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ON p-HARMONIC MAPS INTO SPHERES

By

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Abstract. We study the topology of p-harmonic maps $\phi : M \to S^\nu$ from a compact Riemannian manifold M into a sphere S^ν. We also show that any p-energy minimizing map $\phi : M \to S^\nu$ omitting a totally geodesic submanifold of codimension two $\Sigma \subset S^\nu$ is of class C^1. This extends results by B. Solomon, [13], from harmonic to p-harmonic maps.

1. Introduction

Let (M, g) and S be Riemannian manifolds and $2 \le p < \infty$. Let $\iota : S \hookrightarrow \mathbf{R}^N$ be an isometric immersion in some Euclidean space $\mathbf{R}^N$ and let $W^{1,p}(M, S)$ be the Sobolev space

$$W^{1,p}(M, S) = \{\phi \in W^{1,p}(M, \mathbf{R}^N) : \phi(x) \in S \text{ for a.e. } x \in S\}.$$

Let us consider the *p-energy* integral

$$E_p(\phi) = \int_M \|d\phi\|^p \, d\,\mathrm{vol}(g), \quad \phi \in W^{1,p}_{\mathrm{loc}}(M, S).$$

We study *p-harmonic* maps i.e. C^∞ maps $\phi : M \to N$ which are critical points of E_p (with respect to any variation of compact support). The theory of p-harmonic maps has undergone an ample development both from the point of view of partial differential equations (cf. e.g. [2]–[3], [5], [7]–[9] and [15]) and differential geometry (cf. e.g. [6], [10], [14] and [16]). When $S = S^\nu \subset \mathbf{R}^{\nu+1}$ is the standard sphere we show that any nonconstant p-harmonic map $\phi : M \to S^\nu$ either links or meets $\Sigma = \{x = (x_1, \dots, x_{\nu+1}) \in S^\nu : x_1 = x_2 = 0\}$ (cf. Section 2 for definitions). This generalizes a result by B. Solomon, [13], where the previous statement

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was established for $p = 2$. A map $\phi \in W^{1,p}(M, S)$ is *p-energy minimizing* if $E_p(\phi) \le E_p(\psi)$ for any compact set $K \subset M$ and any map $\psi \in W^{1,p}_{\mathrm{loc}}(M, S)$ such that $\psi = \phi$ a.e. in $M \backslash K$. Building on the results of N. Nakauchi, [9], we show that any p-energy minimizing map $\phi \in W^{1,p}_{\mathrm{loc}}(M, S^\nu)$ which omits a neighborhood of Σ in S^ν is C^1. Given a subset $E \subset M$ we set

$$H_\sigma(E) = \lim_{\varepsilon \to 0^+} \inf\left\{ \sum_{\nu=1}^{\infty} \mathrm{diam}(X_\nu)^\sigma : \bigcup_{\nu=1}^{\infty} X_\nu \supset E, \mathrm{diam}(X_\nu) \le \varepsilon \right\}$$

where diameters are meant with respect to the Riemannian distance function on M. Then $h(E) = \inf\{\sigma \ge 0 : H_\sigma(E) = 0\}$ is the *Hausdorff dimension* of E. For each map $\phi : M \to S$ let $\mathrm{Reg}(\phi)$ consist of all points $x \in M$ such that ϕ is continuous at x. Then $\mathrm{Sing}(\phi) = M \backslash \mathrm{Reg}(\phi)$ is the *singular set* of ϕ. We establish an upper bound on the Hausdorff dimension $h[\mathrm{Sing}(\phi)]$ of a p-energy minimizing p-harmonic map $\phi \in W^{1,p}_{\mathrm{loc}}(M, S)$ whose target manifold S is covered by a warped product manifold (cf. Theorem 3 below).

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2. *p*-Harmonic Maps into Warped Products

Let (L, g_L) be a Riemannian manifold and $S = L \times_w \mathbf{R}$ a warped product where $w \in C^\infty(S)$, $w > 0$, is the warping function. S carries the Riemannian metric $h = \Pi_1^* g_L + w^2\, dt \otimes dt$ where $\Pi_1 : S \to L$ is the natural projection. Let M be a compact orientable n-dimensional Riemannian manifold. By a result of B. Solomon, [13], for any harmonic map $\phi : M \to S$ there is $t_\phi \in \mathbf{R}$ such that $\phi(M) \subset L \times \{t_\phi\}$. The purpose of this section is to establish a similar result for p-harmonic maps.

