TSUKUBA J. MATH.
Vol. 35 No. 1 (2011), 115–130

ON THE QUASI-IRREDUCIBILITY AND COMPLETE QUASI-REDUCIBILITY OF SOME REDUCTIVE PREHOMOGENEOUS VECTOR SPACES

By

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Abstract. In this paper, we investigate the Q-irreducibility and complete Q-reducibility of prehomogeneous vector spaces and classify such prehomogeneous vector spaces in some cases.

Introduction

First we recall the definition and some basic facts about prehomogeneous vector spaces. For the detail, see [K2]. Let ρ be a rational representation of an algebraic group G on a finite-dimensional vector space V where everything is defined over the complex number field $\mathbf{C}$. If V has a Zariski-dense G-orbit $\mathbf{O}$, the triplet (G, ρ, V) is called a prehomogeneous vector space (abbrev. PV). A non-zero rational function $f(x)$ is called a relative invariant if there exists a character $\chi : G \to GL(1)$ satisfying $f(\rho(g)x) = \chi(g)f(x)$ for all $g \in G$. Then we can define the map $\varphi_f = grad \log f : \mathbf{O} \to V^*$ which satisfies $\varphi_f(\rho(g)x) = \rho^*(g)\varphi_f(x)$ $(g \in G, x \in \mathbf{O})$. If φ_f is dominant, i.e., the image $\varphi_f(\mathbf{O})$ is a Zariski-dense orbit of the dual triplet (G, ρ^*, V^*), we call such f a non-degenerate relative invariant. A relative invariant f of degree ≥ 2 is non-degenerate if and only if its Hessian $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is not identically zero. A PV (G, ρ, V) is called regular if there exists a non-degenerate relative invariant. When G is reductive, then (G, ρ, V) is regular if and only if a generic isotropy subgroup $G_x = \{g \in G \mid \rho(g)x = x\}$ $(x \in \mathbf{O})$ is reductive.

For a PV (G, ρ, V), there are relative invariants $f_1(x), \ldots, f_N(x)$ which are algebraically independent irreducible polynomials, and any relative invariant

2010 Mathematics Subject Classification(s): Primary 11S90; Secondary 20G20.
Key words: Prehomogeneous vector spaces, Classification.
Received November 19, 2010.

$f(x)$ can be written uniquely as $f(x) = cf_1(x)^{m_1} \cdots f_N(x)^{m_N}$ with $c \in \mathbf{C}^\times$ and $(m_1 \cdots m_N) \in \mathbf{Z}^N$ (see [K2, Theorem 2.9]). We call $f_1(x), \dots, f_N(x)$ the basic relative invariants of (G, ρ, V) which are unique up to a constant multiple. For $G = GL(1)^l \times H$ with a semisimple algebraic group H, let $\varphi : G \to GL(1)^l$ be the projection. Assume that $\varphi(G_x) \cong GL(1)^r$ for a generic isotropy subgroup G_x. Then the number N of the basic relative invariants is given by $N = l - r$ (cf. [K2, Proposition 2.12]).

H. Rubenthaler introduced the following notion of quasi-irreducibility and complete quasi-reducibility of reductive regular prehomogeneous vector spaces in [R2].

A reductive regular PV (G, ρ, V) is called quasi-irreducible (abbrev. Q-irreducible) if for any proper invariant subspace $U \subset V$, the PV (G, ρ, U) is not regular.

A reductive regular PV with only one relative invariant is always Q-irreducible. In particular an irreducible regular PV is always Q-irreducible.

A reductive regular PV (G, ρ, V) is called completely quasi-reducible (abbrev. completely Q-reducible) if there exists a decomposition $\rho = \bigoplus_{i=1}^n \rho_i$, $V = \bigoplus_{i=1}^n V_i$ where the V_i's are G-invariant subspaces such that (G, ρ_i, V_i) is Q-irreducible. The spaces V_i are then called a Q-irreducible component of (G, ρ, V). In this paper, we assume that $n \geq 2$ for the completely Q-reducible PV to distinguish from the Q-irreducible PV.

In this paper, we give the construction of Q-irreducible and completely Q-reducible PV's which are called "general type". Then we give the list of the non-general type of Q-irreducible and completely Q-reducible PV's among simple PV's, 2-simple PV's of type I, 3-simple PV's of nontrivial type, and PV's which appear in the M. Sato's classification. We give the proof only for the difficult cases. Note that (G, ρ, V) is not Q-irreducible nor completely Q-reducible if and only if there exists some decomposition $\rho = \rho_1 \oplus \rho_2$ and $V = V_1 \oplus V_2$ such that (G, ρ_1, V_1) is a regular PV and (G, ρ_2, V_2) is a non-regular PV.

NOTATION. We denote by $M(m, n)$ (resp. $M(n)$) the totality of $m \times n$ (resp. $n \times n$) matrices. For the classical algebraic groups, we denote by $GL(n)$ (resp. $SL(n)$, $Sp(n)$, $SO(n)$, $Spin(n)$) the general linear group (resp. the special linear group, the symplectic group, the special orthogonal group, the spin group).

For the exceptional simple algebraic group of rank 2, we denote by (G_2) instead of G_2 to distinguish from the second group in G_i $(i = 1, \dots, m)$. We denote by E_i (resp. F_4) the exceptional simple algebraic group of rank i $(6 \leq i \leq 8)$ (resp. 4).

We denote by Λ_1 the standard representation of $GL(n)$ on $\mathbf{C}^n$. For a subgroup H of $GL(n)$, the restriction $\Lambda_1|_H$ (= the inclusion $H \hookrightarrow GL(n)$) is also simply denoted by Λ_1. More generally, Λ_k $(k = 1, \dots, r)$ denotes the fundamental irreducible representation of a simple algebraic group of rank r.

Since $\otimes$ and $\oplus$ are sometimes difficult to distinguish, we use the notation $+$ for the direct sum $\oplus$. Let $\rho_i : G_i \to GL_{m_i}$ be a rational representation of an algebraic group G_i $(i = 1, \dots, m)$. Then we denote the representation $\rho = (\rho_1 \otimes 1 \otimes \cdots \otimes 1) + \cdots + (1 \otimes \cdots \otimes 1 \otimes \rho_m)$ of $G_1 \times \cdots \times G_m$ by $\rho_1 \boxplus \cdots \boxplus \rho_m$.

In general, we denote by ρ^* the dual representation of a rational representation ρ. We denote by $V(n)$ an n-dimensional vector space in general. If $V(n)$ and $V(n)^*$ appear at the same time, $V(n)^*$ denotes the dual space of $V(n)$.

