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METRIC SPHERES IN THE PROJECTIVE SPACES WITH CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

By

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To the memory of our esteemed friend, Hisao Nakagawa

Abstract. We discuss codimension one isometric immersions of complete Riemannian manifolds into the projective spaces with constant holomorphic sectional curvature. Here, the shape operator and the curvature transformation with respect to the normal unit have the same eigenspaces. We then characterize the metric spheres in terms of the shape operator.

1. Introduction and the Statement of Results

Let $\tilde{M}^n(c) = \mathbf{KP}^k(c)$ be a projective space of constant holomorphic sectional curvature $c > 0$. Here, we set $\mathbf{K} = \mathbf{C}$ for $\lambda = 1$, $\mathbf{K} = \mathbf{Q}$ for $\lambda = 3$ and $\mathbf{K} = \mathbf{C}a$ for $\lambda = 7$ and $k = 2$, and the real dimension n of $\tilde{M}^n(c)$ is $n = (\lambda + 1)k$, $k \geq 2$. Let M be a connected and complete Riemannian $(n-1)$-manifold and $\iota : M \to \tilde{M}^n(c)$ an isometric immersion.

Let $\tilde{p} \in \tilde{M}^n(c)$ be an arbitrary fixed point and ξ a unit vector in the tangent space $\tilde{M}^n(c)_{\tilde{p}}$ to $\tilde{M}^n(c)$ at $\tilde{p}$. We then have the unique orthogonal decomposition:

$$\tilde{M}^n(c)_{\tilde{p}} = \operatorname{span}\{\xi\} \oplus \mathscr{H}_\xi \oplus \mathscr{A}\mathscr{H}_\xi.$$

Here, we denote by $\operatorname{span}\{u_1, \ldots, u_m\}$ the span of vectors $u_1, \ldots, u_m \in \tilde{M}^n(c)_{\tilde{p}}$. The $\mathscr{H}_\xi$ and $\mathscr{A}\mathscr{H}_\xi$ are the eigenspaces of the curvature transformation $u \mapsto R(u, \xi)\xi$, $u \in \tilde{M}^n(c)_{\tilde{p}}$ corresponding to the maximum and minimum eigenvalues, respectively. Each point $p \in M$ has a neighborhood where a unit normal field N along ι

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is well defined and the tangent space M_p to M at p admits a unique orthogonal decomposition:

$$M_p = \mathscr{H}_{N(p)} \oplus \mathscr{A}\mathscr{H}_{N(p)}.$$

Here, M_p is naturally identified with $\iota_* M_p$, and the sectional curvature $K_{\tilde{M}^n(c)}$ of $\tilde{M}^n(c)$ satisfies $\frac{c}{4} \le K_{\tilde{M}^n(c)} \le c$ and $\mathscr{H}_{N(p)} = \{u \in M_p \mid K_{\tilde{M}^n(c)}(N(p), u) = c\} \cup \{0\}$, $\mathscr{A}\mathscr{H}_{N(p)} = \{v \in M_p \mid K_{\tilde{M}^n(c)}(N(p), v) = c/4\} \cup \{0\}$.

The shape operator $A : M_p \to M_p$ for ι at a point $p \in M$ is given as

$$A_p X = -\nabla_X N, \quad X \in M_p,$$

where ∇ is the Levi-Civita connection on $\tilde{M}^n(c)$.

In a recent work [4], it has been proved that $\iota(M)$ is a metric ρ-sphere $S^{n-1}(\tilde{p}, \rho) \subset \tilde{M}^n(c)$ centered at a point $\tilde{p} \in \tilde{M}^n(c)$, if there are constants κ_1, κ_2 satisfying

$$A|_{\mathscr{H}_{N(p)}} = \kappa_1 \mathrm{I}_\lambda, \quad A|_{\mathscr{A}\mathscr{H}_{N(p)}} = \kappa_2 \mathrm{I}_{n-\lambda-1}, \tag{1}$$

$$c = 4\kappa_2^2 - 4\kappa_2\kappa_1. \tag{2}$$

Here we have

$$\kappa_1 = \sqrt{c} \cot \sqrt{c}\rho, \quad \kappa_2 = \frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}}{2}\rho. \tag{3}$$

The purpose of this article is to establish a new rigidity theorem for metric spheres in $\tilde{M}^n(c)$ under weaker assumption than stated above. In order to avoid the case by case discussion on the choice of coefficient fields, we employ the matrix Jacobi equation along the unit speed geodesics fitting N. We shall prove the following

THEOREM 1. *Let $\iota : M \to \tilde{M}^n(c)$ be an isometric immersion of a connected and complete Riemannian $(n-1)$-manifold into a projective space $\tilde{M}^n(c) = \mathbf{K}\mathrm{P}^k(c)$ with constant holomorphic sectional curvature c. If the shape operator of ι satisfies* (1) *for smooth functions κ_1 and κ_2, we then have*:

(1) *Both κ_1 and κ_2 are constant.*
(2) *The principal curvatures satisfy* (2).
(3) *$\iota(M)$ is a metric sphere in $\tilde{M}^n(c)$.*
(4) *ι is an embedding.*