LEMMA 1. *Let $\phi : M \to L \times_w \mathbf{R}$ be a p-harmonic map $(p \ge 2)$. Let $u = \Pi_2 \circ \phi$ where $\Pi_2 : S \to \mathbf{R}$ is the projection. Then u is a solution to the elliptic equation*

$$\mathrm{div}[\|d\phi\|^{p-2}(w \circ \phi)^2 \nabla u] = \|d\phi\|^{p-2}(w \circ \phi)(w_t \circ \phi). \tag{1}$$

In particular if w is a function on L (i.e. $\partial w/\partial t = 0$) then $\phi(M) \subset L \times \{t_\phi\}$ for some $t_\phi \in \mathbf{R}$.

Proof. Let $F = \Pi_1 \circ \phi$. Let $\varphi \in C^\infty(M)$ and let us set

$$\phi_s(x) = (F(x), u(x) + s\varphi(x)), \quad x \in M, \ |s| < \varepsilon,$$

so that $\{\phi_s\}_{|s|<\varepsilon}$ is a smooth 1-parameter variation of ϕ. Let g be the Riemannian metric on M. The Hilbert-Schmidt norm of the differential $d\phi_s$ is given by

$$\|d\phi_s\|^2 = \|dF\|^2 + (w \circ \phi_s)^2\{\|\nabla u\|^2 + 2sg(\nabla u, \nabla\varphi) + O(s^2)\}. \tag{2}$$

As $\phi : M \to S$ is p-harmonic $\{dE_p(\phi_s)/ds\}_{s=0} = 0$ hence

$$\int_M \|d\phi\|^{p-2}\{(w \circ \phi)(w_t \circ \phi)\varphi\|\nabla u\|^2 + (w \circ \phi)^2 g(\nabla u, \nabla\varphi)\}\, d\,\mathrm{vol}(g) = 0. \tag{3}$$

The identity

$$\|d\phi\|^{p-2}(w \circ \phi)^2 g(\nabla u, \nabla\varphi) = \mathrm{div}[\varphi\|d\phi\|^{p-2}(w \circ \phi)^2\nabla u] - \varphi\, \mathrm{div}[(w \circ \phi)^2\|d\phi\|^{p-2}\nabla u]$$

together with Green's lemma yields

$$\int_M \varphi\{\|d\phi\|^{p-2}(w \circ \phi)(w_t \circ \phi) - \mathrm{div}[\|d\phi\|^{p-2}(w \circ \phi)^2\nabla u]\}\, d\,\mathrm{vol}(g) = 0$$

hence (as $\varphi \in C^\infty(M)$ is arbitrary) the equation (1) holds. If $w_t = 0$ then (1) becomes $\mathrm{div}[\|d\phi\|^{p-2}(w \circ \phi)^2\nabla u] = 0$. Consequently

$$\mathrm{div}[\|d\phi\|^{p-2}(w \circ \phi)^2 u\nabla u] = \|d\phi\|^{p-2}(w \circ \phi)^2\|\nabla u\|^2$$

and integrating over M leads (by Green's lemma) to $\nabla u = 0$ hence u is constant. Q.e.d.

Let $S^\nu = \{x = (x_1, \dots, x_{\nu+1}) \in \mathbf{R}^{\nu+1} : x_1^2 + \cdots + x_{\nu+1}^2 = 1\}$ and $\Sigma = \{x \in S^\nu : x_1 = x_2 = 0\}$. We recall that $S^\nu \backslash \Sigma$ is isometric to the warped product $S_+^{\nu-1} \times_f S^1$ where

$$S_+^{\nu-1} = \{y = (y', y_\nu) \in \mathbf{R}^\nu : y \in S^{\nu-1}, y_\nu > 0\},$$

$$f : S_+^{\nu-1} \times S^1 \to (0, +\infty), \quad f(y, \zeta) = y_\nu, \quad y \in S_+^{\nu-1}, \ \zeta \in S^1 \subset \mathbf{C}.$$

Indeed the map

$$I(y, \zeta) = (y_\nu u, y_\nu v, y'), \quad y \in S_+^{\nu-1}, \ \zeta = u + iv \in S^1,$$

is an isometry of $S_+^{\nu-1} \times_f S^1$ endowed with the Riemannian metric $\pi_1^* g_{\nu-1} + f^2\pi_2^* g_1$ onto $(S^\nu \backslash \Sigma, g_\nu)$. Here $\pi_1 : S_+^{\nu-1} \times_f S^1 \to S_+^{\nu-1}$ and $\pi_2 : S_+^{\nu-1} \times_f S^1 \to S^1$ are the natural projections. Also g_N denotes the canonical Riemannian metric on the sphere $S^N \subset \mathbf{R}^{N+1}$. The next section is devoted to p-harmonic maps from M into $S^\nu \backslash \Sigma$.