1. Q-irreducible and Completely Q-reducible PV's of General Type

First recall the castling transformation. For $m > n$, a triplet $(G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))$ is a (resp. regular) PV if and only if a triplet $(G \times GL(m-n), \rho^* \otimes \Lambda_1, V(m)^* \otimes V(m-n))$ is a (resp. regular) PV (See [K2], [SK]). In this case, one is called the castling transform of the other. Their generic isotropy subgroups are isomorphic and the number of relative invariants are the same.

If $m < n$, a triplet $(G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))$ is always a non-regular PV for any $(G, \rho, V(m))$. We call such a triplet a non-regular trivial PV.

If $m = n$, a triplet $(G \times GL(n), \rho \otimes \Lambda_1, V(m) \otimes V(n))$ is always a regular PV for any $(G, \rho, V(m))$. We call such a triplet a regular trivial PV.

Theorem 1.1. *Let (G, ρ, V) be any reductive regular PV. Assume that $\rho = \rho_1 \oplus \cdots \oplus \rho_l$, $V = V_1 \oplus \cdots \oplus V_l$ $(l \geq 2)$ with $\dim V_i \geq 2$ where each ρ_i is irreducible $(i = 1, \dots, l)$. Put $M = \dim V_1 + \cdots + \dim V_l - 1$. Then a triplet $(G \times GL(M), \rho_1 \otimes \Lambda_1 + \cdots + \rho_l \otimes \Lambda_1, V_1 \otimes V(M) + \cdots + V_l \otimes V(M))$ is a Q-irreducible PV.*

Proof. Since a triplet $(G \times GL(1), \rho_1 \otimes \Lambda_1 + \cdots + \rho_l \otimes \Lambda_1, V_1 \otimes V(1) + \cdots + V_l \otimes V(1))$ is a reductive regular PV, its castling transform $(G \times GL(M), \rho_1 \otimes \Lambda_1 + \cdots + \rho_l \otimes \Lambda_1, V_1 \otimes V(M) + \cdots + V_l \otimes V(M))$ is also a reductive regular PV. For any proper subset $\{i_1, \dots, i_t\}$ of $\{1, \dots, l\}$, we have $\dim V_{i_1} + \cdots + \dim V_{i_t} < M$. Hence $(G \times GL(M), \rho_{i_1} \otimes \Lambda_1 + \cdots + \rho_{i_t} \otimes \Lambda_1, V_{i_1} \otimes V(M) + \cdots + V_{i_t} \otimes V(M))$ is a non-regular trivial PV. This implies that a triplet $(G \times GL(M), \rho_1 \otimes \Lambda_1 + \cdots + \rho_l \otimes \Lambda_1, V_1 \otimes V(M) + \cdots + V_l \otimes V(M))$ is a Q-irreducible PV. ■

Definition 1.2. We call a regular PV $(G, \rho_1 \oplus \cdots \oplus \rho_r, V(n_1) \oplus \cdots \oplus V(n_r))$ a Q-irreducible PV of general type if there exist $m_1, \dots, m_r$ such that $(G, \rho_i, V(n_i)) \cong (H_i \times GL(m_1 + \cdots + m_r - 1), \sigma_i \otimes \Lambda_1, V(m_i) \otimes V(m_1 + \cdots + m_r - 1))$ where σ_i is an irreducible representation of H_i of the reductive subgroup of G $(i = 1, \dots, r)$. Then $(G, \rho_1 \oplus \cdots \oplus \rho_r, V(n_1) \oplus \cdots \oplus V(n_r))$ is Q-irreducible. It is not reduced because by the castling trasformation the dimension of its representation space can be reduced. However this castling transform may not be Q-irreducible.

Example 1.3. By [KKIY], $(GL(4) \times GL(2), \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1, V(6) \otimes V(2) + V(4) \otimes V(2))$ is a regular PV which has 2 basic relative invariants. For their explicit form, see [KKS; Theorem 5.4]. This PV is not Q-irreducible nor completely Q-reducible. However by Theorem 1.1, its castling transform $(GL(4) \times GL(2) \times GL(19), \Lambda_2 \otimes \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(6) \otimes V(2) \otimes V(19) + V(4) \otimes V(2) \otimes V(19))$ is a Q-irreducible PV of general type. Since the number of the basic relative invariants does not change under the castling transformations, this PV has also the 2 basic relative invariants.

Example 1.4. Let (G_i, ρ_i, V_i) be an irreducible regular PV $(i = 1, \dots, l)$ with $\dim V_i \geq 2$ and $l \geq 2$. Then $(G_1 \times \cdots \times G_l, \rho_1 \boxplus \cdots \boxplus \rho_l, V_1 \oplus \cdots \oplus V_l)$ is a reductive regular PV, and $((G_1 \times \cdots \times G_l) \times GL(M), (\rho_1 \boxplus \cdots \boxplus \rho_l) \otimes \Lambda_1, (V_1 \oplus \cdots \oplus V_l) \otimes V(M))$ with $M = \dim V_1 + \cdots + \dim V_l - 1$ is a Q-irreducible PV of general type. This PV has l irreducible relative invariants.

For example, $(GL(2) \times GL(6) \times GL(23), 3\Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes \Lambda_3 \otimes \Lambda_1, V(4) \otimes V(23) + V(20) \otimes V(23))$ is a Q-irreducible PV of general type which has 2 irreducile relative invariants.

Definition 1.5. For any PV (G_i, ρ_i, V_i) $(i = 1, \dots, l)$, the triplet $(G_1 \times \cdots \times G_k, \rho_1 \boxplus \cdots \boxplus \rho_k, V_1 \oplus \cdots \oplus V_k)$ is a PV. This is called the direct sum of (G_i, ρ_i, V_i) $(i = 1, \dots, l)$, and we denote it by $\bigoplus_{i=1}^{k} (G_i, \rho_i, V_i)$. If each (G_i, ρ_i, V_i) $(i = 1, \dots, l)$ is a regular PV, then its direct sum is also a regular PV (See Proposition 2.13 in [R2]). In particular, if (G_i, ρ_i, V_i) $(i = 1, \dots, l)$ are Q-irreducible, then its direct sum $\bigoplus_{i=1}^{k} (G_i, \rho_i, V_i)$ is always completely Q-reducible.

We shall give another construction of completely Q-reducible PV's.

