TSUKUBA J. MATH.
Vol. 35 No. 1 (2011), 13–52

PROPAGATION OF ANALYTICITY IN THE C^{∞} SOLUTIONS OF QUASI-LINEAR WEAKLY HYPERBOLIC WAVE EQUATIONS

By

R. Manfrin

Abstract. We study the propagation of the analytic regularity of the C^{∞} solutions of the quasi-linear, weakly hyperbolic wave equation $u_{tt} - a(u)u_{xx} = 0$, where $a(u)$ is a bounded, nonnegative analytic function.

1. Introduction

The question of the propagation of analyticity in the C^{∞} solutions of analytic nonlinear strictly hyperbolic equations (or systems) was satisfactorily solved in [2], [18]. In the context of weakly hyperbolic equations only partial results are known.

The first results in this direction were proved by Spagnolo [29, 30] for the analytic semi-linear weakly hyperbolic equation

$$\partial_t^2 u - \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x,t)\partial_{x_j}u) = f(u), \quad (x,t) \in \mathbf{R}^n \times [0,T), \tag{1.1}$$

under one of the following additional conditions:

a) the coefficients a_{ij} have the form $a_{ij}(x,t) = b(t)a_{ij}^{o}(x)$;

b) the solution $u(x,t)$ is *a priori* assumed in a Gevrey class of order $s < 2$.

Afterwards, the problem of the analytic regularity of C^{∞} solutions was considered, among the others, in [5], [6, 7, 8], [21, 22], [14], [19] for suitable classes of nonlinear weakly hyperbolic equations and systems. In all this papers the solution $u(x,t)$ was *a priori* assumed to belong to a space $X \subset C^{\infty}$ where the Cauchy problem for the linearized differential operator is well posed.

2000 *Mathematics Subject Classification*. Primary 35L70; Secondary 35L80, 35B65.
Key words and phrases. Weakly hyperbolic equations, analytic regularity.
Received June 28, 2010.
Revised January 17, 2011.

Here we consider a situation in which the linearized equation (see (1.8)–(1.9) below) may present phenomena of non existence or non-uniqueness. Namely, we investigate the propagation of analyticity in the C^∞ solutions of the Cauchy problem

$$u_{tt} - a(u)u_{xx} = 0, \quad (x,t) \in \mathbf{R} \times [0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.3}$$

where $a : \mathbf{R} \to [0,\infty)$ is merely a bounded analytic function, i.e.,

$$a \in \mathscr{A}(\mathbf{R}) \quad \text{and} \quad 0 \le a(s) \le \lambda \quad (s \in \mathbf{R}), \tag{1.4}$$

for a suitable $\lambda > 0$. Given $T \in (0, +\infty]$ and

$$u : \mathbf{R} \times [0,T) \to \mathbf{R}, \tag{1.5}$$

a C^∞ solution of (1.2), (1.3), we prove the following:

Theorem 1.1. *Let* $a : \mathbf{R} \to [0,\infty)$ *satisfy* (1.4). *If* u_0, u_1 *are analytic in* $(x_o - \varepsilon, x_o + \varepsilon)$, *for some* $x_o \in \mathbf{R}$ *and* $\varepsilon > 0$, *then* $u(x,t)$ *is analytic in*

$$D = \{(x,t) : |x - x_o| < \varepsilon - \sqrt{\lambda}t, 0 \le t < T\}. \tag{1.6}$$

In particular, $u(x,t)$ *is analytic in* $\mathbf{R} \times [0,T)$ *if* u_0, u_1 *are analytic in* $\mathbf{R}$.

To demonstrate Theorem 1.1 we combine energy estimates in influence domains with the results of [23, 24] (cf. Theorem 9.1, Corollary 9.2) of local *well-posedness* and *representation* of solutions of weakly hyperbolic equations of type (1.2):

if, for instance, $a : \mathbf{R} \to [0,\infty)$ *satisfies* (1.4), $a(0) = 0$, $a(s) > 0$ *for* $s \neq 0$, *and* $u_0, u_1 \in C_0^\infty$, *then problem* (1.2), (1.3) *has a unique solution* $u \in C^\infty(\mathbf{R} \times [0,T))$, *for some* $T = T(u_0,u_1) > 0$. *Furthermore, there exist* $g, h \in C^\infty$ *s.t.*

$$u(x,t) = g(x,t)u_0(x) + h(x,t)u_1(x) \quad \textit{in } \mathbf{R} \times [0,T), \tag{1.7}$$

with (*in some sense*) $g \approx 1$ *and* $h \approx t$.

Then, using the representation (1.7), we can apply suitable energy estimates proving, in this way, that the analyticity of Cauchy data propagates according to the geometry of influence domains. This argument circumvents the difficulties due to the fact that the linearization of the operator

$$u \mapsto F(u^{(\alpha)})|_{|\alpha| \le 2} \stackrel{\text{def}}{=} \partial_t^2 u - a(u)\partial_x^2 u, \tag{1.8}$$

at a generic C^∞ function, say $\tilde{u}$, gives the *linear* operator $P = \sum_{|\alpha|\le m} \frac{\partial^\alpha F}{\partial u^{(\alpha)}}(\tilde{u}^{(\alpha)})\partial^\alpha$,

$$u \overset{P}{\mapsto} \partial_t^2 u - a(\tilde{u})\partial_x^2 u - (a'(\tilde{u})\tilde{u}_{xx})u, \tag{1.9}$$

where the coefficient $a(\tilde{u})$ is *almost* an arbitrary nonnegative C^∞ function (since $a(\cdot)$ is analytic and nonnegative, $a(\tilde{u})$ may be, at least locally, the square of an arbitrary C^∞ function). Indeed, even the Cauchy problem for a linear weakly hyperbolic equation such as $u_{tt} - k(t)u_{xx} = 0$ (with $k \in C^\infty$, $k(t) \ge 0$) is, in general, not locally well-posed in C^∞, as the classical examples of [9], [10] show.

Remark 1.2. By the *nonlinear Cauchy-Kowalewski* theorem we know that if $a(s)$, $u_0(x)$, $u_1(x)$ are analytic then problem (1.2), (1.3) has a unique analytic solution, say $u^*(x,t)$, for t small and this statement is true without any hyperbolic assumption.

Hence, it is natural to ask if the result of Theorem 1.1 can be proved as a consequence of the *Cauchy-Kowalewski* theorem assuming, merely, $a(s)$ analytic and $u(x,t)$ a C^∞ *complex-valued* solution in $\mathbf{R} \times [0,T)$ with analytic data for $t = 0$. Without additional information this seems to be difficult for many reasons:

i) First of all, the step-by-step reasoning could not be used directly, if we wished to prove the existence in large of the analytic solution $u^*(x,t)$, that is for any $(x,t) \in D$, because the size of each step (with respect to t) in the argument depends on the radius of convergence of the Cauchy data obtained by the previous step. See [26, §1], [20].