3. p-Harmonic Maps Omitting a Totally Geodesic Submanifold

Let $\Sigma \subset S^\nu$ be a codimension 2 totally geodesic submanifold. A continuous map $\phi : M \to S^\nu$ *meets* Σ if $\phi(M) \cap \Sigma \neq \emptyset$. Let $\phi : M \to S^\nu$ be a continuous map that doesn't meet Σ. Then ϕ is said to *link* Σ if the map $\phi : M \to S^\nu \backslash \Sigma$ is not null-homotopic.

Theorem 1. *Let $\phi : M \to S^\nu$ be a nonconstant p-harmonic $(p \geq 2)$ map. Then ϕ either links or meets Σ.*

For $p = 2$ this is B. Solomon's Theorem 1 in [13], p. 155. The proof of Theorem 1 is by contradiction. We wish to apply Lemma 1 above for $L = S_+^{\nu-1}$ and $w \in C^\infty(S_+^{\nu-1})$ given by $w(y) = y_\nu$. Let $\phi : M \to S^\nu \backslash \Sigma$ be a null-homotopic p-harmonic map. Let $E : \mathbf{R} \to S^1$ be the exponential map $E(t) = e^{it}$. We endow $S_+^{\nu-1} \times \mathbf{R}$ with the warped product metric $\Pi_1^* g_{\nu-1} + (w \circ \Pi_1)^2 \, dt \otimes dt$. Then $\pi = (\mathrm{id}, E)$ is a local isometry of $S_+^{\nu-1} \times_w \mathbf{R}$ onto $S_+^{\nu-1} \times_f S^1$. There is a coordinate system on $\mathbf{R}^{\nu+1}$ such that $\Sigma = \{x \in S^\nu : x_1 = x_2 = 0\}$. Let then $\tilde{\psi} = I^{-1} \circ \phi$ and let us set $F = \pi_1 \circ \tilde{\psi}$ and $\tilde{u} = \pi_2 \circ \tilde{\psi}$. Let $x_0 \in M$ and $\zeta_0 = \tilde{u}(x_0) \in S^1$. Let $t_0 \in \mathbf{R}$ such that $E(t_0) = \zeta_0$. As $\tilde{\psi} : M \to S_+^{\nu-1} \times S^1$ is null-homotopic it follows that $\tilde{u}_* \pi_1(M, x_0) = 0$ hence by standard homotopy theory (cf. e.g. Proposition 5.3 in [4], p. 43) there is a unique function $u : M \to \mathbf{R}$ such that $u(x_0) = t_0$ and $E \circ u = \tilde{u}$. As $\tilde{u}$ is smooth it follows that $u \in C^\infty(M)$ as well. It is an elementary matter that

Lemma 2. *Let S and $\tilde{S}$ be Riemannian manifolds and $\pi : S \to \tilde{S}$ a local isometry. Let $\tilde{\phi} : M \to \tilde{S}$ be a p-harmonic map $(p \geq 2)$ from a compact orientable Riemannian manifold M into $\tilde{S}$. Then any smooth map $\phi : M \to S$ such that $\pi \circ \phi = \tilde{\phi}$ is p-harmonic.*

By Lemma 2 it follows that $\psi = (F, u) : M \to S_+^{\nu-1} \times_w \mathbf{R}$ is a p-harmonic map. Then (by Lemma 1) $\psi(M) \subset S_+^{\nu-1} \times \{t_\psi\}$ for some $t_\psi \in \mathbf{R}$. To end the proof of Theorem 1 we establish

Lemma 3. *Any p-harmonic $(p \geq 2)$ map $\phi : M \to S_+^{\nu-1}$ of a compact orientable Riemannian manifold M into an open upper hemisphere $S_+^{\nu-1}$ is constant.*

Proof. As ϕ is p-harmonic

$$\mathrm{div}(\|d\Phi\|^{p-2}\nabla\Phi^{\alpha}) + \|d\Phi\|^{p}\Phi^{\alpha} = 0 \quad (1 \le \alpha \le \nu)$$

(the p-harmonic map system, cf. e.g. [1]) where $\Phi = \iota \circ \phi$ and $\iota : S^{\nu-1} \to \mathbf{R}^{\nu}$ is the inclusion. Integration over M leads (by Green's lemma) to $\int_M \|d\Phi\|^p \Phi^\alpha \, d\,\mathrm{vol}(g) = 0$ hence (by $\Phi^\nu > 0$) $\|d\Phi\| = 0$ and Φ is constant. Q.e.d.