Proposition 1.6. *The following assertions are equivalent.*

1. $(GL(1)^2 \times G, \rho_1 + \rho_2, V(m_1) + V(m_2))$ *is a regular PV where* $GL(1)^2$ *acts on each irreducible component as a scalar multiplication.*
2. $(G \times GL(m_1 - 1) \times GL(1), \rho_1^* \otimes \Lambda_1 \otimes 1 + \rho_2 \otimes 1 \otimes \Lambda_1, V(m_1)^* \otimes V(m_1 - 1) + V(m_2))$ *is a regular PV.*
3. $(G \times GL(m_1 - 1) \times GL(m_2 - 1),\ \rho_1^* \otimes \Lambda_1 \otimes 1 + \rho_2^* \otimes 1 \otimes \Lambda_1,\ V(m_1)^* \otimes V(m_1 - 1) + V(m_2)^* \otimes V(m_2 - 1))$ *is a regular PV.*

Proof. It is enough to prove the equivalence of 1 and 2. Let H be a generic isotropy subgroup of $(G, \rho_2, V(m_2))$. Then, 1 is PV-equivalent to $(H, \rho_1|_H, V(m_1))$, and its castling transform is $(H \times GL(m_1 - 1), \rho_1^* \otimes \Lambda_1, V(m_1)^* \otimes V(m_1 - 1))$ which is PV-equivalent to 2. Note that the regularity does not change under the castling transformations (see [KKTI; Theorem 1.30]). ∎

By Theorem 1.1 and Proposition 1.6, we can construct a completely Q-reducible PV.

We call a completely Q-reducible PV $(G, \sigma_1 \oplus \cdots \oplus \sigma_s, V(t_1) \oplus \cdots \oplus V(t_s))$ of "general type" if at least one of Q-irreducible components $(G, \sigma_i, V(t_i))$ is of general type.

Example 1.7. By [KKIY], a triplet $(GL(1)^3 \times Spin(7) \times SL(2)$, the vector rep. $\otimes\Lambda_1 + 1 \otimes \Lambda_1 +$ the spin rep. $\otimes 1$, $V(7) \otimes V(2) + V(2) + V(8))$ is a regular PV with the 3 basic relative invariants. Here $GL(1)^3$ acts on each irreducible component as scalar multiplications. This is not Q-irreducible nor completely Q-reducible. Put $\rho_1 =$ the vector rep. $\otimes\Lambda_1 + 1 \otimes \Lambda_1$, and $\rho_2 =$ the spin rep. $\otimes 1$. Then by Proposition 1.6, a triplet $(GL(1)^3 \times Spin(7) \times SL(2) \times GL(15)$, the vector rep. $\otimes\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 +$ the spin rep. $\otimes 1 \otimes 1$, $V(7) \otimes V(2) \otimes V(15) + V(2) \otimes V(15) + V(8))$ is a regular PV. By Theorem 1.1, $(GL(1)^2 \times Spin(7) \times SL(2) \times GL(15)$, the vector rep. $\otimes\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1$, $V(7) \otimes V(2) \otimes V(15) + V(2) \otimes V(15))$ is a Q-irreducible PV. Since $(GL(1) \times Spin(7) \times SL(2) \times GL(15)$, the spin rep. $\otimes 1 \otimes 1$, $V(8))$ is also Q-irreducible, a triplet $(GL(1)^3 \times Spin(7) \times SL(2) \times GL(15)$, the vector rep. $\otimes\Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 +$ the spin rep. $\otimes 1 \otimes 1$, $V(7) \otimes V(2) \otimes V(15) + V(2) \otimes V(15) + V(8))$ is a completely Q-reducible PV with 3 irreducible relative invariants. Note that $(Spin(7)$, the vector rep.) and $(SL(2), \Lambda_1)$ are self-dual.

Example 1.8. Assume that $(G, \rho_1 \oplus \cdots \oplus \rho_r \oplus \sigma_1 \oplus \cdots \oplus \sigma_s, V(m_1) \oplus \cdots \oplus V(m_r) \oplus V(n_1) \oplus \cdots \oplus V(n_s))$ is a regular PV where $(G, \rho_1 \oplus \cdots \oplus \rho_r, V(m_1)$

$\oplus \cdots \oplus V(m_r))$ and $(G, \sigma_1 \oplus \cdots \oplus \sigma_s, V(n_1) \oplus \cdots \oplus V(n_s))$ are regular PV's and ρ_i and σ_j are irreducible with $m_i \geq 2$, $n_j \geq 2$ and $r \geq 2$, $s \geq 2$. Here we do not assume any Q-irreducibility nor complete Q-reducibility. Then by Proposition 1.6, a triplet $(G \times GL(m_1 + \cdots + m_r - 1) \times GL(n_1 + \cdots + n_s - 1), (\rho_1^* + \cdots + \rho_r^*) \otimes \Lambda_1 \otimes 1 + (\sigma_1^* + \cdots + \sigma_s^*) \otimes 1 \otimes \Lambda_1, V(m_1)^* \otimes V(m_1 + \cdots + m_r - 1) + \cdots + V(n_s)^* \otimes V(n_1 + \cdots + n_s - 1))$ is a regular PV. By the assumption and Theorem 1.1, both $(G \times GL(m_1 + \cdots + m_r - 1), (\rho_1^* + \cdots + \rho_r^*) \otimes \Lambda_1, V(m_1)^* \otimes V(m_1 + \cdots + m_r - 1) + \cdots + V(m_r)^* \otimes V(m_1 + \cdots + m_r - 1)$ and $(G \times GL(n_1 + \cdots + n_s - 1), (\sigma_1^* + \cdots + \sigma_s^*) \otimes \Lambda_1, V(n_1)^* \otimes V(n_1 + \cdots + n_s - 1) + \cdots + V(n_s)^* \otimes V(n_1 + \cdots + m_s - 1)$ are Q-irreducible PV's. Therefore a triplet $(G \times GL(m_1 + \cdots + m_r - 1) \times GL(n_1 + \cdots + n_s - 1), (\rho_1^* + \cdots + \rho_r^*) \otimes \Lambda_1 \otimes 1 + (\sigma_1^* + \cdots + \sigma_s^*) \otimes 1 \otimes \Lambda_1, V(m_1)^* \otimes V(m_1 + \cdots + m_r - 1) + \cdots + V(n_s)^* \otimes V(n_1 + \cdots + n_s - 1))$ is a completely Q-reducible PV.

2. Simple Q-irreducible and Completely Q-reducible PV's

In [K3, p93–p97], the list of the simple regular PV's is given. We shall pick up the Q-irreducible PV's and completely Q-reducible PV's among these 21 regular PV's.