REMARK 1. There are many local works on the real hypersurfaces in $\tilde{M}^n(c)$ with constant principal curvatures, for instance [3], [5], [1] and others. Complete real hypersurfaces in $\tilde{M}^n(c)$ have first been discussed in [4]. Real hypersurfaces in the Cayley projective planes have not been discussed even locally, but in [4]. Berndt [1] first introduced the notion of *curvature adapted* for a real hypersurface in $\tilde{M}^n(c)$, which states that the shape operator commutes with the curvature transformation with respect to a unit normal at each point on it. Thus the hypersurfaces in Theorem 1 are curvature adapted. We employ the two-parameter geodesic variation to prove that all the principal curvatures with multiplicities greater than 1 are constant. The matrix Jacobi tensor provides us with a discussion independent of the choice of coefficient fields. As a direct consequence of Theorem 1:(2), we observe that $\tilde{M}^n(c)$ admits neither totally umbilic nor totally geodesic hypersurfaces.

Let $\mathrm{Inj}(q)$ for $q \in \tilde{M}$ be the injectivity radius of the exponential map at q. Let $S^{n-1}(q,t) \subset \tilde{M}$ for $t \in (0, \mathrm{Inj}(q))$ be the metric t-sphere centered at q and $A_p(q,t)$ for $p \in S^{n-1}(q,t)$ the shape operator at p of $S^{n-1}(q,t)$. From the point of view of Riemannian geometry, Proposition 1 gives a general guiding principle to characterize metric spheres in a complete Riemannian manifold. The proof of Proposition 1 is straightforward from the standard variation technic, and left to readers.

Proposition 1. *Let $\iota : M \to \tilde{M}$ be an isometric immersion of a complete Riemannian $(n-1)$-manifold into a complete Riemannian n-manifold $\tilde{M}$. Let $r : M \to \mathbf{R}$ be a smooth function such that $r(p) \in (0, \mathrm{Inj}(\iota(p)))$ for all $p \in M$. If the shape operator A_q of ι at every point $q \in M$ coincides with $A_q(\gamma_{N(q)}(r(q)), r(q))$ at q of $S^{n-1}(\gamma_{N(q)}(r(q)), r(q)) \subset \tilde{M}$, then r is constant, say r_0, and $\gamma_{N(q)}(r_0)$ is a fixed point in $\tilde{M}$. Further, $\iota(M) = S^{n-1}(\gamma_{N(q)}(r_0), r_0)$ and ι is an embedding.*

2. Preliminaries

The global behavior of geodesics on $\tilde{M}^n(c)$ is referred to [2]. For a point $\tilde{p} \in \tilde{M}^n(c)$ and for a unit vector $v \in \tilde{M}^n(c)_{\tilde{p}}$, we denote by $\gamma_v : [0, \infty) \to \tilde{M}^n(c)$ a geodesic with $\gamma_v(0) := \tilde{p}$ and $\dot{\gamma}_v(0) := v$. There are three submanifolds determined by $v \in \tilde{M}^n(c)_{\tilde{p}}$ which are usefull for the proof of our Theorem. They are the $\mathbf{K}$-line $S_v^{\lambda+1}(c)$, the cut locus $\mathrm{Cut}(\tilde{p})$ to $\tilde{p}$, which is the hyperplane at infinity with respect to $\tilde{p}$, and the real projective $(n-\lambda)$-space $\mathbf{RP}_v^{n-\lambda}(c/4)$ of constant sectional curvature $c/4$. Clearly, $\mathbf{K}$-line and $\mathrm{Cut}(\tilde{p})$ are totally geodesic. The $S_v^{\lambda+1}(c)$ is isometric to the standard $(\lambda+1)$-sphere of constant sectional curvature c, and its tangent space at $\gamma_v(t)$ is expressed by

$$S_v^{\lambda+1}(c)_{\gamma_v(t)} = \mathrm{span}\{\dot{\gamma}_v(t)\} \oplus \mathscr{H}_{\dot{\gamma}_v(t)}.$$

The tangent space to $\mathbf{RP}_v^{n-\lambda}(c/4)_{\gamma_v(t)}$ at $\gamma_v(t)$ is expressed by

$$\mathbf{RP}_v^{n-\lambda}(c/4)_{\gamma_v(t)} = \mathrm{span}\{\dot\gamma_v(t)\} \oplus \mathscr{A}\mathscr{H}_{\dot\gamma_v(t)}.$$

The vector $\dot\gamma_v(\pi/\sqrt{c})$ is normal to $\mathrm{Cut}(\tilde p)_{\gamma_v(\pi/\sqrt{c})} = \mathscr{A}\mathscr{H}_{\dot\gamma_v(\pi/\sqrt{c})}$. Let X be a unit parallel field defined on a small neighborhood $U \subset \mathrm{Cut}(\tilde p)$ around $\gamma_v(\pi/\sqrt{c})$ such that $X(\gamma_v(\pi/\sqrt{c})) = -\dot\gamma_v(\pi/\sqrt{c})$. We then have a portion of $\mathbf{RP}_v^{n-\lambda}(c/4)$ as follows:

$$\{\exp_{\tilde q} tX(\tilde q) \mid t \in [0, \pi/\sqrt{c}], \tilde q \in U\} \quad \text{and} \quad \tilde p = \exp_{\tilde q} \frac{\pi}{\sqrt{c}} X(\tilde q), \quad \tilde q \in U.$$