We recall that $S_+^{\nu-1} \times S^1 \simeq S^1$ (a homotopy equivalence). Therefore a continuous map $\phi : M \to S_+^{\nu-1} \times S^1$ is null homotopic if and only if $\pi_2 \circ \phi : M \to S^1$ is null-homotopic. The homotopy classes of continuous maps $M \to S^1$ form an abelian group $\pi^1(M)$ (the Bruschlinski group of M, cf. e.g. [4], p. 48). Also (by Theorem 7.1 in [4], p. 49) there is a natural isomorphism $\pi^1(M) \approx H^1(M, \mathbf{Z})$. Then we may state the following

Corollary 1. *Let M be a compact orientable Riemannian manifold with $H^1(M, \mathbf{Z}) = 0$. Then any nonconstant p-harmonic map $\phi : M \to S^\nu$ meets Σ.*

4. p-Energy Minimizing Maps

Let M be a compact orientable n-dimensional Riemannian manifold, $n \ge 4$. By a result of B. Solomon, [13], any energy minimizing map $\phi : M \to S^\nu$ which omits a neighborhood of a codimension two totally geodesic submanifold $\Sigma \subset S^\nu$ is everywhere smooth. The main ingredient in the proof is R. Schoen & K. Uhlenbeck's Theorem IV in [11], p. 310. For the case of p-harmonic maps the corresponding regularity result was recovered by N. Nakauchi, [9]. Cf. also [8]. We establish

Theorem 2. *Let M be a compact orientable n-dimensional Riemannian manifold and $\phi : M \to S^\nu$ a p-energy minimizing map which omits a neighborhood of a codimension two totally geodesic submanifold $\Sigma \subset S^\nu$. Then $\phi : M \to S^\nu$ is of class C^1.*

Proof. Let S be a Riemannian manifold and let $P(S, k)$ be the following statement: any p-energy minimizing map $\phi \in C^1(S^{d-1}, S)$, $[p] + 1 \le d \le k$ is constant. We need the following

Lemma 4. *Let S be a Riemannian manifold. Let us assume that there is an integer $k \ge [p] + 1$ such that $P(S, k)$ holds true. Then $h[\mathrm{Sing}(\phi)] \le n - k - 1$ for any p-energy minimizing map $\phi : M \to S$. If $n < k + 1$ then $\mathrm{Sing}(\phi) = \varnothing$.*

This is Proposition 1 in [9], p. 1052. Here h indicates the Hausdorff dimension (cf. §1). Let $\phi : M \to S^\nu$ be a p-energy minimizing map such that $\phi(M) \subset S^\nu \backslash V$ for some open set $V \subset S^\nu$ with $\Sigma \subset V$. It is well known that $H^1(S^{d-1}, \mathbf{Z}) = 0$ for any integer $d \geq 3$. By Corollary 1 above any nonconstant p-harmonic map $\phi \in C^1(S^{d-1}, S^\nu)$ meets Σ. Consequently $P(S^\nu \backslash \Sigma, k)$ holds true for any $k \geq \max\{[p]+1, n\}$ so that (by Lemma 4) the singular set of ϕ is empty.

Only the last statement in Lemma 4 was actually used to prove Theorem 2. The following more general result may be established

THEOREM 3. *Let M and S be Riemannian manifolds where M is compact, orientable and n-dimensional. Let $p \geq 2$ and let us assume that* i) *the Riemannian universal covering manifold of S is a warped product manifold $\tilde{S} = L \times_w \mathbf{R}$ with the warping factor $w \in C^\infty(L)$ and* ii) *there is an integer $k \geq [p]+1$ such that $P(L, k)$ holds true. Then $h[\mathrm{Sing}(\phi)] \leq n - k - 1$ for any p-energy minimizing map $\phi \in W^{1,p}_{\mathrm{loc}}(M, S)$.*

PROOF. As $\pi_1(S^{d-1}) = 0$ for each $d \geq 3$, any p-harmonic map $\psi \in C^1(S^{d-1}, S)$ lifts to a map $\tilde{\psi} \in C^1(S^{d-1}, \tilde{S})$ which is p-harmonic (by Lemma 2) hence factors through a p-harmonic map $\bar{\psi} \in C^1(M, L)$ (by Lemma 1). Thus $P(L, k)$ implies $P(S, k)$ and we may apply Lemma 4. Q.e.d.

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