Theorem 2.1. *The Q-irreducible non-irreducible simple PV's are given in the following list. The number of basic relative invariants is denoted by N.*

1. $(GL(1)^2 \times SL(n), \Lambda_1 + \Lambda_1^*, V(n) + V(n)^*)$, $(n \geq 2)$, $N = 1$.
2. $(GL(1)^n \times SL(n), \overbrace{\Lambda_1 + \cdots + \Lambda_1}^{n}, \overbrace{V(n) + \cdots + V(n)}^{n})$, $(n \geq 2)$, $N = 1$.
3. $(GL(1)^2 \times SL(2m+1), \Lambda_2 + \Lambda_1, V(m(2m+1)) + V(2m+1))$, $(m \geq 1)$, $N = 1$.
4. $(GL(1)^2 \times Spin(10)$, *the even half-spin rep.* + *the even half-spin rep.*, $V(16) + V(16))$, $N = 1$.
5. $(GL(1)^2 \times Sp(n), \Lambda_1 + \Lambda_1, V(2n) + V(2n))$, $N = 1$.

Theorem 2.2. *The completely Q-reducible simple PV's are given in the following list. The number of basic relative invariants is denoted by N.*

1. $(GL(1)^3 \times SL(2m), \Lambda_2 + \Lambda_1 + \Lambda_1^*, V(m(2m-1)) + V(2m) + V(2m)^*)$, $N = 2$.
2. $(GL(1)^2 \times Spin(8)$, *the vector rep.* + *a half-spin rep.*, $V(8) + V(8))$, $N = 2$.
3. $(GL(1)^2 \times Spin(7)$, *the even half-spin rep.* + *the even half-spin rep.*, $V(16) + V(16))$, $N = 2$.

4. $(GL(1)^2 \times Spin(12)$, *the vector rep.* + *a half-spin rep.*, $V(12) + V(32))$, $N = 2$.

3. 2-simple Q-irreducible and Completely Q-reducible PV's of Type I

The definition of the 2-simple PV's of type I is given in [KKIY, p369]. In [KKIY, p395–p398], the list of the 2-simple regular PV's of type I is given. We shall pick up the Q-irreducible PV's and completely Q-reducible PV's among these 46 2-simple regular PV's of type I.

THEOREM 3.1. *The Q-irreducible non-irreducible* 2*-simple PV's of type I are given in the following list. The number of basic relative invariants is denoted by N.*

1. $(GL(1)^3 \times SL(5) \times SL(2),\ \Lambda_2 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1^*) \otimes 1,\ V(10) \otimes V(2) + (V(5)^* + V(5)^*) \otimes V(1))$, $N = 3$.
2. $(GL(1)^2 \times SL(5) \times SL(8),\ \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*,\ V(10) \otimes V(8) + V(1) \otimes V(8)^*)$, $N = 2$.
3. $(GL(1)^2 \times SL(5) \times SL(9),\ \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*,\ V(10) \otimes V(9) + V(1) \otimes V(9)^*)$, $N = 1$.
4. $(GL(1)^2 \times Sp(n) \times SL(2m+1), \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1, V(2n) \otimes V(2m+1) + V(2n) \otimes V(1))$, $N = 1$.
5. $(GL(1)^2 \times Spin(10) \times SL(15)$, *a half-spin rep.* $\otimes \Lambda_1 + 1 \otimes \Lambda_1^*$, $V(16) \otimes V(15) + V(1) \otimes V(15)^*)$, $N = 1$.

LEMMA 3.2. (10) $(GL(1)^3 \times Sp(n) \times SL(2m), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1^{(*)} + \Lambda_1^{(*)}))$ *in the list of* [KKIY] *is not Q-irreducible nor completely Q-reducible if and only if* $m \geq 2$ *and* $1 \otimes (\Lambda_1^{(*)} + \Lambda_1^{(*)}) = 1 \otimes (\Lambda_1 + \Lambda_1)^{(*)}$.

PROOF. Since $(GL(1) \times Sp(n) \times SL(2m), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$ is a regular PV and $(GL(2m), (\Lambda_1 + \Lambda_1)^*)$ with $m \geq 2$ is a non-regular PV, we have our result in this case. Note that if $m = 1$, we have $\Lambda_1 = \Lambda_1^*$ for $SL(2)$, and hence $(GL(2), (\Lambda_1 + \Lambda_1)^*)$ is a regular PV. ■

THEOREM 3.3. *The completely Q-reducible* 2*-simple PV's of type I are given in the following list. The number of basic relative invariants is denoted by N.*

1. $(GL(1)^3 \times Sp(n) \times SL(2m), \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1^*), V(2n) \otimes V(2m) + V(1) \otimes (V(2m) + V(2m)^*))$, $N = 2$.
2. $(GL(1)^2 \times Sp(n) \times SL(2), \Lambda_1 \otimes \Lambda_1 + 1 \otimes 2\Lambda_1, V(2n) \otimes V(2) + V(1) \otimes V(3))$, $N = 2$.

3. $(GL(1)^2 \times Sp(n) \times SL(2), \Lambda_1 \otimes \Lambda_1 + 1 \otimes 3\Lambda_1, V(2n) \otimes V(2) + V(1) \otimes V(4)),\ N = 2.$
4. $(GL(1)^2 \times Spin(7) \times SL(2),$ *the vector rep.* $\otimes\Lambda_1 +$ *the spin rep.* $\otimes 1,$ $V(7) \otimes V(2) + V(8) \otimes V(1)),\ N = 2.$
5. $(GL(1)^2 \times Spin(8) \times SL(2),$ *the vector rep.* $\otimes\Lambda_1 +$ *a half-spin rep.* $\otimes 1,$ $V(8) \otimes V(2) + V(8) \otimes V(1)),\ N = 2.$
6. $(GL(1)^2 \times Spin(8) \times SL(3),$ *the vector rep.* $\otimes\Lambda_1 +$ *a half-spin rep.* $\otimes 1,$ $V(8) \otimes V(3) + V(8) \otimes V(1)),\ N = 2.$
7. $(GL(1)^2 \times Spin(10) \times SL(2),$ *a half-spin rep.* $\otimes\Lambda_1 + 1 \otimes 2\Lambda_1,\ V(16) \otimes V(2) + V(1) \otimes V(3)),\ N = 2.$
8. $(GL(1)^2 \times Spin(10) \times SL(2),$ *a half-spin rep.* $\otimes\Lambda_1 + 1 \otimes 3\Lambda_1,\ V(16) \otimes V(2) + V(1) \otimes V(4)),\ N = 2.$
9. $(GL(1)^3 \times Spin(10) \times SL(2),$ *a half-spin rep.* $\otimes\Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1),$ $V(16) \otimes V(2) + V(1) \otimes (V(2) + V(2))),\ N = 2.$

4. 3-simple Q-irreducible and Completely Q-reducible PV's of Nontrivial Type

The definition of the 3-simple PV's of nontrivial type is given in [KUY, p159]. In [KUY, p187–p190], the list of the 3-simple regular PV's of nontrivial type is given. We shall pick up the Q-irreducible PV's and completely Q-reducible PV's among these 67 3-simple regular PV's of nontrivial type.