Let $\{E_1, \dots, E_n\}$ be an orthonormal parallel frame field along γ_v with $E_n := \dot\gamma_v(0)$. Let $R(t)$ be the matrix representation of $R(\cdot, \dot\gamma_v(t))\dot\gamma_v(t)$ on $\mathrm{span}\{E_1, \dots, E_{n-1}\}$. We consider the $(n-1) \times (n-1)$-matrix Jacobi equation as follows:

$$D''(t) + R(t)D(t) = 0. \tag{4}$$

If $D(t)$ is a solution of (4) and if a vector $x \in \tilde M^n(c)_{\tilde p}$, $x \perp v$ is identified with the parallel field along γ_v generated by $x = x(0)$, then $Y(t) = D(t)x$ is the representation of a Jacobi field. Let $D_0(t)$ and $D_1(t)$ be the fundamental solution of (4) such that

$$D_0(0) = 0, \quad D_0'(0) = \mathrm{I}_{n-1}, \quad \text{and} \quad D_1(0) = \mathrm{I}_{n-1}, \quad D_1'(0) = 0. \tag{5}$$

Here I_m is the $m \times m$ identity matrix. Then, every Jacobi field Y along γ_v is expressed uniquely by

$$Y(t) = D_0(t)x + D_1(t)y, \quad x := Y'(0), \quad y := Y(0).$$

Clearly both $\mathscr{H}_{\dot\gamma_v(t)}$ and $\mathscr{A}\mathscr{H}_{\dot\gamma_v(t)}$ are the eigenspaces of the curvature transformation corresponding to the eigenvalues c and $c/4$ respectively.

From now on, let $\{E_1, \dots, E_\lambda\}$ *and* $\{E_{\lambda+1}, \dots, E_n\}$ *be parallel orthonormal frame fields of* $\mathscr{H}_{\dot\gamma_v}$ *and* $\mathscr{A}\mathscr{H}_{\dot\gamma_v}$ *along* γ_v *respectively*. We then have

$$R(t) = \begin{bmatrix} c\mathrm{I}_\lambda & \\ & \frac{c}{4}\mathrm{I}_{n-\lambda-1} \end{bmatrix}, \quad t \geq 0.$$

Therefore we have

$$D_0(t) = \begin{bmatrix} \frac{1}{\sqrt{c}} \sin \sqrt{c}t\mathrm{I}_\lambda & \\ & \frac{2}{\sqrt{c}} \sin \frac{\sqrt{c}}{2} t\mathrm{I}_{n-\lambda-1} \end{bmatrix}$$

and

$$D_1(t) = \begin{bmatrix} \cos\sqrt{c}t\mathrm{I}_\lambda & \\ & \cos\frac{\sqrt{c}}{2}t\mathrm{I}_{n-\lambda-1} \end{bmatrix}.$$

Let $r_0 \in (0, \pi/\sqrt{c})$ be a fixed number and take a point $\tilde{q} \in S^{n-1}(\tilde{p}, r_0)$. For a smooth curve $c : (-\varepsilon, \varepsilon) \to S^{n-1}(\tilde{p}, r_0)$ such that $c(0) := \tilde{q}$ and $\dot{c}(0) := x \in S^{n-1}(\tilde{p}, r_0)_{\tilde{q}}$, we set $v(s) \in \tilde{M}^n(c)_{\tilde{p}}$, $s \in (-\varepsilon, \varepsilon)$ such that $c(s) := \exp_{\tilde{p}} r_0 v(s)$. Then the Jacobi field $Y(t)$ along γ_v associated with the geodesic variation $(t, s) \mapsto \exp_{\tilde{p}} tv(s)$ is an N-Jacobi field for this metric sphere. We then have

$$Y(t) = D_0(t)v'(0), \quad v'(0) \perp v.$$

From construction, we have $Y(r_0) = x$ and hence the shape operator $A_{\tilde{q}}(\tilde{p}, r_0)$ at $\tilde{q}$ of $S^{n-1}(\tilde{p}, r_0)$ is given as

$$A_{\tilde{q}}(\tilde{p}, r_0) = D_0'(r_0)D_0(r_0)^{-1}.$$

Let $\iota : M \to \tilde{M}^n(c)$ be an isometric immersion and N a unit normal field along ι defined in a small neighborhood around $q \in M$. Let A_q be the shape operator at q of ι. An N-Jacobi field Y along $\gamma_{N(q)}$ is a Jacobi field associated with the geodesic variation whose variational geodesics at $t = 0$ are fitting N, and is written as

$$Y(t) = -D_0(t)A_q x + D_1(t)x,$$

where $Y(0) = x \in M_q$ and $Y'(0) = -A_q x$. Therefore, the matrix N-Jacobi tensor is written as

$$D(t) = -D_0(t)A_q + D_1(t), \quad t \geq 0.$$

3. Metric Spheres

We now discuss the shape operator of smooth metric spheres with radius $r \in (0, \pi/\sqrt{c})$. When our discussion is local, we may identify points and vectors on M with those on $\iota(M) \subset \tilde{M}^n(c)$.