As a matter of fact, given any kowalewskian linear equation with analytic coefficients, a necessary condition for the global well-posedness in the space of real analytic functions is the weak hyperbolicity, i.e. the reality of the characteristic roots. See [27], [28]. On the other hand, by the Bony-Schapira's theorem [3, 4] the Cauchy problem for *linear* weakly hyperbolic equations is globally well-posed in the space of real analytic functions, provided the coefficients of the equations are analytic. See also [12, 13].

ii) Secondly, to prove the analyticity of the given C^∞ solution u, we need some kind of uniqueness, i.e. we need to know that $u(x,t) = u^*(x,t)$ where both the solutions are defined. But an example of nonuniqueness for the analytic nonlinear Cauchy problem due to Métivier [25], see also Hörmander [16], shows that Hölmgren's uniqueness theorem does not extend in general to higher order nonlinear equations, nor systems (for first order scalar equation uniqueness is known, see [25] and the references therein). For instance, uniqueness fails for the following equation

$$(\partial_t + \partial_z)(\partial_t^2 u + \partial_x^2 u - \partial_y^2 u + (\partial_t u)^2 + (\partial_x u)^2 - (\partial_y u)^2) = 0, \tag{1.10}$$

which is a semilinear analytic equation of *kowalewskian* type, whose principal part is $(\partial_t + \partial_z)(\partial_t^2 + \partial_x^2 - \partial_y^2)$ and $\{t = 0\}$ is non characteristic.

iii) Finally, since (1.2) is quasi-linear, we can also recall the result of [17], where it was proved that well posed Cauchy problems for complex nonlinear equations must be semilinear. More precisely, given $\Omega \subset \mathbf{R}^n$, $\Gamma \subset \mathbf{C} \times \mathbf{C}^n$ $(n \geq 2)$ open sets, $G : \Omega \times \Gamma \to \mathbf{C}$ depending smoothly on $x \in \Omega$ and holomorphically on $(\zeta, \xi) \in \Gamma$, let us consider the first order, complex nonlinear equation

$$G(x, v, \nabla v) = 0, \quad x \in \Omega, \tag{1.11}$$

where $v : \Omega \to \mathbf{C}$ is an unknown function. Then, studying the solvability of the non characteristic Cauchy problem for equation (1.11), in [17, Theorem 1] it is proved that the existence of a unique local C^∞ solution for all complex data close to a given one, implies that equation (1.11) is locally equivalent to a *hyperbolic*, *semilinear* equation, i.e., locally in $(x, \zeta, \xi) \in \Omega \times \Gamma$ there exist smooth functions $f(x, \zeta)$, $\mu_i(x)$ such that

$$G(x, \zeta, \xi) = 0 \quad \Leftrightarrow \quad \sum_{j=1}^{n} \mu_j(x)\xi_j + f(x, \zeta) = 0, \tag{1.12}$$

and the functions $\mu_j(x)$ are real.

In conclusion, the considerations above indicate that in order to prove the propagation of the analytic regularity in the C^∞ solutions of the *quasi-linear* equation (1.2) it is natural to consider *real-valued* solutions and that we need also some hyperbolic assumption, such as $a(s) \geq 0$ for all $s \in \mathbf{R}$. Namely, equation (1.2) must be weakly hyperbolic.

Remark 1.3. Finally, we observe that Theorem 1.1 could be easily extended to higher space dimensions. Namely, it is possible to prove a similar statement for the equation

$$u_{tt} - a(u)\Delta u = 0, \quad (x, t) \in \mathbf{R}^n \times [0, T), \tag{1.13}$$

assuming that $a(\cdot)$ satisfies (1.4). Here we confine ourselves to the *one* dimensional case to reduce the technicalities of the proof.

2. Notation

2.1. Main Notation

In what follows C, Λ (or, occasionally, $C_0, C_1, C_2, \ldots$ and $\Lambda_0, \Lambda_1 \ldots$) will stand for generic nonnegative constants.

Given $a, b \in \mathbf{R}$, we use the symbol $a \vee b$ for $\max\{a, b\}$; $a \wedge b$ denotes $\min\{a, b\}$.

We use the standard multi-index notation: a multi-index $\alpha = (\alpha_1, \dots \alpha_n)$ is a n-tuple of integers $\alpha_i \geq 0$, i.e. $\alpha \in (\mathbf{Z}^+)^n$ with $\mathbf{Z}^+ = \{0, 1, 2, \dots\}$. As usual

$$\alpha! = \alpha_1! \cdots \alpha_n!, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n, \tag{2.1}$$

Given $\alpha, \beta \in (\mathbf{Z}^+)^n$, we say that $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for $1 \leq i \leq n$. We also say that $\beta < \alpha$ if $\beta \leq \alpha$ and $|\beta| < |\alpha|$. For $\alpha, \beta \in (\mathbf{Z}^+)^n$ with $\beta \leq \alpha$, we set

$$\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha - \beta)!\beta!}. \tag{2.2}$$

In this work we always consider multi-indices $\alpha = (\alpha_1, \alpha_2) \in (\mathbf{Z}^+)^2$ and write

$$\partial^\alpha \stackrel{\text{def}}{=} \partial_x^{\alpha_1} \partial_t^{\alpha_2}. \tag{2.3}$$

Let $I \subset \mathbf{R}$ be an open interval. Given $f : I \to \mathbf{R}$, we write $f \in AC(I)$ if f is absolutely continuous in I.

Given $\Omega \subset \mathbf{R} \times \mathbf{R}$ and $g : \Omega \to \mathbf{R}$, we say that g is analytic in Ω if there exists an open set $\tilde{\Omega} \supset \Omega$ and $\tilde{g} : \tilde{\Omega} \to \mathbf{R}$ analytic such that

$$\tilde{g}|_\Omega = g. \tag{2.4}$$

Furthermore, if g is analytic in Ω, we say that g is *uniformly* analytic if there exist constants $C, \Lambda \geq 0$ such that, for all $\alpha \in (\mathbf{Z}^+)^2$,

$$|\partial^\alpha \tilde{g}(x, t)| \leq C \Lambda^{|\alpha|} \alpha! \quad \text{in } \Omega. \tag{2.5}$$

2.2. Notation for Influence Domains

Given $T > 0$ and $\tau_1, \tau_2 \in [0, T)$, with $\tau_1 < \tau_2$, let

$$\gamma_1, \gamma_2 : [\tau_1, \tau_2] \to \mathbf{R} \tag{2.6}$$

be C^1 functions such that

$$\gamma_1(t) < \gamma_2(t), \quad \gamma_2'(t) \leq 0 \leq \gamma_1'(t) \quad \text{for } t \in (\tau_1, \tau_2). \tag{2.7}$$

For $t \in [\tau_1, \tau_2]$, we introduce the domains:

$$B_t \stackrel{\text{def}}{=} \{x \in \mathbf{R} : \gamma_1(t) \leq x \leq \gamma_2(t)\}, \tag{2.8}$$

$$\Gamma_t \stackrel{\text{def}}{=} \{(x, s) : x \in B_s,\ \tau_1 \leq s \leq t\}, \tag{2.9}$$

$$\Gamma \stackrel{\text{def}}{=} \Gamma_{\tau_2}. \tag{2.10}$$

Let $v : \mathbf{R} \times [0, T) \to \mathbf{R}$ be a sufficiently regular function. Then, for $q \in [1, \infty]$ and $\alpha \in (\mathbf{Z}^+)^2$, we set

$$\|\partial^\alpha v\|_q = \|\partial^\alpha v(t)\|_q \stackrel{\text{def}}{=} \|\partial^\alpha v(\cdot, t)\|_{L^q(B_t)} \quad \text{for } t \in [\tau_1, \tau_2), \tag{2.11}$$

where $\partial^\alpha v(\cdot, t) = (\partial^\alpha v)(\cdot, t)$. Besides, given $j \in \mathbf{Z}^+$, we also define

$$\|\partial^j v\|_q = \|\partial^j v(t)\|_q \stackrel{\text{def}}{=} \sum_{|\alpha|=j} \|\partial^\alpha v(t)\|_q \quad \text{for } t \in [\tau_1, \tau_2). \tag{2.12}$$

3. Energy Estimates in a Influence Domain

Let $u(x, t)$ be a C^∞ solution of (1.2) in $\mathbf{R} \times [0, T)$. Besides, let $\Gamma \subset \mathbf{R} \times [0, T)$ be defined according to (2.6)–(2.10) above. From now on we assume the following:

Assumption 3.1. *The functions $\gamma_1(t)$, $\gamma_2(t)$ and $a(u(x, t))$ satisfy the conditions*:

i) $a(u(\gamma_i(t), t)) \le \gamma_i'(t)^2$ *for all* $t \in [\tau_1, \tau_2]$ *and* $i = 1, 2$;

ii) *there exists* $C_1 = C_1(\Gamma) \ge 0$ *such that* $\partial_t a(u) \le C_1 a(u)$ *in* Γ.