Lemma 4.1. (12) $(GL(1)^4 \times Spin(10) \times SL(2) \times SL(5),$ *a half-spin rep.* $\otimes\Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_2 + 1 \otimes 1 \otimes (\Lambda_1^* + \Lambda_1^{(*)}))$ *in the list of* [KUY] *is completely Q-reducible if and only if* $\Lambda_1^{(*)} = \Lambda_1^*$.

Proof. First note that $(Spin(10) \times GL(2),$ *a half-spin rep.* $\otimes\Lambda_1)$ is a regular PV. If $\Lambda_1^{(*)} = \Lambda_1^*$, by 1 of Theorem 3.1, we have our result. If $\Lambda_1^{(*)} = \Lambda_1$, then $(GL(1)^2 \times SL(5), \Lambda_1^* + \Lambda_1)$ is a regular PV and $(GL(2) \times SL(5), \Lambda_1 \otimes \Lambda_2)$ is a non-regular PV, and hence it is not Q-irreducible nor completely Q-reducible. ■

Lemma 4.2. (67) $(GL(1)^4 \times Sp(n) \times SL(2) \times SL(2n-1), \Lambda_1 \otimes \Lambda_1 \otimes 1 + \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^* + 1 \otimes \Lambda_1 \otimes 1)$ *in the list of* [KUY] *is not Q-irreducible nor completely Q-reducible.*

Proof. Since $(Sp(n) \times GL(2), \Lambda_1 \otimes \Lambda_1)$ is a regular PV, it is enough to show that $(GL(1)^3 \times Sp(n) \times SL(2) \times SL(2n-1), \Lambda_1 \otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes \Lambda_1^* +$

$1 \otimes \Lambda_1 \otimes 1)$ is a non-regular PV. However the $SL(2)$-part of its generic isotropy subgroup is, clearly, not reductive, and hence it is not a regular PV. ∎

THEOREM 4.3. *The Q-irreducible* 3*-simple PV's of nontrivial type are irreducible PV's which are given in the following list. The number of basic relative invariants is denoted by N.*

1. $(GL(1) \times SL(2) \times SL(2) \times SL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(2) \otimes V(2) \otimes V(2))$, $N = 1$.
2. $(GL(1) \times SL(3) \times SL(3) \times SL(2),\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(3) \otimes V(3) \otimes V(2))$, $N = 1$.

THEOREM 4.4. *The completely Q-reducible* 3*-simple PV's of nontrivial type are given in the following list. The number of basic relative invariants is denoted by N.*

1. $(GL(1)^2 \times Spin(10) \times SL(2) \times Spin(10)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes$ *a half-spin rep.*, $V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(16))$, $N = 2$.
2. $(GL(1)^3 \times Spin(10) \times SL(2) \times Spin(10)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes$ *a half-spin rep.* $+1 \otimes 2\Lambda_1 \otimes 1,\ V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(16) + V(1) \otimes V(3) \otimes V(1))$, $N = 3$.
3. $(GL(1)^3 \times Spin(10) \times SL(2) \times Spin(10)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes$ *a half-spin rep.* $+1 \otimes 3\Lambda_1 \otimes 1,\ V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(16) + V(1) \otimes V(4) \otimes V(1))$, $N = 3$.
4. $(GL(1)^4 \times Spin(10) \times SL(2) \times Spin(10)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes$ *a half-spin rep.* $+1 \otimes (\Lambda_1 + \Lambda_1) \otimes 1,\ V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(16) + V(1) \otimes (V(2) + V(2)) \otimes V(1))$, $N = 3$.
5. $(GL(1)^4 \times Spin(10) \times SL(2) \times SL(5)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_2 + 1 \otimes 1 \otimes (\Lambda_1^* + \Lambda_1^*),\ V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(10) + V(1) \otimes V(1) \otimes (V(5)^* + V(5)^*))$, $N = 4$.
6. $(GL(1)^2 \times Spin(10) \times SL(2) \times Sp(n)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1, V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(2n))$, $N = 2$.
7. $(GL(1)^3 \times Spin(10) \times SL(2) \times Sp(n)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 2\Lambda_1 \otimes 1,\ V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(2n) + V(1) \otimes V(3) \otimes V(1))$, $N = 3$.
8. $(GL(1)^3 \times Spin(10) \times SL(2) \times Sp(n)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 3\Lambda_1 \otimes 1,\ V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(2n) + V(1) \otimes V(4) \otimes V(1))$, $N = 3$.

