j_1}^{p_1} \cdots x_{j_k}^{p_k}$, define $u \prec v$ if one of the following holds: $i_1 <_* j_1$; $i_1 = j_1$, $m_1 < p_1$; $i_1 = j_1$, $m_1 = p_1$, $i_2 <_* j_2$; and so on. (By convention, let us consider $1 \in K$ as an (empty) monomial, and let $1 \prec u$ for any other monomial u). We note that for monomials u, v with $u \prec v$, and $w(\succ 0)$, $wu \prec wv$.

(iii) Any polynomial $\alpha_1 w_1 + \cdots + \alpha_m w_m$ in F is arranged by $w_1 \prec w_2 \prec \cdots \prec w_n$, where $\alpha_i \in K - \{0\}$, and w_i are monomials in F. (By convention, let $0u = 0$ for any monomial u). Let us define a linearly order $\leq$ in F. For $\eta \in F$, let

$\eta = \pm(f/g)$, where $f = a_1u_1 + \cdots + a_mu_m$ and $g = b_1v_1 + \cdots + b_nv_n$ are polynomials with $a_m, b_n > 0$ in K. Define $\eta > 0$ if the sign of the fraction is "+", and $\eta < 0$ if "−". For $\eta, \xi \in F$, define $\eta < \xi$ if $0 < \xi - \eta$. Then $(F, \leq)$ is an ordered field (for example, for $\eta < \xi$, and $\zeta > 0$, $\zeta\eta < \zeta\xi$), but it is not Archimedean (for $x \in \mathbf{X}$, $x > n$ for all $n \in N$).

Now, first, for (b), let $F_2 = \mathbf{Q}(\mathbf{X})$ with $|I| = \aleph$ $(\geq \aleph_1)$. Then F_2 has cardinality $\aleph$. To show that F_2 is not metrizable, let $L = \{\eta_n : n \in N\} \subset F_2$ with $\eta_n > 0$. Let $S = \{x_i : i \in N\} \subset \mathbf{X}$ be all variables appeared in denominators of η_n $(n \in N)$. Since S is countable, there exists $x_k \in \mathbf{X}$ such that $x_i \prec x_k$ for all $x_i \in S$ by (i) and (ii). Let $\eta_n = +(f_n/g_n)$ for each $n \in N$. Then, for any $n \in N$, $1/x_k < f_n$ and $g_n < x_k$ by (iii), thus $0 < 1/x_k^2 < +(f_n/g_n) = \eta_n$. Hence, the sequence L does not accumulate to 0. Thus, F_2 is not metrizable by Proposition 2.2.

Next, for (a), let $K = \mathbf{Q}$ for $\aleph = \aleph_0$, and let $K = F_2$ for $\aleph \neq \aleph_0$, for example. Let $\mathbf{X} = \{x\}$. Then $F_1 = K(\mathbf{X})$ has cardinality $\aleph$. To see F_1 is metrizable, let $L = \{1/x^n : n \in N\}$. For $\eta = +(f/g) > 0$, let $n = max\{deg(f), deg(g)\}$. Then $0 < 1/x^{n+1} < \eta$. Thus L converges to 0. Hence, F_1 is metrizable by Proposition 2.2. □

Remark 3.1. In [2, Remark 2.7(b)], it is shown that $F = \mathbf{R}(x_i : i \in \mathbf{R})$ is not metrizable. The *coefficients-set* $\mathbf{R}$ is the usual ordered field, but for the index-set $\mathbf{R}$, we consider it as a well-ordered set satisfying the condition (*) in (i) of the proof of Theorem 3.1, then F is not metrizable in view of Theorem 3.1. While, if the index-set $\mathbf{R}$ is considered as the usual order set $\mathbf{R}$, then F would be metrizable, because the sequence $\{1/x_i : i \in N\}$ converges to 0 in F. Similarly, any $F = K(\{x_i : i \in I\})$ would be metrizable if we take an order on I defined by replacing $[0, \omega_1)$ with $[0, \omega_0)$ in (i) of the proof of Theorem 3.1 (assuming I is not finite), because I has the countable subset $[0, \omega_0)$ which is cofinal in I with respect to this order.

Let us give a characterization for $K(\{x_i : i \in I\})$ to be metrizable (or separable metrizable), here we consider the index-set I as in (i) of the proof of Theorem 3.1.

Theorem 3.2. *For $F = K(\{x_i : i \in I\})$, F is metrizable if and only if the index set I is countable.*

Proof. This holds in view of the proof of Theorem 3.1. Indeed, for the "only if" part, suppose I is not countable, then F is not metrizable. For the "if" part, if the countable set I is infinite, then $\{1/x_i : i \in I\}$ converges to 0. If I is finite, for $m = max\, I$, $\{1/x_m^n : n \in N\}$ also converges to 0. Hence, F is metrizable by Proposition 2.2. □

Lemma 3.1. *For $F = K(\mathbf{X})$, K is a closed discrete subset of F.*

Proof. Let $x \in F - K$. Then for each $\eta \in F$, $V(\eta) = (\eta - 1/x, \eta + 1/x)$ is a neighborhood at η in F such that $|V(\eta) \cap K| \leq 1$. Indeed, suppose that there exist $\alpha, \beta \in V(\eta) \cap K$ with $\alpha \neq \beta$. Then $0 < \gamma = |\alpha - \beta| < 2/x$. But, $\gamma > 2/x$ since $\gamma \in K$. This is a contradiction. Hence, K is a closed discrete subset of F. □

Remark 3.2. Any ordered field has no isolated points by its order topology. Thus, by Lemma 3.1, for any $F = K(\mathbf{X})$, the ordered field K is not a *subspace* in F (namely, the order topology of K is not the relative topology from F).

Corollary 3.1. *For $F = K(\mathbf{X})$, F is separable metrizable if and only if F is countable.*

Proof. The "if" part holds by a fact that every countable ordered field is separable metrizable (see [2], etc.). For the "only if" part, F is Lindelöf, then K is countable by Lemma 3.1. While, F is metrizable, then $\mathbf{X}$ is countable by Theorem 3.2. Thus, F is countable. □

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