Remark 3.2. If (1.2) is strictly hyperbolic (that is $a(\cdot) \ge \eta > 0$), condition ii) is always verified. Besides, condition ii) holds if $a = a_0 a_1$ with $a_i : \mathbf{R} \to [0, \infty)$ $(i = 0, 1)$ differentiable functions such that:

$$\begin{gathered} \min_{(x,t)\in\Gamma} a_0(u(x, t)) > 0, \\ \partial_t a_1(u(x, t)) \le C a_1(u(x, t)) \quad \text{in } \Gamma. \end{gathered} \tag{3.1}$$

Proof. If $a = a_0 a_1$, we have $\partial_t a(u) = a_0(u)\partial_t a_1(u) + a_1(u)\partial_t a_0(u)$. Then (3.1) implies that ii) is verified with $C_1 = C + |\max_\Gamma a_0'(u)u_t|(\min_\Gamma a_0(u))^{-1}$. □

Definition 3.3. *Given $j \ge 1$, we introduce the j-th energies of the solution $u(x, t)$ by setting, for $\alpha \in (\mathbf{Z}^+)^2$, $|\alpha| = j - 1$,*

$$E_\alpha(t) \stackrel{\text{def}}{=} \int_{B_t} \{a(u)|\partial^\alpha u_x|^2 + |\partial^\alpha u_t|^2 + j^2|\partial^\alpha u|^2\}\, dx, \quad t \in [\tau_1, \tau_2) \tag{3.2}$$

and then

$$\sqrt{E_j(t)} \stackrel{\text{def}}{=} \sum_{|\alpha|=j-1} \sqrt{E_\alpha(t)}, \tag{3.3}$$

$$F_j(t) \stackrel{\text{def}}{=} E_j(t) + \int_{\tau_1}^{t} E_j(s)\, ds. \tag{3.4}$$

Lemma 3.4. *Let* $u \in C^\infty$ *be a solution of* (1.2) *in* $\mathbf{R} \times [0, T)$. *Besides, let us suppose that Assumption* 3.1 *holds. Then* $\sqrt{E_\alpha} \in AC(\tau_1, \tau_2)$ *and there exists* $C \geq 0$ *such that*

$$\frac{d}{dt}\sqrt{E_\alpha} \leq (C + j)\sqrt{E_\alpha} + \left(\int_{B_t} |G_\alpha|^2\, dx\right)^{1/2} \quad \text{a.e. in } (\tau_1, \tau_2), \tag{3.5}$$

for all $\alpha \in (\mathbf{Z}^+)^2$, *with*

$$G_\alpha \stackrel{\text{def}}{=} \partial^\alpha(a(u)u_{xx}) - a(u)\partial^\alpha u_{xx}. \tag{3.6}$$

Proof. Differentiating E_α, for $\tau_1 < t < \tau_2$, we find

$$\begin{aligned} \frac{d}{dt}E_\alpha = {} & \int_{B_t} \partial_t a(u)|\partial^\alpha u_x|^2\, dx \\ & + 2\int_{B_t} \{a(u)\partial^\alpha u_x \partial^\alpha u_{xt} + \partial^\alpha u_t \partial^\alpha u_{tt} + j^2 \partial^\alpha u \partial^\alpha u_t\}\, dx \\ & + \{a(u)|\partial^\alpha u|^2 + |\partial^\alpha u_t|^2 + j^2|\partial^\alpha u|^2\}|_{(\gamma_2(t), t)}\gamma_2'(t) \\ & - \{a(u)|\partial^\alpha u|^2 + |\partial^\alpha u_t|^2 + j^2|\partial^\alpha u|^2\}|_{(\gamma_1(t), t)}\gamma_1'(t). \end{aligned} \tag{3.7}$$

Integrating by parts, we have

$$\begin{aligned} \int_{B_t} a(u)\partial^\alpha u_x \partial^\alpha u_{xt}\, dx = {} & -\int_{B_t} a(u)\partial^\alpha u_{xx}\partial^\alpha u_t\, dx - \int_{B_t} \partial_x a(u)\partial^\alpha u_x \partial^\alpha u_t\, dx \\ & + a(u)\partial^\alpha u_x \partial^\alpha u_t|_{(\gamma_2(t), t)} - a(u)\partial^\alpha u_x \partial^\alpha u_t|_{(\gamma_1(t), t)}. \end{aligned} \tag{3.8}$$

Noting that

$$|a(u)\partial^\alpha u_x \partial^\alpha u_t| \leq \frac{\sqrt{a(u)}}{2}[a(u)|\partial^\alpha u_x|^2 + |\partial^\alpha u_t|^2], \tag{3.9}$$

and using i) of Assumption 3.1, it follows that in (3.7) the total contribution of the boundary terms is ≤ 0. Then, since

$$\partial^\alpha u_{tt} - a(u)\partial^\alpha u_{xx} = G_\alpha, \tag{3.10}$$

it follows that

$$\frac{d}{dt}E_\alpha \le \int_{B_t} \partial_t a(u)|\partial^\alpha u_x|^2\,dx$$
$$+ 2\int_{B_t} \{j^2 \partial^\alpha u \partial^\alpha u_t - \partial_x a(u)\partial^\alpha u_x \partial^\alpha u_t\}\,dx$$
$$+ 2\int_{B_t} G_\alpha \partial^\alpha u_t\,dx. \tag{3.11}$$

Furthermore, since $a(u(x,t)) \ge 0$ in $\mathbf{R}\times[0,T)$ and $\Gamma \subset \mathbf{R}\times[0,T)$ is compact, using the Gleaser inequality [15] it is easy to see that there exists $C_2 \ge 0$ such that

$$|\partial_x a(u)| \le C_2\sqrt{a(u)} \quad \text{for } (x,t)\in\Gamma. \tag{3.12}$$

Hence, we have

$$|\partial_x a(u)\partial^\alpha u_x \partial^\alpha u_t| \le C_2\sqrt{a(u)}|\partial^\alpha u_x|\,|\partial^\alpha u_t|$$
$$\le 2^{-1}C_2\{a(u)|\partial^\alpha u_x|^2 + |\partial^\alpha u_t|^2\}. \tag{3.13}$$

Then, using also ii) of Assumption 3.1, we obtain that

$$\frac{d}{dt}E_\alpha \le (C_1 + C_2)\int_{B_t}\{a(u)|\partial^\alpha u_x|^2 + |\partial_t u|^2\}\,dx$$
$$+ 2j\left(\int_{B_t}|\partial^\alpha u_t|^2\,dx\right)^{1/2}\left(\int_{B_t} j^2|\partial^\alpha u|^2\,dx\right)^{1/2}$$
$$+ 2\left(\int_{B_t}|G_\alpha|^2\,dx\right)^{1/2}\left(\int_{B_t}|\partial^\alpha u_t|^2\,dx\right)^{1/2}$$
$$\le 2(C_3 + j)E_\alpha + 2\sqrt{E_\alpha}\left(\int_{B_t}|G_\alpha|^2\,dx\right)^{1/2}, \tag{3.14}$$

with $C_3 = \frac{1}{2}(C_1 + C_2)$. This gives (3.5) when $E_\alpha > 0$. To conclude, we apply Lemma 8.1 (§8.1, Appendix A) and observe that $(\sqrt{E_\alpha})' = 0$ a.e. in $\{t \in (\tau_1, \tau_2) : E_\alpha = 0\}$. □

