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ON WEAKLY s-QUASINORMALLY EMBEDDED AND ss-QUASINORMAL SUBGROUPS OF FINITE GROUPS*

By

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Abstract. Suppose G is a finite group and H is a subgroup of G. H is called weakly s-quasinormally embedded in G if there are a subnormal subgroup T of G and an s-quasinormally embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$; H is called ss-quasinormal in G if there is a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B. We investigate the influence of weakly s-quasinormally embedded and ss-quasinormal subgroups on the structure of finite groups. Some recent results are generalized.

1. Introduction

All groups considered in this paper are finite. A subgroup H of a group G is said to be s-quasinormal in G if H permutes with every Sylow subgroups of G. This concept was introduced by Kegel in [1]. Morerecently, Ballester-Bolinches and Pedraza-Aguilera [2] generalized s-quasinormal subgroups to s-quasinormally embedded subgroups. H is said to be s-quasinormally embedded in a group G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-quasinormal subgroup of G. In recent years, it has been of interest to use supplementation properties of subgroups to characterize properties of a group. For example, Yanming Wang [3] introduced the concept of c-normal subgroup (a subgroup H of a group G is said to be c-normal in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the maximal normal subgroup of G contained in H). In 2009, Yangming Li [4]

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introduced the concept of weakly s-quasinormally embedded subgroup (a subgroup H of a group G is called weakly s-quasinormally embedded in G if there are a subnormal subgroup T of G and an s-quasinormally embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$). In 2008, Shirong Li [5] introduced the concept of ss-quasinormal subgroup (a subgroup H of a group G is said to be an ss-quasinormal subgroup of G if there is a subgroup B such that $G = HB$ and H permutes with every Sylow subgroup of B). There are examples to show that weakly s-quasinormally embedded subgroups are not ss-quasinormal subgroups and in general the converse is also false. The aim of this article is to unify and improve some earlier results using weakly s-quasinormally embedded and ss-quasinormal subgroups.

2. Preliminaries

Lemma 2.1 ([4], Lemma 2.5). *Let H be a weakly s-quasinormally embedded subgroup of a group G.*

(1) *If $H \leq L \leq G$, then H is weakly s-quasinormally embedded in L.*

(2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is weakly s-quasinormally embedded in G/N.*

(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-quasinormally embedded in G/N.*

(4) *Suppose H is a p-group for some prime p and H is not s-quasinormally embedded in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = HM$.*

Lemma 2.2 ([5], Lemma 2.1). *Let H be an ss-quasinormal subgroup of a group G.*

(1) *If $H \leq L \leq G$, then H is ss-quasinormal in L.*

(2) *If $N \trianglelefteq G$, then HN/N is ss-quasinormal in G/N.*

Lemma 2.3 ([5], Lemma 2.2). *Let H be a nilpotent subgroup of G. Then the following statements are equivalent*:

(1) *H is s-quasinormal in G.*

(2) *$H \leq F(G)$ and H is ss-quasinormal in G.*

(3) *$H \leq F(G)$ and H is s-quasinormally embedded in G.*

Lemma 2.4 ([15], Lemma 2.7). *Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.*

(1) *If* N *is normal in* G *of order* p, *then* $N \leq Z(G)$.

(2) *If* G *has cyclic Sylow* p-*subgroup, then* G *is* p-*nilpotent.*

(3) *If* $M \leq G$ *and* $|G : M| = p$, *then* $M \trianglelefteq G$.

Lemma 2.5 ([4], Theorem 4.7). *Let* P *be a Sylow* p-*subgroup of a group* G, *where* p *is a prime divisor of* $|G|$ *with* $(|G|, p-1) = 1$. *If every maximal subgroup of* P *is weakly* s-*quasinormally embedded in* G, *then* G *is* p-*nilpotent*.

Lemma 2.6 ([8], Lemma 2.3). *Let* G *be a group and* $N \leq G$.

(1) *If* $N \trianglelefteq G$, *then* $F^*(N) \leq F^*(G)$.

(2) *If* $G \neq 1$, *then* $F^*(G) \neq 1$. *In fact,* $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G))$.

(3) $F^*(F^*(G)) = F^*(G) \geq F(G)$. *If* $F^*(G)$ *is Solvable, then* $F^*(G) = F(G)$.