Let γ be a geodesic with $\gamma(0) = \tilde{p}$ and $\gamma(t) = \tilde{q}$. As is observed at the end of §2, the shape operator $A_{\tilde{q}}(\tilde{p}, t)$ of $S^{n-1}(\tilde{p}, t) \subset \tilde{M}^n(c)$ at a point $\tilde{q}$ is expressed as

$$A_{\tilde{q}} = D_0'(t)D_0(t)^{-1} = \begin{bmatrix} \sqrt{c}\cot\sqrt{c}t\mathrm{I}_\lambda & \\ & \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}t\mathrm{I}_{n-\lambda-1} \end{bmatrix}$$

with respect to $\{E_1(t), \dots, E_n(t)\}$. The principal curvatures $\kappa_1(t)$ and $\kappa_2(t)$ of $S^{n-1}(\tilde{p}, t)$ with respect to the inward unit normal at $\tilde{q}$ are given as

$$\kappa_1(t) = \sqrt{c} \cot \sqrt{c}t \quad \text{and} \quad \kappa_2(t) = \frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}}{2} t.$$

Clearly, they satisfy

$$c = 4\kappa_2(t)^2 - 4\kappa_1(t)\kappa_2(t).$$

3.1. Principal Curvatures and Zeros of N-Jacobi Field

Let M be a hypersurface in $\tilde{M}^n(c)$ and $q \in M$. Let $v \in \tilde{M}^n(c)_q$ be a unit normal vector to M at q and $\gamma_v(t) := \exp_q tv$, $t \geq 0$. Then the matrix N-Jacobi tensor along γ_v is expressed as

$$D(t) = -\begin{bmatrix} \frac{1}{\sqrt{c}} \sin \sqrt{c}t \mathrm{I}_\lambda & \\ & \frac{2}{\sqrt{c}} \sin \frac{\sqrt{c}}{2} t \mathrm{I}_{n-\lambda-1} \end{bmatrix} A_q + \begin{bmatrix} \cos \sqrt{c}t \mathrm{I}_\lambda & \\ & \cos \frac{\sqrt{c}}{2} t \mathrm{I}_{n-\lambda-1} \end{bmatrix}$$

with respect to $E_1(t), \dots, E_{n-1}(t)$. If $x \in M_q$ is an eigenvector corresponding to an eigenvalue k and if $x = x_1 + x_2$, $x_1 \in \mathscr{H}_v$ and $x_2 \in \mathscr{A}\mathscr{H}_v$, we then have

$$\begin{aligned} D(t)x &= -D_0(t)A_q x + D_1(t)x \\ &= \left(\cos \sqrt{c}t - \frac{k}{\sqrt{c}} \sin \sqrt{c}t \right) x_1 + \left(\cos \frac{\sqrt{c}}{2} t - \frac{2k}{\sqrt{c}} \sin \frac{\sqrt{c}}{2} t \right) x_2. \end{aligned}$$

Therefore, $D(t)x = 0$ holds for some t if and only if one of the following is true.

(1) $x = x_1 \in \mathscr{H}_v$ and $k = \sqrt{c} \cot \sqrt{c}t$.

(2) $x = x_2 \in \mathscr{A}\mathscr{H}_v$ and $k = \frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}}{2} t$.

LEMMA 1. *Let M be a hypersurface in $\tilde{M}^n(c)$ and $q \in M$. If there exists for every $x \in M_q$ a number $t(x)$ such that $D(t(x))x = 0$, then $t(x)$ is independent of the choice of $x \in M_q$. Setting $t_q := t(x)$ for $x \in M_q$, the shape operator $A_q := D_0(t_q)^{-1} D_1(t_q)$ at q satisfies*

$$A_q = \begin{bmatrix} \sqrt{c} \cot \sqrt{c}t_q \mathrm{I}_\lambda & \\ & \frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}}{2} t_q \mathrm{I}_{n-\lambda-1} \end{bmatrix}.$$

In particular, A_q has two distinct eigenvalues $\kappa_1(q) = \sqrt{c} \cot \sqrt{c}t_q$ and $\kappa_2(q) = \frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}}{2} t_q$, satisfying $c = 4\kappa_2(q)^2 - 4\kappa_1(q)\kappa_2(q)$.