9. $(GL(1)^4 \times Spin(10) \times SL(2) \times Sp(n)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1) \otimes 1$, $V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(2n) + V(1) \otimes (V(2) + V(2)) \otimes V(1))$, $N = 3$.
10. $(GL(1)^2 \times Spin(10) \times SL(2) \times SO(n)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1$, $V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(n))$, $N = 2$.
11. $(GL(1)^3 \times Spin(10) \times SL(2) \times Spin(7)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes$ *the vector rep.* $+1 \otimes 1 \otimes$ *the spin rep.*, $V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(7) + V(1) \otimes V(1) \otimes V(8))$, $N = 3$.
12. $(GL(1)^3 \times Spin(10) \times SL(2) \times Spin(8)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes$ *the vector rep.* $+1 \otimes 1 \otimes$ *a half-spin rep.*, $V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(8) + V(1) \otimes V(1) \otimes V(8))$, $N = 3$.
13. $(GL(1)^2 \times Spin(10) \times SL(2) \times Spin(7)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes$ *the spin rep.*, $V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(8))$, $N = 2$.
14. $(GL(1)^2 \times Spin(10) \times SL(2) \times (G_2)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_2$, $V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(7))$, $N = 2$.
15. $(GL(1)^2 \times Spin(10) \times SL(2) \times SL(2)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes 2\Lambda_1 \otimes \Lambda_1$, $V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(3) \otimes V(2))$, $N = 2$.
16. $(GL(1)^2 \times Spin(10) \times SL(2) \times SL(6)$, *a half-spin rep.* $\otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_2$, $V(16) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(15))$, $N = 2$.
17. $(GL(1)^4 \times SL(5) \times SL(2) \times Sp(n), \Lambda_2 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 + (\Lambda_1^* + \Lambda_1^*) \otimes 1 \otimes 1, V(10) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(2n) + (V(5)^* + V(5)^*) \otimes V(1) \otimes V(1))$, $N = 4$.
18. $(GL(1)^2 \times Sp(n) \times SL(2) \times Sp(m), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(2m))$, $N = 2$.
19. $(GL(1)^3 \times Sp(n) \times SL(2) \times Sp(m), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 2\Lambda_1 \otimes 1, V(2n) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(2m) + V(1) \otimes V(3) \otimes V(1))$, $N = 3$.
20. $(GL(1)^3 \times Sp(n) \times SL(2) \times Sp(m), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes 3\Lambda_1 \otimes 1, V(2n) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(2m) + V(1) \otimes V(4) \otimes V(1))$, $N = 3$.
21. $(GL(1)^4 \times Sp(n) \times SL(2) \times Sp(m), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1 + 1 \otimes (\Lambda_1 + \Lambda_1) \otimes 1, V(2n) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(2m) + V(1) \otimes (V(2) + V(2)) \otimes V(1))$, $N = 3$.
22. $(GL(1)^2 \times Sp(n) \times SL(2) \times SO(m), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(m))$, $N = 2$.
23. $(GL(1)^3 \times Sp(n) \times SL(2) \times Spin(7)$, $\Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes$ *the vector rep.* $+1 \otimes 1 \otimes$ *the spin rep.*, $V(2n) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(7) + V(1) \otimes V(1) \otimes V(8))$, $N = 3$.

24. $(GL(1)^3 \times Sp(n) \times SL(2) \times Spin(8),\ \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes$ *the vector rep.* $+1 \otimes 1 \otimes$ *a half-spin rep.*, $V(2n) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(8) + V(1) \otimes V(1) \otimes V(8)),\ N = 3$.
25. $(GL(1)^2 \times Sp(n) \times SL(2) \times Spin(7),\ \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes$ *the spin rep.*, $V(2n) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(8)),\ N = 2$.
26. $(GL(1)^2 \times Sp(n) \times SL(2) \times (G_2), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_2, V(2n) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(7)),\ N = 2$.
27. $(GL(1)^2 \times Sp(n) \times SL(2) \times SL(2), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes 2\Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2) \otimes V(1) + V(1) \otimes V(3) \otimes V(2)),\ N = 2$.
28. $(GL(1)^2 \times Sp(n) \times SL(2) \times SL(6), \Lambda_1 \otimes \Lambda_1 \otimes 1 + 1 \otimes \Lambda_1 \otimes \Lambda_2, V(2n) \otimes V(2) \otimes V(1) + V(1) \otimes V(2) \otimes V(15)),\ N = 2$.

5. M. Sato's Classification

In the 1960s, Professor Mikio Sato considered the reductive PV's of the form $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ with a connected semisimple subgroup G_0 of $SL(n)$. Here $\rho : G \to GL(V)$ is a d-dimensional representation of a connected reductive algebraic group G. Then we have $\rho = \rho_1 + \cdots + \rho_m$ and $V = V_1 + \cdots + V_m$ where $\rho_\mu : G \to GL(V_\mu)$ is an irreducible representation $(1 \leq \mu \leq m)$. For each μ, we have $V_\mu = V_{\mu 1} \otimes \cdots \otimes V_{\mu k_\mu}$ where some simple component of G acts on $V_{\mu\nu}$ irreducibly. Put $d_\mu = \dim V_\mu$ and $d_{\mu\nu} = \dim V_{\mu\nu}$. Then we have $d = d_1 + \cdots + d_m$ and $d_\mu = d_{\mu 1} \cdots d_{\mu k_\mu}$. Here if $d_\mu = 1$, we put $k_\mu = 0$. If $d_\mu \geq 2$, we have $k_\mu \geq 1$ and we may assume $d_{\mu\nu} \geq 2$ $(1 \leq \nu \leq k_\mu)$. Now put $\delta = \max\{d_{\mu\nu}\}$. We may assume that $\delta = d_{11}$ by changing the numbers if necessary. Then $k_1 = 0$ implies that $\delta = 1$.

Professor Mikio Sato proved that if $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ is a PV for $\delta \leq n \leq d - \delta$, then k_1 must be one of 0, 1, 2, and classified such PV's when $k_1 = 2$ as follows.

Proposition 5.1 (M. Sato). *Assume that* $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ *is a PV with* $\delta \leq n \leq d - \delta$ *and* $k_1 = 2$. *Then it is one of the following regular PV's.*

1. $(SL(n) \times ((GL(2) \times SL(2)) \times \overbrace{GL(2) \times \cdots \times GL(2)}^{m-1}),\ \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-1}), V(n) \otimes (V(2) \otimes V(2) + \overbrace{V(2) + \cdots + V(2)}^{m-1}))$ *with* $m \geq 1$ *and* $n = 2$ *or* $n = 2m (= d - 2)$.
2. $(SL(n) \times ((GL(3) \times SL(2)) \times \overbrace{GL(3) \times \cdots \times GL(3)}^{m-1}),\ \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-1}), V(n) \otimes (V(3) \otimes V(2) + \overbrace{V(3) + \cdots + V(3)}^{m-1}))$ *with* $m \geq 1$ *and* $n = 3$ *or* $n = 3m (= d - 3)$.

3. $(SL(3) \times ((GL(2) \times SL(2)) \times GL(2)), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1), V(3) \otimes (V(2) \otimes V(2) + V(2)))$.
4. $(SL(n) \times ((GL(3) \times SL(2)) \times GL(3)), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1), V(n) \otimes (V(3) \otimes V(2) + V(3)))$ $(n = 4, 5)$.
5. $(SL(n) \times ((GL(3) \times SL(2)) \times GL(k) \times \overbrace{GL(3) \times \cdots \times GL(3)}^{m-2}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1 \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-2}), V(n) \otimes (V(3) \otimes V(2) + V(k) + \overbrace{V(3) + \cdots + V(3)}^{m-2}))$ *with* $m \geq 2$; $n = 3$ *or* $n = k + 3m - 3 (= d - 3)$; $k = 1$ *or* 2.
6. $(SL(n) \times ((GL(2) \times SL(2)) \times GL(1) \times \overbrace{GL(2) \times \cdots \times GL(2)}^{m-2}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1 \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-2}), V(n) \otimes (V(2) \otimes V(2) + V(1) + \overbrace{V(2) + \cdots + V(2)}^{m-2}))$ *with* $m \geq 2$; *and* $n = 2$ *or* $n = 2m - 1 (= d - 2)$.