Lemma 2.7 ([13], Lemma 2.3). *Suppose that* H *is* s-*quasinormal in* G, P *a Sylow* p-*subgroup of* H, *where* p *is a prime. If* $H_G = 1$, *then* P *is* s-*quasinormal in* G.

Lemma 2.8 ([13], Lemma 2.2). *If* P *is an* s-*quasinormal* p-*subgroup of* G *for some prime* p, *then* $N_G(P) \geq O^p(G)$.

3. p-nilpotentcy

Theorem 3.1. *Let* P *be a Sylow* p-*subgroup of a group* G, *where* p *is a prime divisor of* $|G|$ *with* $(|G|, p-1) = 1$. *If every maximal subgroup of* P *is either weakly* s-*quasinormally embedded or* ss-*quasinormal in* G, *then* G *is* p-*nilpotent*.

Proof. Let H be a maximal subgroup of P. We will prove H is weakly s-quasinormally embedded in G.

If H is ss-quasinormal in G, then there is a subgroup $B \leq G$ such that $G = HB$ and $HX = XH$ for all $X \in \mathrm{Syl}(B)$. From $G = HB$, we obtain $|B : H \cap B|_p = |G : H|_p = p$, and hence $H \cap B$ is of index p in B_p, a Sylow p-subgroup of B containing $H \cap B$. Thus $S \not\leq H$ for all $S \in \mathrm{Syl}_p(B)$ and $HS = SH$ is a Sylow p-subgroup of G. In view of $|P : H| = p$ and by comparison of orders, $S \cap H = B \cap H$, for all $S \in \mathrm{Syl}_p(B)$. So $B \cap H = \bigcap_{b \in B} (S^b \cap H) =\leq \bigcap_{b \in B} S^b = O_p(B)$.

We claim that B has a Hall p'-subgroup. Because $|O_p(B) : B \cap H| = p$ or 1, it follows that $|B/O_p(B)|_p = p$ or 1. As $(|G|, p-1) = 1$, then $B/O_p(B)$ is

p-nilpotent by Lemma 2.4, and hence B is p-solvable. So B has a Hall p'-subgroup. Thus the claim holds.

Now, let K be a p'-subgroup of B, $\pi(K) = \{p_2, \dots, p_s\}$ and $P_i \in \mathrm{Syl}_{p_i}(K)$. By the condition, H permutes with every P_i and so H permutes with the subgroup $\langle P_2, \dots, P_s \rangle = K$. Thus $HK \leq G$. Obviously, K is a Hall p'-subgroup of G and HK is a subgroup of index p in G. Let $M = HK$ and so $M \trianglelefteq G$ by Lemma 2.4. It follows that H is s-quasinormally embedded, and so weakly s-quasinormally embedded in G.

Since every maximal subgroup of P is weakly s-quasinormally embedded in G, we have G is p-nilpotent by Lemma 2.5.

Corollary 3.2. *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and H a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is either weakly s-quasinormally embedded or ss-quasinormal in G, then G is p-nilpotent.*

Proof. By Lemmas 2.1 and 2.2, every maximal subgroup of P is either weakly s-quasinormally embedded or ss-quasinormal in H. By Theorem 3.1, H is p-nilpotent. Now, let $H_{p'}$ be the normal p-complement of H. Then $H_{p'} \lhd G$. If $H_{p'} \neq 1$, then we consider $G/H_{p'}$. It is easy to see that $G/H_{p'}$ satisfies all the hypotheses of our Corollary for the normal subgroup $H/H_{p'}$ of $G/H_{p'}$ by Lemmas 2.1 and 2.2. Now by induction, we see that $G/H_{p'}$ is p-nilpotent and so G is p-nilpotent. Hence we assume $H_{p'} = 1$ and therefore $H = P$ is a p-group. Since G/H is p-nilpotent, let K/H be the normal p-complement of G/H. By Schur-Zassenhaus's theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = HK_{p'}$. By Theorem 3.1, K is p-nilpotent and so $K = H \times K_{p'}$. Hence $K_{p'}$ is a normal p-complement of G. This completes the proof.

Corollary 3.3. *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P is either weakly s-quasinormally embedded or ss-quasinormal in G, then G is p-nilpotent.*

Proof. It is clear that $(|G|, p-1) = 1$ if p is the smallest prime dividing the order of G and therefore Corollary 3.3 follows immediately from Theorem 3.1.