PROOF. Let $x_1, \dots, x_{n-1} \in T_qM$ be an orthonormal basis consisting of eigenvectors of A_q such that $A_q x_i = k_i x_i$ for $i = 1, \dots, n-1$. From what we have discussed above, we may suppose without loss of generality that $x_1, \dots, x_\lambda \in \mathscr{H}_v$ and $x_{\lambda+1}, \dots, x_{n-1} \in \mathscr{A}\mathscr{H}_v$. By assuming $\lambda > 1$, we assert that $k_i = k_j$ holds for $1 \le i \ne j \le \lambda$ and for $\lambda + 1 \le i \ne j \le n-1$. Let $x = x_i + x_j$ for $1 \le i \ne j \le \lambda$. Since $A_q x = k_i x_i + k_j x_j$, we have

$$\begin{aligned} D(t(x))x &= -D_0(t(x))A_q x + D_1(t(x))x \\ &= \left(\cos\sqrt{c}t(x) - \frac{k_i}{\sqrt{c}}\sin\sqrt{c}t(x)\right)x_i + \left(\cos\sqrt{c}t(x) - \frac{k_j}{\sqrt{c}}\sin\sqrt{c}t(x)\right)x_j \\ &= 0. \end{aligned}$$

We therefore have $k_i = \sqrt{c}\cot\sqrt{c}t(x)$ and $k_j = \sqrt{c}\cot\sqrt{c}t(x)$. If $\lambda > 1$, we then have $k_i = k_j$ for all $i, j = 1, \dots, \lambda$, say, κ_1. In the same way we can prove that $k_i = k_j$ for all $\lambda = 1, 3, 7$ and for all $i, j = \lambda + 1, \dots, n-1$, say, κ_2. Apply the same method to any $x = x_1' + x_2'$ with $x_1' \in \mathscr{H}_v$ and $x_2' \in \mathscr{A}\mathscr{H}_v$. We then have

$$\kappa_1 = \sqrt{c}\cot\sqrt{c}t(x) \quad \text{and} \quad \kappa_2 = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}t(x).$$

Therefore $t(x)$ is independent of the choice of $x \in M_q$, say t_q, and $c = 4\kappa_2^2 - 4\kappa_1\kappa_2$. $\square$

3.2. Decomposition of the Shape Operator

Let $\iota : M \to \tilde{M}^n(c)$ be an isometric immersion and N a unit normal field along ι defined in a small neighborhood of q. Let $\gamma_{N(q)} : [0, \infty) \to \tilde{M}^n(c)$ for a point $q \in M$ be a geodesic with $\gamma_{N(q)}(0) = q$ and $\dot{\gamma}_{N(q)} = N(q)$. We recall that $\mathscr{H}_{\dot{\gamma}_{N(q)}(t)} = \operatorname{span}\{E_1(t), \dots, E_\lambda(t)\}$ and $\mathscr{A}\mathscr{H}_{\dot{\gamma}_{N(q)}(t)} = \operatorname{span}\{E_{\lambda+1}(t), \dots, E_{n-1}(t)\}$.

Assume that the shape operator A_q for ι satisfies

$$A_q|_{\mathscr{H}_{N(q)}} = \kappa_1(q)\mathrm{I}_\lambda \quad \text{and} \quad A_q|_{\mathscr{A}\mathscr{H}_{N(q)}} = \kappa_2(q)\mathrm{I}_{n-\lambda-1}.$$

Then the matrix N-Jacobi tensor along $\gamma_{N(q)}$ is written as

$$D(q,t) = \begin{bmatrix} D_1(q,t) & \\ & D_2(q,t) \end{bmatrix},$$

where

$$D_1(q,t) = \left(\cos\sqrt{c}t - \frac{\kappa_1(q)}{\sqrt{c}}\sin\sqrt{c}t\right)\mathrm{I}_\lambda,$$

$$D_2(q,t) = \left(\cos\frac{\sqrt{c}}{2}t - \frac{2\kappa_2(q)}{\sqrt{c}}\sin\frac{\sqrt{c}}{2}t\right)\mathrm{I}_{n-\lambda-1}.$$

Let $\Phi : M \times [0,\infty) \to \tilde{M}^n(c)$ be defined by

$$\Phi(q,t) := \exp_q tN(q), \quad q \in M,\ t \geq 0,$$

and the t-parallel submanifold $\tilde{M}(t)$ of $\iota(M)$, by $\tilde{M}(t) := \{\Phi(q,t) \mid q \in M\}$. If $\det D(q,t) \neq 0$, then the shape operator A_{qt} of $\tilde{M}(t)$ at a point $\gamma_{N(q)}(t)$ is given as

$$A_{qt} = \begin{bmatrix} A_{qt,1} & \\ & A_{qt,2} \end{bmatrix},$$

where

$$A_{qt,1} = \frac{-\sqrt{c}\sin\sqrt{c}t - \kappa_1(q)\cos\sqrt{c}t}{\cos\sqrt{c}t - \frac{\kappa_1(q)}{\sqrt{c}}\sin\sqrt{c}t}\mathrm{I}_\lambda,$$

$$A_{qt,2} = \frac{-\frac{\sqrt{c}}{2}\sin\frac{\sqrt{c}}{2}t - \kappa_2(q)\cos\frac{\sqrt{c}}{2}t}{\cos\frac{\sqrt{c}}{2}t - \frac{2\kappa_2(q)}{\sqrt{c}}\sin\frac{\sqrt{c}}{2}t}\mathrm{I}_{n-\lambda-1}.$$

LEMMA 2. *The decomposition of A_q and $D(q,t)$ has the following properties*:
(1) $D_1(q,t) = 0$ *if and only if* $\kappa_1(q) = \sqrt{c}\cot\sqrt{c}t$.
(2) $D_2(q,t) = 0$ *if and only if* $\kappa_2(q) = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}t$.
(3) $A_{qt,1} = 0$ *if and only if* $\kappa_1(q) = -\sqrt{c}\tan\sqrt{c}t$.
(4) $A_{qt,2} = 0$ *if and only if* $\kappa_2(q) = -\frac{\sqrt{c}}{2}\tan\frac{\sqrt{c}}{2}t$.