PROOF. See p. 239 in [K1]. ■

We shall pick up the Q-irreducible PV's and completely Q-reducible PV's among the list of Proposition 5.1.

THEOREM 5.2. *The Q-irreducible PV's among the list of Proposition* 5.1 *are given as follows. The number of basic relative invariants is denoted by N.*

1. $(SL(2) \times (GL(2) \times SL(2)), \Lambda_1 \otimes (\Lambda_1 \otimes \Lambda_1), V(2) \otimes (V(2) \otimes V(2)))$, $N = 1$.
2. $(SL(3) \times (GL(3) \times SL(2)), \Lambda_1 \otimes (\Lambda_1 \otimes \Lambda_1), V(3) \otimes (V(3) \otimes V(2)))$, $N = 1$.
3. $(SL(5) \times ((GL(3) \times SL(2)) \times GL(3)), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1), V(5) \otimes (V(3) \otimes V(2) + V(3)))$, $N = 2$.
4. $(SL(5) \times ((GL(3) \times SL(2)) \times GL(2)), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1), V(5) \otimes (V(3) \otimes V(2) + V(2)))$, $N = 2$.

THEOREM 5.3. *The completely Q-reducible PV's among the list of Proposition* 5.1 *are given as follows. The number of basic relative invariants is denoted by N.*

1. $(SL(2) \times ((GL(2) \times SL(2)) \times \overbrace{GL(2) \times \cdots \times GL(2)}^{m-1}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-1}), V(n) \otimes (V(2) \otimes V(2) + \overbrace{V(2) + \cdots + V(2)}^{m-1}))$ *with* $m \geq 2$, $N = m$.

2. $(SL(3) \times ((GL(3) \times SL(2)) \times \overbrace{GL(3) \times \cdots \times GL(3)}^{m-1}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-1}), V(n) \otimes (V(3) \otimes V(2) + \overbrace{V(3) + \cdots + V(3)}^{m-1}))$ *with* $m \geq 2$, $N = m$.

The remaining case $k_1 = 1$ implies that $(G, \rho_1, V_1) = ((GL(1) \times) G_s, (\Lambda_1 \otimes)\sigma, V(\delta))$ where G_s is a simple algebraic group.

In [KIRHKOK], the complete classification of these PV's when G_s is an exceptional simple algebraic group is given as follows.

Proposition 5.4. *Assume that* $(G_0 \times G, \Lambda_1 \otimes \rho, V(n) \otimes V)$ *with* $k_1 = 1$ *is a PV with* $(G, \rho_1, V_1) = ((GL(1) \times) G_s, (\Lambda_1 \otimes)\sigma, V(\delta))$ *where* G_s *is an exceptional simple algebraic group. Then it is one of the following regular PV's.*

1. $(SL(n) \times ((GL(1) \times G_s) \times \overbrace{GL(\delta) \times \cdots \times GL(\delta)}^{m-1}), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-1}), V(n) \otimes (V(\delta) + \overbrace{V(\delta) + \cdots + V(\delta)}^{m-1}))$ *with* $m \geq 2$; $n = \delta$ *or* $n = (m-1)\delta$ *where* σ *is any irreducible representation of* G_s *with* $\deg \sigma = \delta$.
2. $(SL(n) \times ((GL(1) \times (G_2)) \times GL(t) \times \overbrace{GL(7) \times \cdots \times GL(7)}^{m-2}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_2) \boxplus \Lambda_1 \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-2}), V(n) \otimes (V(7) + V(t) + \overbrace{V(7) + \cdots + V(7)}^{m-2}))$ $(m \geq 3)$ *with* $t = 1, 2, 5, 6$ *where* $n = 7$ *or* $n = t + 7(m-2)$.
3. $(SL(n) \times ((GL(1) \times E_6) \times GL(t) \times \overbrace{GL(27) \times \cdots \times GL(27)}^{m-2}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1 \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-2}), V(n) \otimes (V(27) + V(t) + \overbrace{V(27) + \cdots + V(27)}^{m-2}))$ $(m \geq 3)$ *with* $t = 1, 2, 25, 26$ *where* $n = 27$ *or* $n = t + 27(m-2)$.
4. $(SL(n) \times ((GL(1) \times E_7) \times GL(t) \times \overbrace{GL(56) \times \cdots \times GL(56)}^{m-2}), \Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_6) \boxplus \Lambda_1 \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-2}), V(n) \otimes (V(56) + V(t) + \overbrace{V(56) + \cdots + V(56)}^{m-2}))$ $(m \geq 3)$ *with* $t = 1, 55$ *where* $n = 56$ *or* $n = t + 56(m-2)$.

Theorem 5.5. *There is no Q-irreducible PV's among the list of Proposition* 5.4.

Theorem 5.6. *The completely Q-reducible PV's among the list of Proposition* 5.4 *are given as follows. The number of basic relative invariants is denoted by N.*

1. $(SL(n) \times ((GL(1) \times G_s) \times \overbrace{GL(n) \times \cdots \times GL(n)}^{m-1}),\ \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-1}), V(n) \otimes (V(n) + \overbrace{V(n) + \cdots + V(n)}^{m-1}))$ *with* $m \geq 2$; σ *is any irreducible representation of* G_s *with* $\deg \sigma = n$, $N = m$.

The regular PV's of the extreme case $n = \delta$ or $n = d - \delta$ when G_s is a classical simple algebraic group with $G_s \neq SL(\delta)$ is also given as follows in [KIRHKOK].