Corollary 3.4. *Suppose that every maximal subgroup of any Sylow subgroup of a group G is either weakly s-quasinormally embedded or ss-quasinormal in G, then G is a Sylow tower group of supersolvable type.*

Proof. Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. By Corollary 3.3, G is p-nilpotent. Let U be the normal p-complement of G. By Lemmas 2.1 and 2.2, U satisfies the hypothesis of the Corollary. It follows by induction that U, and hence G is a Sylow tower group of supersolvable type.

Corollary 3.5 ([6], Theorem 3.1). *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is either c-normal or s-quasinormally embedded in G, then G is p-nilpotent.*

Corollary 3.6 ([9], Theorem 3.1). *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P is either c-normal or ss-quasinormal in G, then G is p-nilpotent.*

Theorem 3.7. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is either weakly s-quasinormally embedded or ss-quasinormal in G, then G is p-nilpotent.*

Proof. It is easy to see that the theorem holds when $p = 2$ by Corollary 3.3, so it suffices to prove the theorem for the case when p is odd. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, we consider $G/O_{p'}(G)$. By Lemmas 2.1 and 2.2, it is easy to see that every maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$ is either weakly s-quasinormally embedded or ss-quasinormal in $G/O_{p'}(G)$. Since

$$N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$$

is p-nilpotent, $G/O_{p'}(G)$ satisfies all the hypotheses of our theorem. The minimality of G yields that $G/O_{p'}(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction.

(2) If M is a proper subgroup of G with $P \leq M < G$, then M is p-nilpotent.

It is clear to see $N_M(P) \leq N_G(P)$ and hence $N_M(P)$ is p-nilpotent. Applying Lemmas 2.1 and 2.2, we immediately see that M satisfies the hypotheses of our theorem. Now, by the minimality of G, M is p-nilpotent.

(3) $G = PQ$ is solvable, where Q is a Sylow q-subgroup of G with $p \neq q$.

Since G is not p-nilpotent, by a result of Thompson [11, Corollary], there exists a non-trivial characteristic subgroup T of P such that $N_G(T)$ is not p-nilpotent. Choose T such that the order of T is as large as possible. Since $N_G(P)$ is p-nilpotent, we have $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P satisfying $T < K \leq P$. Now, $T \operatorname{char} P \trianglelefteq N_G(P)$, which gives $T \trianglelefteq N_G(P)$. So $N_G(P) \leq N_G(T)$. By (2), we get $N_G(T) = G$ and $T = O_P(G)$. Now, applying the result of Thompson again, we have that $G/O_p(G)$ is p-nilpotent and therefore G is p-solvable. Then for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup of Q such that PQ is a subgroup of G [12, Theorem 6.3.5]. If $PQ < G$, then PQ is p-nilpotent by (2), contrary to the choice of G. Consequently, $PQ = G$, as desired.

(4) G has a unique minimal normal subgroup N such that G/N is p-nilpotent. Moreover $\Phi(G) = 1$.

By (3), G is solvable. Let N be a minimal subgroup of G. Then $N \leq O_p(G)$ by (1). Consider G/N. It is easy to see that every maximal subgroup of P/N is either weakly s-quasinormally embedded or ss-quasinormal in G/N. Since $N_{G/N}(P/N) = N_G(P)/N$ is p-nilpotent, we have G/N satisfies the hypothesis of the theorem. The choice of G yields that G/N is p-nilpotent. Consequently the uniqueness of N and the fact that $\Phi(G) = 1$ are obvious.

(5) The final contradiction.

By step (4), there exists a maximal subgroup M of G such that $G = MN$ and $M \cap N = 1$. Since N is elementary abelian p-group, $N \leq C_G(N)$ and $C_G(N) \cap M \trianglelefteq G$. By the uniqueness of N, we have $C_G(N) \cap M = 1$ and $N = C_G(N)$. But $N \leq O_p(G) \leq F(G) \leq C_G(N)$, hence $N = O_p(G) = C_G(N)$. If $|N| = p$, then $\mathrm{Aut}(N)$ is a cyclic group of order $p - 1$. If $q > p$, then NQ is p-nilpotent and therefore $Q \leq C_G(N) = N$, a contradiction. On the other hand, if $q < p$, then, since $N = C_G(N)$, we see that $M \cong G/N = N_G(N)/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$ and therefore M, and in particular Q, is cyclic. Since Q is a cyclic group and $q < p$, we know that G is q-nilpotent and therefore P is normal in G. Hence $N_G(P) = G$ is p-nilpotent, a contradiction. So we may assume N is not a cyclic subgroup of order p. Obviously $P = P \cap NM = N(P \cap M)$. Since $P \cap M < P$, we take a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. By our hypotheses, P_1 is either weakly s-quasinormally embedded or ss-quasinormal in G. If P_1 is weakly s-quasinormally embedded, then there are a subnormal subgroup T of G and an s-quasinormally embedded subgroup $(P_1)se$ of G contained in P_1 such that $G = P_1T$ and $P_1 \cap T \leq (P_1)se$. So there is an s-quasinormal subgroup K of G such that $(P_1)se$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then