Setting

$$G_j(t) \overset{\text{def}}{=} \sum_{|\alpha| = j-1}\left(\int_{B_t}|G_\alpha(x,t)|^2\,dx\right)^{1/2}, \tag{3.15}$$

from the definitions of E_j and F_j we easily have:

Corollary 3.5. *Under the assumptions of Lemma* 3.4, $\sqrt{E_j}, \sqrt{F_j} \in AC(\tau_1, \tau_2)$ *and their derivatives satisfy*

$$(\sqrt{E_j})' \le (C + j)\sqrt{E_j} + G_j, \tag{3.16}$$

$$(\sqrt{F_j})' \le \left(C + \frac{1}{2} + j\right)\sqrt{E_j} + G_j \tag{3.17}$$

a.e. in (τ_1, τ_2), *where* $C \ge 0$ *is the same constant of* (3.5).

Proof. From (3.5) and the definition of E_j we immediately have

$$\begin{aligned}(\sqrt{E_j})' &= \sum_{|\alpha|=j-1} (\sqrt{E_\alpha})' \\ &\le \sum_{|\alpha|=j-1} (C + j)(\sqrt{E_\alpha}) + \sum_{|\alpha|=j-1} \left(\int_{B_t} |G_\alpha|^2 \, dx\right)^{1/2} \\ &= (C + j)\sqrt{E_j} + G_j, \end{aligned} \tag{3.18}$$

a.e. in (τ_1, τ_2). By Definition 3.3 it is clear that $F_j \in AC(\tau_1, \tau_2)$. Besides, a.e. in $\{E_j > 0\}$, we have the inequality

$$\begin{aligned}(\sqrt{F_j})' &= \frac{E_j' + E_j}{2\sqrt{F_j}} = \frac{E_j'}{2\sqrt{E_j}} \frac{\sqrt{E_j}}{\sqrt{F_j}} + \frac{E_j}{2\sqrt{F_j}} \\ &\le \left(C + \frac{1}{2} + j\right)\sqrt{E_j} + G_j. \end{aligned} \tag{3.19}$$

Finally, applying Lemma 8.1 (§8.1, Appendix A), and noting that $(\sqrt{F_j})' = 0$ a.e. in $\{E_j = 0\}$, we obtain the inequality (3.17) a.e. in (τ_1, τ_2). □

4. Estimate of the Terms G_j

Using the analyticity of $a : \mathbf{R} \to [0, \infty)$, we will estimate, for $j \ge 5$, the terms G_j defined in (3.15). To begin with, for $|\alpha| = j - 1 \ge 2$, we write

$$G_\alpha = \sum_{\mu < \alpha} \binom{\alpha}{\mu} \partial^{\alpha-\mu} a(u) \partial^\mu u_{xx} = I_\alpha + J_\alpha, \tag{4.1}$$

where

$$I_\alpha \overset{\text{def}}{=} \sum_{\beta<\alpha,\,|\beta|=1} \binom{\alpha}{\alpha-\beta} \partial^\beta a(u) \partial^{\alpha-\beta} u_{xx}, \tag{4.2}$$

$$J_\alpha \overset{\text{def}}{=} \sum_{\mu<\alpha,\,|\mu|\le|\alpha|-2} \binom{\alpha}{\mu} \partial^{\alpha-\mu} a(u) \partial^\mu u_{xx}. \tag{4.3}$$

Estimate of the terms I_α.

For $|\alpha| = j-1$ and $\beta \le \alpha$, $|\beta| = 1$, one has

$$\binom{\alpha}{\alpha-\beta} \le j-1. \tag{4.4}$$

Besides, applying condition ii) of Assumption 3.1 (in the case $\partial^\beta = \partial_t$) and the inequality (3.12) (if $\partial^\beta = \partial_x$) we deduce that there exists $C = C(C_1, C_2) > 0$ such that

$$|\partial^\beta a(u)| \le C\sqrt{a(u)} \quad \text{in } \Gamma, \tag{4.5}$$

for $|\beta| = 1$. This means that

$$|I_\alpha| \le Cj \sum_{\beta<\alpha,\,|\beta|=1} \sqrt{a(u)} |\partial^{\alpha-\beta+e_1} u_x|, \tag{4.6}$$

where $\partial^{e_1} = \partial_x$, i.e. $e_1 = (1,0)$. Hence, we have

$$\begin{aligned} \sum_{|\alpha|=j-1} \left(\int_{B_t} |I_\alpha|^2 \, dx \right)^{1/2} &\le Cj \sum_{|\alpha|=j-1} \sum_{\beta<\alpha,\,|\beta|=1} \sqrt{E_{\alpha-\beta+e_1}} \\ &\le 2Cj \sum_{|\alpha|=j-1} \sqrt{E_\alpha} = 2Cj\sqrt{E_j}. \end{aligned} \tag{4.7}$$

Estimate of the terms J_α.

To estimate $\sum_{|\alpha|=j-1} \|J_\alpha\|_{L^2(B_t)}$, we will suppose that:

Assumption 4.1. *There exist $C, M > 0$ such that, for all integers $\nu \ge 0$, one has*

$$|a^{(\nu)}(s)| \le CM^\nu \nu! \quad \textit{for all } s \in u(\Gamma), \tag{4.8}$$

where $u(\Gamma) = \{s \mid s = u(x,t) \text{ with } (x,t) \in \Gamma\}$.

Remark 4.2. In view of the analyticity of the function $a(s)$, it is not restrictive to assume that Assumption 4.1 holds.

For $\alpha \in (\mathbf{Z}^+)^2$, $|\alpha| \geq 4$ (that is $j \geq 5$) we can write

$$J_\alpha = H_\alpha + K_\alpha + L_\alpha, \tag{4.9}$$

where

$$H_\alpha \stackrel{\text{def}}{=} \partial^\alpha a(u) u_{xx}, \tag{4.10}$$

$$K_\alpha \stackrel{\text{def}}{=} \sum_{\mu<\alpha,\, |\mu|=1} \binom{\alpha}{\mu} \partial^{\alpha-\mu} a(u) \partial^\mu u_{xx}, \tag{4.11}$$

$$L_\alpha \stackrel{\text{def}}{=} \sum_{\substack{2 \leq |\mu| \leq |\alpha|-2 \\ \mu \leq \alpha}} \binom{\alpha}{\mu} \partial^{\alpha-\mu} a(u) \partial^\mu u_{xx}. \tag{4.12}$$