The following Lemma 3 is usefull for the proof of Theorem 1.

LEMMA 3. *Let v be a unit tangent vector in $\tilde{M}^n(c)$. Let $\mathscr{E}$ be an equator hypersphere in a* **K**-*line $S_v^{\lambda+1}(c)$. Then the shape operator of every t-parallel hypersurface $M \subset \tilde{M}^n(c)$ for $t \in (0, \pi/(2\sqrt{c}))$ of $\mathscr{E}$ does not satisfy* (1).

PROOF. Recall that both **K**-line and $\mathbf{RP}_v^{n-\lambda}(c/4)$ have constant sectional curvature. Without loss of generality we may assume that $\mathscr{H}_v$ is the tangent space to $\mathscr{E}$ at a point $\tilde{p} \in \mathscr{E}$. Then $M \cap S_v^{\lambda+1}(c)$ is a small sphere of dimension λ and $M \cap \mathbf{RP}_v^{n-\lambda}(c/4)$ is a metric t-sphere centered at $\tilde{p}$. Suppose that the shape operator of M satisfies (1) at each point of M. Take a unit vector $v_1 \in \tilde{M}^n(c)_{\tilde{p}}$ such that

(1) v_1 makes a small positive angle with v

(2) v_1 is normal to $\mathscr{E}_{\tilde{p}}$.

Setting $q := \gamma_v(t)$ and $q_1 := \gamma_{v_1}(t)$, we observe that the inward unit normal field N to M satisfies $N(q) = -\dot{\gamma}_v(t)$, $N(q_1) = -\dot{\gamma}_{v_1}(t)$, and hence

$$\operatorname{Ker} d\Phi|_{(q,t)} = \mathscr{A}\mathscr{H}_{N(q)}, \quad \operatorname{Ker} d\Phi|_{(q_1,t)} = \mathscr{A}\mathscr{H}_{N(q_1)}.$$

The tangent space to the focal submanifold $\mathscr{E}$ of M at $\tilde{p}$ is given as

$$d\Phi\mathscr{H}_{N(q)} = d\Phi\mathscr{H}_{N(q_1)} = \mathscr{E}_{\tilde{p}}.$$

Since $\mathscr{H}_{\dot{\gamma}_v(t)}$ is parallel along γ_v, we have $\mathscr{E}_{\tilde{p}} = \mathscr{H}_v = \mathscr{H}_{v_1}$, a contradiction. $\square$

3.3. Integral Submanifolds

We prove that the distributions $\mathscr{H}_N$ and $\mathscr{A}\mathscr{H}_N$ for a unit normal field N along ι are integrable. Let x and y be vector fields defined in a small neighborhood of $q \in M$ such that $x(q), y(q) \in \mathscr{H}_{N(q)}$, (or $x(q), y(q) \in \mathscr{A}\mathscr{H}_{N(q)}$, respectively). Let X and Y be parallel fields along $\gamma_{N(q)}$ such that $X(0) = x$, $Y(0) = y$. We assert that $[x, y](q) \in \mathscr{H}_{N(q)}$, (or $[x, y](q) \in \mathscr{A}\mathscr{H}_{N(q)}$, respectively) for $q \in M$. Let $r : M \to \mathbf{R}$ be a smooth function such that $\kappa_1(q) = \sqrt{c}\cot\sqrt{c}r(q)$ (or $\kappa_2(q) = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}r(q)$, respectively) for $q \in M$. Setting $\varphi(q) := \Phi(q, r(q))$, $q \in M$, we have

$$d\varphi_q(x) = (x(q)r)\dot{\gamma}_{N(q)}(r(q)), \quad d\varphi_q(y) = (y(q)r)\dot{\gamma}_{N(q)}(r(q))$$
$$d\varphi_q([x, y]) = D(q, r(q))([x, y](q)) + ([x, y](q)r)\dot{\gamma}_{N(q)}(r(q)),$$

for every point $q \in M$. Since

$$\nabla_{d\varphi_q(x)}\, d\varphi_q(y) = x(q)(yr)\dot{\gamma}_{N(q)}(r(q)) + (x(q)r)D'(q, r(q))Y$$
$$\nabla_{d\varphi_q(y)}\, d\varphi_q(x) = y(q)(xr)\dot{\gamma}_{N(q)}(r(q)) + (y(q)r)D'(q, r(q))X,$$