$N \le K_G \le K$. It follows that $N \le (P_1)_{se} \le P_1$, and so $P = N(P \cap M) = NP_1 = P_1$, a contradiction. If $K_G = 1$, by Lemma 2.7, $(P_1)_{se}$ is s-quasinormal in G. From Lemma 2.8 we have $O^p(G) \le N_G((P_1)_{se})$. Since $(P_1)_{se}$ is subnormal in G, $P_1 \cap T \le (P_1)_{se} \le O_p(G) = N$. Thus, $(P_1)_{se} \le P_1 \cap N$ and $(P_1)_{se} \le ((P_1)_{se})^G = ((P_1)_{se})^{O^p(G)P} = ((P_1)_{se})^P \le (P_1 \cap N)^P = P_1 \cap N \le N$. It follows that $((P_1)_{se})^G = 1$ or $((P_1)_{se})^G = P_1 \cap N = N$. If $((P_1)_{se})^G = P_1 \cap N = N$, then $N \le P_1$ and so $P = P_1$, a contradiction. So we may assume $((P_1)_{se})^G = 1$. Then $P_1 \cap T = 1$. Since $|G : T|$ is a number of p-power and $T \lhd\lhd G$, $O^p(G) \le T$. From the fact that N is the unique minimal normal subgroup of G, we have $N \le O^p(G) \le T$. Hence $N \cap P_1 \le T \cap P_1 = 1$. Since $|N : P_1 \cap N| = |NP_1 : P_1| = |P : P_1| = p$, $P_1 \cap N$ is a maximal of N. Therefore $|N| = p$, a contradiction. Now we assume P_1 is ss-quasinormal in G. By [5, Lemma 2.5], $P_1 Q$ is a subgroup of G. As $N \trianglelefteq G$, we have $P_1 \cap N = N \cap P_1 Q \trianglelefteq P_1 Q$, and it follows that $P_1 \cap N \trianglelefteq \langle P_1 Q, N \rangle = G$. Moreover, since N is a minimal normal subgroup of G, we have $P_1 \cap N = 1$ and N is a cyclic subgroup of order p, a contradiction.

Corollary 3.8. *Let p be a prime dividing the order of a group G and H a normal subgroup of G such that G/H is p-nilpotent. If $N_G(P)$ is p-nilpotent and there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is either weakly s-quasinormally embedded or ss-quasinormal in G, then G is p-nilpotent.*

Proof. By Theorem 3.7, H is p-nilpotent. If N is a normal Hall p'-subgroup of H, then N is normal in G. By the using the arguments as in the proof of Corollary 3.2, we may assume $N = 1$ and $H = P$. In the case, by our hypotheses, $N_G(P) = G$ is p-nilpotent.

Corollary 3.9 ([13], Theorem 3.2). *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is s-quasinormally embedded in G, then G is p-nilpotent.*

Corollary 3.10 ([14], Theorem 3.1). *Let P be a Sylow p-subgroup of a group G, where p is an odd prime divisor of $|G|$. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

4. Supersolvability

Theorem 4.1. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersoluble groups. A group $G \in \mathscr{F}$ if and only if there is a normal subgroup H*

of G such that $G/H \in \mathscr{F}$ and every maximal subgroup of any Sylow subgroup of H is either weakly s-quasinormally embedded or ss-quasinormal in G.

PROOF. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order.