Then, by Leibniz' formula (§ 8.2, Appendix A), we have

$$\begin{aligned}
\sum_{|\alpha|=j-1} \|H_\alpha\|_2 &\leq \|u_{xx}\|_\infty \sum_{|\alpha|=j-1} \|\partial^\alpha a(u)\|_2 \\
&\leq C \sum_{|\alpha|=j-1} \sum_{v=1}^{j-1} \frac{\|a^{(v)}\|_\infty}{v!} \sum_{\substack{\beta_1+\cdots+\beta_v=\alpha \\ |\beta_i| \geq 1}} \frac{\alpha!}{\beta_1! \cdots \beta_v!} \|\partial^{\beta_1} u \cdots \partial^{\beta_v} u\|_2 \\
&\leq C \sum_{|\alpha|=j-1} \sum_{v=1}^{j-1} M^v \sum_{\substack{\beta_1+\cdots+\beta_v=\alpha \\ |\beta_i| \geq 1}} \frac{\alpha!}{\beta_1! \cdots \beta_v!} \|\partial^{\beta_1} u \cdots \partial^{\beta_v} u\|_2 \\
&= C \sum_{v=1}^{j-1} M^v \sum_{|\alpha|=j-1} \sum_{\substack{\beta_1+\cdots+\beta_v=\alpha \\ |\beta_i| \geq 1}} \frac{\alpha!}{\beta_1! \cdots \beta_v!} \|\partial^{\beta_1} u \cdots \partial^{\beta_v} u\|_2,
\end{aligned} \tag{4.13}$$

where, according to (2.11), $\|\cdot\|_q = \|\cdot\|_{L^q(B_t)}$, for $q \in [1, +\infty]$. Now, the function

$$\phi(\beta_1, \ldots, \beta_v) \stackrel{\text{def}}{=} \frac{\|\partial^{\beta_1} u \cdots \partial^{\beta_v} u\|_2}{\beta_1! \cdots \beta_v!} \tag{4.14}$$

is nonnegative and symmetric with respect to $\beta_1, \ldots, \beta_v \in (\mathbf{Z}^+)^2$. Besides, for every fixed $\alpha \in (\mathbf{Z}^+)^2$, $|\alpha| \geq v$, the set $\{(\beta_1, \ldots, \beta_v) : \beta_1 + \cdots + \beta_v = \alpha, |\beta_i| \geq 1\}$ is also symmetric. Hence, we can easily see that

$$\sum_{\substack{\beta_1+\cdots+\beta_v=\alpha \\ |\beta_i| \geq 1}} \phi \leq v \sum_{\substack{\beta_1+\cdots+\beta_v=\alpha \\ 1 \leq |\beta_i| \leq |\beta_v|}} \phi. \tag{4.15}$$

Furthermore, changing the order of summation, we also have

$$\sum_{|\alpha|=j-1}\ \sum_{\substack{\beta_1+\cdots+\beta_\nu=\alpha\\ 1\le|\beta_i|\le|\beta_\nu|}}\phi=\sum_{\substack{h_1+\cdots+h_\nu=j-1\\ 1\le h_i\le h_\nu}}\ \sum_{|\beta_1|=h_1}\cdots\sum_{|\beta_\nu|=h_\nu}\phi. \tag{4.16}$$

See §8.3, Appendix A. Thus, noting that $\beta_1+\cdots+\beta_\nu=\alpha$ implies

$$\frac{\alpha!}{\beta_1!\cdots\beta_\nu!}\le\frac{|\alpha|!}{|\beta_1|!\cdots|\beta_\nu|!}, \tag{4.17}$$

after some calculations, we may write

$$\begin{aligned}
\sum_{|\alpha|=j-1}\|H_\alpha\|_2 &\le C\sum_{\nu=1}^{j-1}M^\nu\nu\sum_{|\alpha|=j-1}\ \sum_{\substack{\beta_1+\cdots+\beta_\nu=\alpha\\ 1\le|\beta_i|\le|\beta_\nu|}}\frac{\alpha!}{\beta_1!\cdots\beta_\nu!}\|\partial^{\beta_1}u\cdots\partial^{\beta_\nu}u\|_2\\
&\le C\sum_{\nu=1}^{j-1}M^\nu\nu\sum_{\substack{h_1+\cdots+h_\nu=j-1\\ 1\le h_i\le h_\nu}}\ \sum_{|\beta_1|=h_1}\cdots\sum_{|\beta_\nu|=h_\nu}\frac{(j-1)!}{|\beta_1|!\cdots|\beta_\nu|!}\|\partial^{\beta_1}u\|_\infty\cdots\\
&\quad\cdots\|\partial^{\beta_{\nu-1}}u\|_\infty\|\partial^{\beta_\nu}u\|_2\\
&\le C(j-1)!\sum_{\nu=1}^{j-1}M^\nu\nu\sum_{\substack{h_1+\cdots+h_\nu=j-1\\ 1\le h_i\le h_\nu}}\frac{1}{h_1!\cdots h_\nu!}\sum_{|\beta_1|=h_1}\|\partial^{\beta_1}u\|_\infty\cdots\\
&\quad\cdots\sum_{|\beta_{\nu-1}|=h_{\nu-1}}\|\partial^{\beta_{\nu-1}}u\|_\infty\sum_{|\beta_\nu|=h_\nu}\|\partial^{\beta_\nu}u\|_2\\
&= C(j-1)!\sum_{\nu=1}^{j-1}M^\nu\nu\sum_{\substack{h_1+\cdots+h_\nu=j-1\\ 1\le h_i\le h_\nu}}\frac{\|\partial^{h_1}u\|_\infty}{h_1!}\cdots\\
&\quad\cdots\frac{\|\partial^{h_{\nu-1}}u\|_\infty}{h_{\nu-1}!}\frac{\|\partial^{h_\nu}u\|_2}{h_\nu!},
\end{aligned} \tag{4.18}$$

where, according to the notation (2.11)–(2.12), $\sum_{|\beta|=h}\|\partial^\beta u\|_q=\|\partial^h u\|_q$.

In the same way, we can estimate $\sum_{|\alpha|=j-1}\|K_\alpha\|_2$. In fact, since $|\alpha|=j-1$, $\mu<\alpha$ with $|\mu|=1$, we have

$$\binom{\alpha}{\mu}\le\frac{|\alpha|!}{|\alpha-\mu|!|\mu|!}\le j-1, \tag{4.19}$$

and

$$\sum_{|\alpha|=j-1} \|K_\alpha\|_2 \le (j-1) \sum_{|\mu|=1} \|\partial^\mu u_{xx}\|_\infty \sum_{|\beta|=j-2} \|\partial^\beta a(u)\|_2$$

$$\le C(j-1) \sum_{|\beta|=j-2} \|\partial^\beta a(u)\|_2. \tag{4.20}$$

Then, as in the previous estimates, from (4.10) to (4.18), with $j-2$ instead of $j-1$, we obtain that the quantity $\sum_{|\alpha|=j-1} \|K_\alpha\|_2$ is majorized by

$$C(j-1)! \sum_{\nu=1}^{j-2} M^\nu \nu \sum_{\substack{h_1+\cdots+h_\nu=j-2 \\ 1\le h_i \le h_\nu}} \frac{\|\partial^{h_1} u\|_\infty}{h_1!} \cdots \frac{\|\partial^{h_{\nu-1}} u\|_\infty}{h_{\nu-1}!} \frac{\|\partial^{h_\nu} u\|_2}{h_\nu!}. \tag{4.21}$$