we have

$$D(q, r(q))([x, y](q)) = (x(q)r)D'(q, r(q))Y - (y(q)r)D'(q, r(q))X.$$

This implies that $[x, y](q) \in \mathscr{H}_{N(q)}$ (or $[x, y](q) \in \mathscr{AH}_{N(q)}$, respectively) for $q \in M$, since

$$D'(q,t)X = (-\kappa_1 \cos \sqrt{c}t - \sqrt{c} \sin \sqrt{c}t)X,$$

$$D'(q,t)Y = (-\kappa_1 \cos \sqrt{c}t - \sqrt{c} \sin \sqrt{c}t)Y.$$

(or similar relations are obtained for $\mathscr{AH}_{N(q)}$ by replacing κ_1 by κ_2, respectively.)

4. Proof of Theorem 1

For the proof of Theorem 1, we only need to show that $r_1 = r_2$ is constant on M, for the rest of the proof is direct from (1). Recall that the matrix N-Jacobi tensor along $\gamma_{N(q)}$ is written as

$$D(q,t) = \begin{bmatrix} \left(\cos \sqrt{c}t - \frac{\kappa_1(q)}{\sqrt{c}} \sin \sqrt{c}t\right)\mathrm{I}_\lambda & \\ & \left(\cos \frac{\sqrt{c}}{2}t - \frac{2\kappa_2(q)}{\sqrt{c}} \sin \frac{\sqrt{c}}{2}t\right)\mathrm{I}_{n-\lambda-1} \end{bmatrix},$$

for any $q \in M$. If $r_1(q)$ is given as $\kappa_1(q) = \sqrt{c} \cot \sqrt{c}r_1(q)$, then

$$D(q, r_1(q)) = \begin{bmatrix} 0 & \\ & \left(\cos \frac{\sqrt{c}}{2} r_1(q) - \frac{2\kappa_2(q)}{\sqrt{c}} \sin \frac{\sqrt{c}}{2} r_1(q)\right)\mathrm{I}_{n-\lambda-1} \end{bmatrix},$$

and also if $r_2(q)$ is given as $\kappa_2(q) = \frac{\sqrt{c}}{2} \cot \frac{\sqrt{c}}{2} r_2(q)$, then

$$D(q, r_2(q)) = \begin{bmatrix} \left(\cos \sqrt{c}r_2(q) - \frac{\kappa_1(q)}{\sqrt{c}} \sin \sqrt{c}r_2(q)\right)\mathrm{I}_\lambda & \\ & 0 \end{bmatrix}.$$

LEMMA 4. *The functions κ_2 and r_2 for all $\lambda = 1, 3, 7$ are constant on M. Moreover, if $\Omega_{\mathscr{AH}}(q)$ is the maximal integral submanifold through $q \in M$, then $\iota(\Omega_{\mathscr{AH}}(q))$ is the metric r_2-sphere in $\mathbf{RP}^{n-\lambda}_{N(q)}(c/4)$ centered at $\gamma_{N(q)}(r_2)$ with radius r_2.*

PROOF. Let $N(q)$ be chosen such that $\kappa_2(q) > 0$. We first prove that κ_2 and r_2 for all $\lambda = 1, 3, 7$ are constant along each integral submanifold $\Omega_{\mathscr{AH}}(q)$, $q \in M$. If x and y are vector fields on M such that $x(q) \in \mathscr{AH}_{N(q)}$ and $y(q) \in \mathscr{AH}_{N(q)}$, and if X and Y are parallel fields along $\gamma_{N(q)}$ such that $x(q) = X(0)$, $y(q) = Y(0)$, we then have

$$D(q, r_2(q))([x, y](q)) = (x(q)r_2)D'(q, r_2(q))Y - (y(q)r_2)D'(q, r_2(q))X = 0.$$

If x and y are linearly independent everywhere on M, we then have

$$x(q)r_2 = 0 \quad \text{and} \quad y(q)r_2 = 0 \quad \text{for all } q \in M.$$

Therefore, the functions r_2 and κ_2 are constant along each $\Omega_{\mathscr{A}\mathscr{H}}(q)$, $q \in M$. We then observe by the completeness of M that $\iota(\Omega_{\mathscr{A}\mathscr{H}}(q))$ is the metric $r_2(q)$-sphere in $\mathbf{RP}^{n-\lambda}_{N(q)}(c/4)$ centered at $\gamma_{N(q)}(r_2(q))$.