By Lemmas 2.1 and 2.2, every maximal subgroup of any Sylow subgroup of H is either weakly s-quasinormally embedded or ss-quasinormal in H. By Corollary 3.4, H is a Sylow tower group of supersolvable type. Let p be the largest prime divisor of $|H|$ and let P be a Sylow p-subgroup of H. Then P is normal in G. We consider G/P. It is easy to see that $(G/P, H/P)$ satisfies the hypothesis of the Theorem. By the minimality of G, we have $G/P \in \mathscr{F}$. If the maximal P_1 of P is ss-quasinormal in G, then P_1 is s-quasinormal in G by Lemma 2.3. Thus every maximal subgroup of P is weakly s-quasinormally embedded in G. By [4, Theorem 3.4], $G \in \mathscr{F}$, a contradiction.

COROLLARY 4.2 ([7], Theorem 3.2). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersoluble groups. A group $G \in \mathscr{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathscr{F}$ and every maximal subgroup of any Sylow subgroup of H is either s-quasinormally embedded or c-normal in G.*

COROLLARY 4.3 ([9], Theorem 3.2). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersoluble groups. A group $G \in \mathscr{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathscr{F}$ and every maximal subgroup of any Sylow subgroup of H is either ss-quasinormal or c-normal in G.*

COROLLARY 4.4 ([5], Theorem 1.5). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersoluble groups. A group $G \in \mathscr{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathscr{F}$ and every maximal subgroup of any Sylow subgroup of H is ss-quasinormal in G.*

COROLLARY 4.5. *Let H be a normal subgroup of a group G such that G/H is supersolvable. If every maximal subgroup of any Sylow subgroup of H is either weakly s-quasinormally embedded or ss-quasinormal in G, then G is supersolvable.*

THEOREM 4.6. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If every maximal* *subgroup of any Sylow subgroup of* $F^*(H)$ *is either weakly s-quasinormally embedded or ss-quasinormal in* G, *then* $G \in \mathcal{F}$.

Proof. By Lemmas 2.1 and 2.2, every maximal subgroup of any Sylow subgroup of $F^*(H)$ is either weakly s-quasinormally embedded or ss-quasinormal in $F^*(H)$. Thus $F^*(H)$ is supersolvable by Corollary 4.4. In particular, $F^*(H)$ is solvable. By Lemma 2.6, $F^*(H) = F(H)$. It follows that every maximal subgroup of any Sylow subgroup of $F^*(H)$ is weakly s-quasinormally embedded in G by Lemma 2.3. Thus the result is a corollary of Theorem 3.5 in [4].

Corollary 4.7 ([6], Theorem 3.9). *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$. *Suppose that* G *is a group with a normal subgroup* H *such that* $G/H \in \mathcal{F}$. *If every maximal subgroup of any Sylow subgroup of* $F^*(H)$ *is either s-quasinormally embedded or c-normal in* G, *then* $G \in \mathcal{F}$.

Corollary 4.8. *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$, *the class of all supersoluble groups. Suppose that* G *is a group with a solvable normal subgroup* H *such that* $G/H \in \mathcal{F}$. *If every maximal subgroup of any Sylow subgroup of* $F(H)$ *are either weakly s-quasinormally embedded or ss-quasinormal in* G, *then* $G \in \mathcal{F}$.

Corollary 4.9 ([6], Theorem 3.7). *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$, *the class of all supersoluble groups. Suppose that* G *is a group with a solvable normal subgroup* H *such that* $G/H \in \mathcal{F}$. *If every maximal subgroup of any Sylow subgroup of* $F(H)$ *is either s-quasinormally embedded or c-normal in* G, *then* $G \in \mathcal{F}$.

Corollary 4.10 ([9], Theorem 3.3). *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$, *the class of all supersoluble groups. Suppose that* G *is a group with a solvable normal subgroup* H *such that* $G/H \in \mathcal{F}$. *If every maximal subgroup of any Sylow subgroup of* $F(H)$ *is either ss-quasinormal or c-normal in* G, *then* $G \in \mathcal{F}$.

Corollary 4.11 ([16], Theorem 3.3). *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$, *the class of all supersoluble groups. Suppose that* G *is a group with a normal subgroup* H *such that* $G/H \in \mathcal{F}$. *If every maximal subgroup of any Sylow subgroup of* $F^*(H)$ *is ss-quasinormal in* G, *then* $G \in \mathcal{F}$.