Finally, we estimate $\sum_{|\alpha|=j-1} \|L_\alpha\|_2$. We have

$$\sum_{|\alpha|=j-1} \|L_\alpha\|_2 \le \sum_{|\alpha|=j-1} \sum_{\substack{2\le|\mu|\le|\alpha|-2 \\ \mu\le\alpha}} \binom{\alpha}{\mu} \|\partial^{\alpha-\mu} a(u) \partial^\mu u_{xx}\|_2$$

$$\le C \sum_{|\alpha|=j-1} \sum_{\substack{2\le|\mu|\le j-3 \\ \mu\le\alpha}} \binom{\alpha}{\mu} \sum_{\nu=1}^{j-|\mu|-1} M^\nu$$

$$\times \sum_{\substack{\beta_1+\cdots+\beta_\nu=\alpha-\mu \\ |\beta_i|\ge 1}} \frac{(\alpha-\mu)!}{\beta_1!\cdots\beta_\nu!} \|\partial^{\beta_1} u \cdots \partial^{\beta_\nu} u \partial^\mu u_{xx}\|_2$$

$$\le C \sum_{\nu=1}^{j-3} M^\nu \sum_{|\alpha|=j-1} \sum_{\substack{\beta_1+\cdots+\beta_\nu+\mu=\alpha \\ |\beta_i|\ge 1,\, 2\le|\mu|\le j-3}}$$

$$\times \frac{\alpha!}{\beta_1!\cdots\beta_\nu!\mu!} \|\partial^{\beta_1} u \cdots \partial^{\beta_\nu} u \partial^\mu u_{xx}\|_2. \tag{4.22}$$

Now, we observe that $\beta_1 + \cdots + \beta_\nu + \mu = \alpha$ and $|\mu| \ge 2 \Rightarrow |\beta_i| \le j-3$. Hence, setting $\mu = \beta_{\nu+1}$, we easily have the inequality

$$\sum_{\substack{\beta_1+\cdots+\beta_\nu+\mu=\alpha \\ |\beta_i|\ge 1,\, 2\le|\mu|\le j-3}} \frac{\alpha!}{\beta_1!\cdots\beta_\nu!\mu!} \|\partial^{\beta_1} u \cdots \partial^{\beta_\nu} u \partial^\mu u_{xx}\|_2$$

$$\le \sum_{\substack{\beta_1+\cdots+\beta_{\nu+1}=\alpha \\ 1\le|\beta_i|\le j-3}} \frac{\alpha!}{\beta_1!\cdots\beta_{\nu+1}!} \left\| \prod_{i=1}^{\nu+1} (|\partial^{\beta_i} u| + |\partial^{\beta_i} u_{xx}|) \right\|_2. \tag{4.23}$$

Thus, we may conclude that

$$\sum_{|\alpha|=j-1} \|L_\alpha\|_2 \le C \sum_{v=2}^{j-2} M^{v-1} \sum_{|\alpha|=j-1} \sum_{\substack{\beta_1+\cdots+\beta_v=\alpha \\ 1\le|\beta_i|\le j-3}} \frac{\alpha!}{\beta_1!\cdots\beta_v!} \left\| \prod_{i=1}^{v} (|\partial^{\beta_i} u| + |\partial^{\beta_i} u_{xx}|) \right\|_2$$

$$\le C(j-1)! \sum_{v=2}^{j-2} M^v v^2 \sum_{|\alpha|=j-1} \sum_{\substack{\beta_1+\cdots+\beta_v=\alpha \\ 1\le|\beta_1|\le|\beta_i|\le|\beta_v|\le j-3}}$$

$$\times \left(\prod_{i=1}^{v-1} \frac{\|\partial^{\beta_i} u\|_\infty + \|\partial^{\beta_i} u_{xx}\|_\infty}{|\beta_i|!} \right) \frac{\|\partial^{\beta_v} u\|_2 + \|\partial^{\beta_v} u_{xx}\|_2}{|\beta_v|!}, \tag{4.24}$$

where, noting that $v \ge 2$, we have applied the argument (4.14)–(4.15) twice. From this we obtain that the quantity $\sum_{|\alpha|=j-1} \|L_\alpha\|_2$ is majorized by

$$C(j-1)! \sum_{v=2}^{j-2} M^v v^2 \sum_{\substack{h_1+\cdots+h_v=j-1 \\ 1\le h_1\le h_i\le h_v\le j-3}} \left(\prod_{i=1}^{v-1} \frac{\|\partial^{h_i} u\|_\infty + \|\partial^{h_i} u_{xx}\|_\infty}{h_i!} \right)$$

$$\cdot \frac{\|\partial^{h_v} u\|_2 + \|\partial^{h_v} u_{xx}\|_2}{h_v!}. \tag{4.25}$$

5. Analytic Energy

To proceed further, we make the following assumption:

Assumption 5.1. *There exist $C > 0$ and $p \in \mathbf{N}$ such that, for all integers $h \ge 0$,*

$$\|\partial^h u(t)\|_\infty \le C \sum_{i=1}^{p} \sqrt{F_{h+i}(t)} \quad \textit{for} \ \ t \in [\tau_1, \tau_2). \tag{5.1}$$

Remark 5.2. In applying our estimates, in the final part of the paper (see Lemma 7.2), we will verify that (5.1) holds with a constant C independent of $h \ge 0$.

Let k be a fixed integer, such that

$$k \ge p + 4, \tag{5.2}$$

and let $\rho : [\tau_1, \tau_2] \to \mathbf{R}$ be a C^1 function (which will be defined in Lemma 6.2) such that

$$\rho(t) \in (0,1], \quad \rho'(t) \le 0 \quad \text{in } [\tau_1, \tau_2]. \tag{5.3}$$

Definition 5.3. *For $N \ge k+1$ we introduce the energy-functions*

$$\mathscr{E}_N \stackrel{\text{def}}{=} \rho + \sum_{j=k+1}^{N} \frac{\rho^{j-k}}{j!} j^k \sqrt{F_j} \quad \textit{for } t \in [\tau_1, \tau_2). \tag{5.4}$$

Then, deriving $\mathscr{E}_N$, from Corollary 3.5 we find

$$\begin{aligned} \mathscr{E}_N' &= \rho' + \sum_{j=k+1}^{N} \frac{\rho^{j-k-1}}{(j-1)!} j^k \frac{j-k}{j} \rho' \sqrt{F_j} + \sum_{k=j+1}^{N} \frac{\rho^{j-k}}{j!} j^k (\sqrt{F_j})' \\ &\le \rho' + \sum_{j=k+1}^{N} \frac{\rho^{j-k-1}}{(j-1)!} j^k \left[\frac{j-k}{j} \rho' + \frac{C+j}{j} \rho \right] \sqrt{F_j} + \sum_{k=j+1}^{N} \frac{\rho^{j-k}}{j!} j^k G_j, \end{aligned} \tag{5.5}$$

where, by the relations (3.15), (4.1) and (4.7),

$$\begin{aligned} \sum_{j=k+1}^{N} \frac{\rho^{j-k}}{j!} j^k G_j &\le \sum_{j=k+1}^{N} \frac{\rho^{j-k}}{j!} j^k \sum_{|\alpha|=j-1} (\|I_\alpha\|_2 + \|J_\alpha\|_2) \\ &\le C \sum_{k=j+1}^{N} \frac{\rho^{j-k}}{(j-1)!} j^k \sqrt{E_j} + \sum_{j=k+1}^{N} \frac{\rho^{j-k}}{j!} j^k \sum_{|\alpha|=j-1} \|J_\alpha\|_2. \end{aligned} \tag{5.6}$$

Recalling (4.9), we have $J_\alpha = H_\alpha + K_\alpha + L_\alpha$. Therefore, we must estimate the quantities:

$$H_N \stackrel{\text{def}}{=} \sum_{j=k+1}^{N} \frac{\rho^{j-k}}{j!} j^k \sum_{|\alpha|=j-1} \|H_\alpha\|_2, \tag{5.7}$$

$$K_N \stackrel{\text{def}}{=} \sum_{j=k+1}^{N} \frac{\rho^{j-k}}{j!} j^k \sum_{|\alpha|=j-1} \|K_\alpha\|_2, \tag{5.8}$$

$$L_N \stackrel{\text{def}}{=} \sum_{j=k+1}^{N} \frac{\rho^{j-k}}{j!} j^k \sum_{|\alpha|=j-1} \|L_\alpha\|_2. \tag{5.9}$$

To this aim, we set:

Definition 5.4.

$$\eta_j \overset{\text{def}}{=} \begin{cases} \rho/k & \text{if } 1 \le j \le k \\ \frac{\rho^{j-k}}{j!} j^k \sqrt{F_j} & \text{if } j \ge k+1 \end{cases} \tag{5.10}$$

Thus, we have

$$\mathscr{E}_N = \sum_{j=1}^{N} \eta_j. \tag{5.11}$$

Besides, we suppose that:

Assumption 5.5.

$$\|\partial^h u(t)\|_\infty \le C, \quad \|\partial^h u(t)\|_2 \le C, \tag{5.12}$$

for $t \in [\tau_1, \tau_2)$ *and* $h \le k+1$.