Proposition 5.7. *Assume that* $(G_0 \times G, \Lambda_1 \otimes (\rho_1 + \cdots + \rho_m), V(n) \otimes (V(\delta) + V(d_2) + \cdots + V(d_m))$ *with* $n = \delta$ *or* $n = d - \delta = d_2 + \cdots + d_m$ *is a regular PV where* $(G, \rho_1, V(\delta)) = (GL(1) \times G_s, \Lambda_1 \otimes \sigma, V(\delta))(\neq (GL(\delta), \Lambda_1, V(\delta)))$ *with a classical simple algebraic group* G_s *and each* $V(d_\mu)$ *has an independent scalar multiplication. Then it is one of the following PV's.*

1. $(SL(n) \times ((GL(1) \times G_s) \times \overbrace{GL(\delta) \times \cdots \times GL(\delta)}^{m-1}),\ \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-1}), V(n) \otimes (V(\delta) + \overbrace{V(\delta) + \cdots + V(\delta)}^{m-1}))$ *with* $m \geq 2$; $n = \delta$ *or* $n = (m-1)\delta$ *where* σ *is any irreducible representation of* G_s *with* $\deg \sigma = \delta$.
2. $(SL(n) \times ((GL(1) \times G_s) \times T \times \overbrace{GL(\delta) \times \cdots \times GL(\delta)}^{m-2}), \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \tau \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-2}), V(n) \otimes (V(\delta) + V(t) + \overbrace{V(\delta) + \cdots + V(\delta)}^{m-2}))$ *with* $m \geq 3$, $\delta > t \geq 1$; $n = \delta$ *or* $n = t + (m-2)\delta$ *where* $(G_s \times T, \sigma \otimes \tau, V(\delta) \otimes V(t))$ *is a nontrivial irreducible regular* 2*-simple PV.*
3. $(SL(n) \times ((GL(1) \times Sp(t)) \times GL(u) \times GL(v) \times \overbrace{GL(2t) \times \cdots \times GL(2t)}^{m-3}),$ $\Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1 \boxplus \Lambda_1 \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-3}), V(n) \otimes (V(2t) + V(u) + V(v) + \overbrace{V(2t) + \cdots + V(2t)}^{m-3}))$ *with* $t \geq 2$; $n = 2t$ *or* $n = u + v + 2t(m-3)$ *where* $(u, v) = (1, 1)$ *or* $(1, k)$ *with* $m \geq 4$, *or* $(u, v) = (1, 2t-1)$, $(2t-1, 2t-1)$ *or* $(k, 2t-1)$ *with* $m \geq 3$. *Here* k *is an odd integer satisfying* $3 \leq k \leq 2t-3$.
4. $(SL(n) \times ((GL(1) \times Spin(10)) \times GL(u) \times GL(u) \times \overbrace{GL(16) \times \cdots \times GL(16)}^{m-3}),$ $\Lambda_1 \otimes ((\Lambda_1 \otimes$ *a half-spin rep.*$) \boxplus \Lambda_1 \boxplus \Lambda_1 \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-3}), V(n) \otimes (V(16) + V(u) + V(u) + \overbrace{V(16) + \cdots + V(16)}^{m-3}))$ *with* $n = 16$ *or* $n = 2u + 16(m-3)$ *where* $u = 1$ *and* $m \geq 4$, *or* $u = 15$ *and* $m \geq 3$.

5. $((SL(2t-1) \times SL(1)) \times ((GL(1) \times Sp(t)) \times \overbrace{GL(2t) \times \cdots \times GL(2t)}^{m-1}),$ $(\Lambda_1 \boxplus \Lambda_1) \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-1}), (V(2t-1) + V(1)) \otimes (V(2t) + \overbrace{V(2t) + \cdots + V(2t)}^{m-1}))$ *with* $t \geq 2$ *and* $m \geq 2$.

Theorem 5.8. *There is no Q-irreducible PV's among the list of Proposition* 5.7.

Theorem 5.9. *The completely Q-reducible PV's among the list of Proposition* 5.7 *are given as follows. The number of basic relative invariants is denoted by N.*

1. $(SL(n) \times ((GL(1) \times G_s) \times \overbrace{GL(n) \times \cdots \times GL(n)}^{m-1}),\ \Lambda_1 \otimes ((\Lambda_1 \otimes \sigma) \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-1}), V(n) \otimes (V(n) + \overbrace{V(n) + \cdots + V(n)}^{m-1}))$ *with* $m \geq 2$; σ *is any irreducible representation of* G_s *with* $\deg \sigma = n$, $N = m$.
2. $(SL(2t) \times ((GL(1) \times Sp(t)) \times GL(1) \times GL(2t-1) \times \overbrace{GL(2t) \times \cdots \times GL(2t)}^{m-3}),$ $\Lambda_1 \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \Lambda_1 \boxplus \Lambda_1 \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-3}), V(2t) \otimes (V(2t) + V(1) + V(2t-1) + \overbrace{V(2t) + \cdots + V(2t)}^{m-3}))$ *with* $t \geq 2$ *and* $m \geq 3$, $N = m - 1$.
3. $((SL(2t-1) \times SL(1)) \times ((GL(1) \times Sp(t)) \times \overbrace{GL(2t) \times \cdots \times GL(2t)}^{m-1}),$ $(\Lambda_1 \boxplus \Lambda_1) \otimes ((\Lambda_1 \otimes \Lambda_1) \boxplus \overbrace{\Lambda_1 \boxplus \cdots \boxplus \Lambda_1}^{m-1}), (V(2t-1) + V(1)) \otimes (V(2t) + \overbrace{V(2t) + \cdots + V(2t)}^{m-1}))$ *with* $t \geq 2$ *and* $m \geq 2$, $N = m$.

Proof. For 2, only we have to note that $(SL(2t) \times (GL(1) \times GL(2t-1)), \Lambda_1 \otimes (\Lambda_1 \boxplus \Lambda_1))$ is castling-equivalent to $(GL(1)^2 \times SL(2t), \Lambda_1 + \Lambda_1^*)$ for which $N = 1$ by 1 of Theorem 2.1

For 3, by the number of $GL(1)$ (see Introduction), we have $N \leq 1$ for $((SL(2t-1) \times SL(1)) \times ((GL(1) \times Sp(t)), (\Lambda_1 \boxplus \Lambda_1) \otimes ((\Lambda_1 \otimes \Lambda_1), M(2t))$ and $((SL(2t-1) \times SL(1)) \times (GL(2t), (\Lambda_1 \boxplus \Lambda_1) \otimes \Lambda_1, M(2t))$. Since $f(x) = \det x (x \in M(2t))$ is its non-generate relative invariant, these PV's are regular PV's with $N = 1$. ■

Acknowledgment

The author would like to express his hearty thanks to Professor Tatsuo Kimura for his enthusiastic supervising and helpful advices. The author dedicates this paper to him. The author also thanks to Dr. Masaya Ouchi for the valuable discussions about this topic.

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