Theorem 4.12. *If every cyclic subgroup of any Sylow subgroup of a group* G *of prime order or order* 4 *is either weakly* s*-quasinormally embedded or* ss*-quasinormal in* G*, then* G *is supersolvable.*

Proof. Assume the theorem is false and let G be a counterexample of minimal order. It is obvious that the hypotheses of the Lemma are inherited for subgroups of G. Our minimal choice yields that G is not supersolvable but every proper subgroup of G is supersolvable. A well-known result of Doerk implies that there exists a normal Sylow p-subgroup of G such that $G = PM$, where M is supersolvable and if $p > 2$ then the exponent of P is p, if $p = 2$, the exponent of P is 2 or 4. Let x be an arbitrary element of P. If $\langle x \rangle$ is ss-quasinormal in G, then $\langle x \rangle$ is s-quasinormally embedded in G by Lemma 2.3. If $\langle x \rangle$ is weakly s-quasinormally embedded in G, there are a subnormal subgroup T of G and an s-quasinormally embedded subgroup $\langle x \rangle_{se}$ of G contained in $\langle x \rangle$ such that $G = HT$ and $H \cap T \leq \langle x \rangle_{se}$. Hence $P = P \cap G = P \cap \langle x \rangle T = \langle x \rangle (P \cap T)$. Since $P/\Phi(P)$ is abelian, we have $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, $P \cap T \leq \Phi(P)$ or $P = (P \cap T)\Phi(P) = P \cap T$. If $P \cap T \leq \Phi(P)$, then $\langle x \rangle = P \trianglelefteq G$ and so $\langle x \rangle$ is s-quasinormally embedded in G. If $P = P \cap T$, then $T = G$ and so $\langle x \rangle$ is also s-quasinormally embedded in G. We have proved that every cyclic subgroup of any Sylow subgroup of G of prime order or order 4 is s-quasinormally embedded in G. Applying Theorem 3.3 in [10], we have G is supersolvable, a contradiction.

Theorem 4.13. *Let* $\mathscr{F}$ *be a saturated formation containing* $\mathscr{U}$*, the class of all supersoluble groups. Suppose that G is a group with a normal subgroup H such that* $G/H \in \mathscr{F}$*. If every cyclic subgroup of any Sylow subgroup of* $F^*(H)$ *of prime order or order* 4 *is either weakly* s*-quasinormally embedded or* ss*-quasinormal in* G*, then* $G \in \mathscr{F}$*.*

Proof. By Lemmas 2.1 and 2.2, every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is weakly s-quasinormally embedded or ss-quasinormal in $F^*(H)$. Thus $F^*(H)$ is supersolvable by Theorem 4.12. In particular, $F^*(H)$ is solvable. By Lemma 2.6, $F^*(H) = F(H)$. Since $G/H \in \mathscr{F}$, we have that $G^{\mathscr{F}}$, the $\mathscr{F}$-residual subgroup of G, is contained in H. Hence, for any cyclic subgroup $\langle x \rangle$ of $F^*(G^{\mathscr{F}}) \leq F^*(H)$ of prime order or order 4, $\langle x \rangle$ is weakly s-quasinormally embedded or ss-quasinormal in G. If $\langle x \rangle$ is weakly s-quasinormally embedded in G, then there are a subnormal subgroup T of G and an s-quasinormally embedded subgroup $\langle x \rangle_{se}$ of G contained in $\langle x \rangle$ such that $G = \langle x \rangle T$ and $\langle x \rangle \cap T \leq \langle x \rangle_{se}$. If $\langle x \rangle$ is not s-quasinormally embedded in G, then G has a normal subgroup K such that $|G : K| = p$ and $G = \langle x \rangle K$ by Lemma 2.1(4). Since G/K is cyclic, it follows that $G/K \in \mathscr{F}$ by the hypotheses. Therefore $G^{\mathscr{F}} \leq K$. This implies that $\langle x \rangle \leq K$, so $G = K$, a contradiction. If $\langle x \rangle$ is ss-quasinormal in G, then $\langle x \rangle$ is also s-quasinormally embedded in G by lemma 2.3. Hence we have proved that every cyclic subgroup of prime order or order 4 of $F^*(G^{\mathscr{F}})$ is s-quasinormally embedded in G. Applying Theorem 1.2 in [10], we have $G \in \mathscr{F}$.

Corollary 4.14 ([6], Theorem 4.3). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersoluble groups. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is either c-normal or s-quasinormally embedded in G, then $G \in \mathscr{F}$.*

Corollary 4.15 ([16], Theorem 3.7). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersoluble groups. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is ss-quasinormal in G, then $G \in \mathscr{F}$.*

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