It is clear that Assumption 5.5 is always verified if we suppose $u(x,t)$ of class C^∞ in $\mathbf{R} \times [0,T)$ and $0 \le \tau_1 < \tau_2 < T$.

Estimate of H_N.

From (4.18) we have

$$H_N \le \sum_{j=k+1}^{N} \rho^{j-k} j^{k-1} \sum_{v=1}^{j-1} M^v v \sum_{\substack{h_1+\cdots+h_v=j-1 \\ 1 \le h_i \le h_v}} \frac{\|\partial^{h_1} u\|_\infty}{h_1!} \cdots \frac{\|\partial^{h_{v-1}} u\|_\infty}{h_{v-1}!} \frac{\|\partial^{h_v} u\|_2}{h_v!}. \tag{5.13}$$

Then, to estimate H_N, we can write

$$H_N \le H_{N,I} + H_{N,II}, \tag{5.14}$$

where $H_{N,I}$ groups the terms, in the right-hand side of (5.13), in which $h_v < k$; $H_{N,II}$ groups the terms with $h_v \ge k$. From (5.12), for $1 \le v \le j-1$, we have

$$\sum_{\substack{h_1+\cdots+h_v=j-1 \\ 1 \le h_i \le h_v < k}} \frac{\|\partial^{h_1} u\|_\infty}{h_1!} \cdots \frac{\|\partial^{h_{v-1}} u\|_\infty}{h_{v-1}!} \frac{\|\partial^{h_v} u\|_2}{h_v!} \le C^v \sum_{\substack{h_1+\cdots+h_v=j-1 \\ 0 \le h_i \le j-1}} \frac{1}{h_1! \cdots h_v!}$$

$$= C^v \frac{v^{j-1}}{(j-1)!} \le C^v \frac{(j-1)^{j-1}}{(j-1)!} \le C^v e^{j-1}. \tag{5.15}$$

Hence, provided $\rho > 0$ is small enough, we can easily see that, $\forall N \geq k+1$,

$$H_{N,I} \leq e^k \rho \sum_{j=k+1}^{N} (\rho e)^{j-k-1} j^{k-1} \sum_{v=1}^{j-1} M^v C^v v \leq C_4 \rho, \tag{5.16}$$

with $C_4 > 0$ a constant independent of N. For instance, (5.16) holds if

$$0 < \rho \leq \frac{1}{e(2+2MC)}. \tag{5.17}$$

Let us estimate $H_{N,II}$, where $h_v \geq k$. Using Assumption 5.1, we have

$$\begin{aligned} H_{N,II} \leq & \sum_{j=k+1}^{N} \rho^{j-k} j^{k-1} \sum_{v=1}^{j-1} M^v v \sum_{\substack{h_1+\cdots+h_v=j-1 \\ 1\leq h_i\leq h_v,\, h_v\geq k}} \prod_{i=1}^{v-1} \left(\frac{C}{h_i!} \sum_{r=1}^{p} \sqrt{F_{h_i+r}} \right) \frac{\sqrt{F_{h_v+1}}}{(h_v+1)!} \\ \leq & \sum_{j=k+1}^{N} j^{k-1} \sum_{v=1}^{j-1} M^v C^{v-1} v \sum_{\substack{h_1+\cdots+h_v=j-1 \\ 1\leq h_i\leq h_v,\, h_v\geq k}} \prod_{i=1}^{v-1} \left(\frac{\rho^{h_i}}{h_i!} \sum_{r=1}^{p} \sqrt{F_{h_i+r}} \right) \\ & \cdot \rho^{h_v+1-k} \frac{\sqrt{F_{h_v+1}}}{(h_v+1)!} \end{aligned} \tag{5.18}$$

where, since $p \leq k-4 \leq h_v - 4$, the terms $h_i + r$ satisfy

$$h_i + r \leq h_i + p \leq h_i + h_v - 4 \leq N. \tag{5.19}$$

Now, by Assumption 5.5 and Definition 5.4 for $h, r \geq 1$ we have

$$\frac{\rho^h}{h!} \sqrt{F_{h+r}} \leq \eta_{h+r} \rho^{k-r} \frac{(h+r)\cdots(h+1)}{(h+r)^k} \quad \text{if } h+r > k, \tag{5.20}$$

$$\frac{\rho^h}{h!} \sqrt{F_{h+r}} \leq \eta_{h+r} C \rho^{h-1} \frac{k}{h!} \quad \text{if } h+r \leq k. \tag{5.21}$$

Since $k \geq p+4$ and $0 < \rho(t) \leq 1$, we certainly have:

$$\frac{\rho^h}{h!} \sqrt{F_{h+r}} \leq C \eta_{h+r} \quad \text{for all } h \geq 1,\ 1 \leq r \leq p, \tag{5.22}$$

with $C > 0$ a suitable constant. Hence, we obtain

$$H_{N,II} \leq C \sum_{j=k+1}^{N} j^{k-1} \sum_{v=1}^{j-1} M^v C^v v \sum_{\substack{h_1+\cdots+h_v=j-1 \\ 1\leq h_i\leq h_v,\, h_v\geq k}} \prod_{i=1}^{v-1} \left(\sum_{r=1}^{p} \eta_{h_i+r} \right) \frac{\eta_{h_v+1}}{(h_v+1)^k}. \tag{5.23}$$

Noting that

$$\frac{j}{h_\nu + 1} \le \nu,$$

and recalling (5.11), changing the order of summation over the indices j, ν, h_ν and using the inequality (8.11) of §8.3, Appendix A, we have

$$\begin{aligned}
H_{N,II} &\le C \sum_{j=k+1}^{N} \sum_{\nu=1}^{j-1} M^\nu C^\nu \nu^k \sum_{\substack{h_1+\cdots+h_\nu=j-1\\ 1\le h_i\le h_\nu,\, h_\nu\ge k}} \frac{\eta_{h_\nu+1}}{h_\nu+1} \prod_{i=1}^{\nu-1}\left(\sum_{r=1}^{p} \eta_{h_i+r}\right)\\
&\le C \sum_{\nu=1}^{N-1} M^\nu C^\nu \nu^k \sum_{j=(\nu+1)\vee(k+1)}^{N} \sum_{h_\nu=k}^{j-\nu} \frac{\eta_{h_\nu+1}}{h_\nu+1} \sum_{\{\star\}} \prod_{i=1}^{\nu-1}\left(\sum_{r=1}^{p} \eta_{h_i+r}\right)\\
&\le C \sum_{\nu=1}^{N-1} M^\nu C^\nu \nu^k \sum_{h_\nu=k}^{N-\nu} \frac{\eta_{h_\nu+1}}{h_\nu+1} \sum_{j=h_\nu+\nu}^{N} \sum_{\{\star\}} \prod_{i=1}^{\nu-1}\left(\sum_{r=1}^{p} \eta_{h_i+r}\right)\\
&\le C \sum_{\nu=1}^{N-1} M^\nu C^\nu \nu^k \sum_{h_\nu=k}^{N-\nu} \frac{\eta_{h_\nu+1}}{h_\nu+1} p^{\nu-1} \mathscr{E}_N^{\nu-1}\\
&\le C \sum_{h_\nu=k}^{N-1} \frac{\eta_{h_\nu+1}}{h_\nu+1} \sum_{\nu=1}^{N-h_\nu} M^\nu C^\nu \nu^k p^{\nu-1} \mathscr{E}_N^{\nu-1},
\end{aligned} \tag{5.24}$$

where $(\nu+1)\vee(k+1) = \max\{\nu+1, k+1\}$ and $\{\star\}$ denotes the set of conditions:

$$\begin{aligned} h_1 + \cdots + h_{\nu-1} &= j - 1 - h_\nu,\\ 1 \le h_i &\le h_\nu. \end{aligned} \tag{5.25}$$