We next prove that r_2 and κ_2 for all $\lambda = 1, 3, 7$ are constant on M. We observe that N is defined on an open set of M containing $\Omega_{\mathscr{A}\mathscr{H}}(q)$. From the above discussion, $\gamma_{N(q)} : [0, \infty) \to \tilde{M}^n(c)$ at each point $q \in M$ has the properties:

$$\dot{\gamma}_{N(q)}(0) = N(q), \quad \gamma_{N(q)}(2r_2(q)) \in \iota(\Omega_{\mathscr{A}\mathscr{H}}(q)) \quad \text{and}$$

$$\dot{\gamma}_{N(q)}(2r_2(q)) = -N(\gamma_{N(q)}(2r_2(q))).$$

Therefore, $\gamma_{N(q)}[0, 2r_2(q)]$ hits $\iota(M)$ orthogonally at its endpoints. Thus the first variation formula for this family of geodesics implies that $r_2(q)$ is constant on an open set of M containing $\Omega_{\mathscr{A}\mathscr{H}}(q)$, and hence on M. □

Lemma 5. *The functions κ_1 and r_1 for all $\lambda = 1, 3, 7$ are constant on M. The r_2-parallel submanifold $\tilde{M}(r_2)$ of $\iota(M)$ degenerates to a point, and we have $r_1 = r_2$.*

Proof. Let $N(q)$ be chosen such that $r_1(q) > 0$.

Assuming $\lambda \neq 1$, we can prove that the functions κ_1 and r_1 are constant on M by the same manner as developed in Lemma 4. Further, if $\Omega_{\mathscr{H}}(q)$ is the maximal integral submanifold through $q \in M$, then $\iota(\Omega_{\mathscr{H}}(q))$ is the metric r_1-sphere in $S^{\lambda+1}_{N(q)}(c)$ centered at $\gamma_{N(q)}(r_1)$ with radius r_1. Setting $\tilde{p} := \gamma_{N(q)}(r_2)$, we observe that a unit vector $\xi \in \tilde{M}^n(c)_{\tilde{p}}$ is normal to $\mathrm{span}\{\dot{\gamma}_{N(q)}(r_2)\} \oplus \tilde{M}(r_2)_{\tilde{p}}$ if and only if $\xi \in \mathscr{A}\mathscr{H}_{\dot{\gamma}_{N(q)}(r_2)}$. We also observe that $\exp_{\tilde{p}} r_2\xi \in \iota(\Omega_{\mathscr{A}\mathscr{H}}(q))$ if and only if $\xi \in \mathscr{A}\mathscr{H}_{\dot{\gamma}_{N(q)}(r_2)}$. The symmetric property of the metric on $\tilde{M}^n(c)$ implies that the shape operator $A_{qr_2,1}$ of $\tilde{M}(r_2)$ with respect to $\xi \in \mathscr{A}\mathscr{H}_{\dot{\gamma}_{N(q)}(r_2)}$ coincides with that with respect to $-\xi$. We then have from Lemma 2 that $A_{qr_2,1} = -A_{qr_2,1}$ and

$$-\sqrt{c}\sin\sqrt{c}r_2 - \kappa_1 \cos\sqrt{c}r_2 = 0. \tag{6}$$

Therefore, $\tilde{M}(r_2)$ is totally geodesic (and a great shpere) in $S^{\lambda+1}_{N(q)}(c)$. However, Lemma 3 implies that $\tilde{M}(r_2)$ degenerate to a point. This proves $r_1 = r_2$.

Assuming $\lambda = 1$, we notice that $\tilde{M}(r_2)$ is the limiting submanifold of t-parallel hypersurfaces $\{\tilde{M}(t)\}_{t \in [0, r_2)}$ as $t \to r_2$ and that each $\tilde{M}(t)$ for $t \in [0, r_2)$ is foliated by geodesics tangent to $\mathscr{H}_{\dot{\gamma}_{N(q)}(t)} = J\dot{\gamma}_{N(q)}(t)$. Here J is the complex structure. Thus we see that $\tilde{M}(r_2)$ is a geodesic in $\tilde{M}^n(c)$. In fact, setting $X(q, t) := J\dot{\gamma}_{N(q)}(t)$, we have

$$\nabla_{X(q,t)} X(q, t) = A_{qt,1}\dot{\gamma}_{N(q)}(t).$$

From $\lim_{t\to r_2} X(q,t) = X(\tilde{p})$ we have

$$\nabla_{X(\tilde{p})}X = A_{qr_2,1}\dot{\gamma}_{N(q)}(r_2).$$

Since the left hand side is independent of the choice of $q \in \Omega_{\mathscr{AH}}(p)$, we have $A_{qr_2,1} = 0$, and hence $\tilde{M}(r_2)$ is a geodesic in $\tilde{M}^n(c)$. However, Lemma 3 implies that $\tilde{M}(r_2)$ degenerate to a point. This proves $r_1 = r_2$. □

Assuming finally that $\tilde{M}(r_1)$ is non-degenerate, we observe that $\tilde{M}(r_1)$ is the hyperplane at infinity with respect to $\gamma_{N(q)}\big(r_1 - \frac{\pi}{\sqrt{c}}\big)$. Then $\iota(M)$ is the r_1-parallel hypersurface around $\tilde{M}(r_1)$, which is nothing but the metric $(\pi/\sqrt{c} - r_1)$-sphere centered at $\gamma_{N(q)}\big(r_1 - \frac{\pi}{\sqrt{c}}\big)$. Here N is defined as the outer normal with respect to this metric sphere. Therefore we have $\kappa_2 < 0$, and conclude the proof of $r_1 = r_2$ for all real numbers $\kappa_2 \neq 0$.

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