If, for instance, $|\mathscr{E}_N| < \frac{1}{pMC}$, then we have

$$H_{N,II} \le \Phi_1(\mathscr{E}_N) \sum_{h_\nu=k}^{N-1} \frac{\eta_{h_\nu+1}}{h_\nu+1} = \Phi_1(\mathscr{E}_N) \sum_{j=k+1}^{N} \frac{\eta_j}{j}, \tag{5.26}$$

where $\Phi_1(\cdot)$ is the analytic function

$$\Phi_1(s) \stackrel{\text{def}}{=} C \sum_{\nu=1}^{\infty} M^\nu C^\nu \nu^k p^{\nu-1} s^{\nu-1}, \quad \text{for } |s| < \frac{1}{pMC}. \tag{5.27}$$

Summarizing up the estimates (5.16), (5.26) we obtain that, $\forall N \geq k+1$,

$$H_N \leq C\rho + \Phi_1(\mathscr{E}_N) \sum_{j=k+1}^{N} \frac{\eta_j}{j}, \tag{5.28}$$

provided $|\mathscr{E}_N| < \frac{1}{pMC}$.

<u>*Estimate of K_N.*</u>

From (4.21) we have

$$K_N \leq \sum_{j=k+1}^{N} \rho^{j-k} j^{k-1} \sum_{v=1}^{j-2} M^v v \sum_{\substack{h_1+\cdots+h_v=j-2 \\ 1 \leq h_i \leq h_v}} \frac{\|\partial^{h_1} u\|_\infty}{h_1!} \cdots \frac{\|\partial^{h_{v-1}} u\|_\infty}{h_{v-1}!} \frac{\|\partial^{h_v} u\|_2}{h_v!}. \tag{5.29}$$

The estimate of K_N is similar to that of H_N and we finally obtain that, $\forall N \geq k+1$,

$$K_N \leq C\rho + \rho\Phi_1(\mathscr{E}_N) \sum_{j=k+1}^{N} \frac{\eta_j}{j} \quad \text{if } |\mathscr{E}_N| < \frac{1}{pMC}. \tag{5.30}$$

<u>*Estimate of L_N.*</u>

For brevity, we will denote with $\{*\}$ the set of conditions:

$$h_1 + \cdots + h_v = j-1, \quad 1 \leq h_1 \leq h_i \leq h_v \leq j-3. \tag{5.31}$$

From (4.25) we have

$$L_N \leq C \sum_{j=k+1}^{N} \rho^{j-k} j^{k-1} \sum_{v=2}^{j-2} M^v v^2 \sum_{\{*\}} \left(\prod_{i=1}^{v-1} \frac{\|\partial^{h_i} u\|_\infty + \|\partial^{h_i+2} u\|_\infty}{h_i!} \right) \cdot \frac{\|\partial^{h_v} u\|_2 + \|\partial^{h_v+2} u\|_2}{h_v!}. \tag{5.32}$$

As above we write

$$L_N \leq L'_N + L''_N,$$

where L'_N groups the terms, in the right-hand side of (5.32), in which $h_v < k$ and L''_N the terms with $h_v \geq k$. If $\rho > 0$ is sufficiently small, using Assumption 5.5 and the same arguments of the estimate of $H_{N,I}$, we obtain that

$$L'_N \leq C_5\rho, \tag{5.33}$$

with $C_5 > 0$ independent of N.

To continue, we can write $L''_N = L''_{N,0} + L''_{N,2}$ where

$$L''_{N,i} = C \sum_{j=k+1}^{N} \rho^{j-k} j^{k-1} \sum_{v=2}^{j-2} M^v v^2 \sum_{\{*\},h_v \geq k} \left(\prod_{i=1}^{v-1} \frac{\|\partial^{h_i} u\|_\infty + \|\partial^{h_i+2} u\|_\infty}{h_i!} \right)$$

$$\cdot \frac{\|\partial^{h_v+i} u\|_2}{h_v!} \quad \text{for } i = 0, 2. \tag{5.34}$$

By Assumption 5.1, we have

$$\|\partial^h u\|_\infty + \|\partial^{h+2} u\|_\infty \leq C \sum_{i=1}^{p+2} \sqrt{F_{h+i}}. \tag{5.35}$$

Thus, $L''_{N,0}$ can be estimated as $H_{N,II}$ because $k \geq p+2$. To estimate $L''_{N,2}$, using (5.35), we can write

$$L''_{N,2} \leq C \sum_{j=k+1}^{N} j^{k-1} \sum_{v=2}^{j-2} M^v C^v v^2 \sum_{\{*\},h_v \geq k} \left(\prod_{i=1}^{v-1} \frac{\rho^{h_i}}{h_i!} \sum_{r=1}^{p+2} \sqrt{F_{h_i+r}} \right)$$

$$\cdot \rho^{h_v+1-k} \frac{\sqrt{F_{h_v+3}}}{h_v!(h_v+3)}, \tag{5.36}$$

where, by (5.2),

$$h_i + r \leq h_i + p + 2 \leq N. \tag{5.37}$$

Hence

$$L''_{N,2} \leq L''_{N,3} + L''_{N,4} + L''_{N,5}, \tag{5.38}$$

where $L''_{N,3}$ groups the terms, in the right-hand side of (5.36), in which $h_1 \geq 3$; $L''_{N,4}$ groups the terms with $h_1 = 2$ and $L''_{N,5}$ the terms with $h_1 = 1$.

Since $k \geq p+4$, by (5.20), (5.21) we easily have

$$\frac{\rho^h}{h!} \sqrt{F_{h+r}} \leq C\rho^2 \eta_{h+r} \quad \text{for all } h \geq 3,\ 1 \leq r \leq p+2. \tag{5.39}$$

Thus, noting that $(h_v + 3)v \geq j$, we obtain

$$L''_{N,3} = C \sum_{j=k+1}^{N} j^{k-1} \sum_{v=2}^{j-2} M^v C^v v^2 \sum_{\{*\},h_1 \geq 3,h_v \geq k} \left(\prod_{i=1}^{v-1} \frac{\rho^{h_i}}{h_i!} \sum_{r=1}^{p+2} \sqrt{F_{h_i+r}} \right)$$

$$\cdot \rho^{h_v+1-k} \frac{\sqrt{F_{h_v+3}}}{h_v!(h_v+3)}$$

$$\leq C \sum_{j=k+1}^{N} j^{k-1} \sum_{v=2}^{j-2} M^v C^v v^2 \sum_{\{*\}, h_1 \geq 3, h_v \geq k} \frac{\eta_{h_v+3}}{(h_v+3)^{k-2}} \left(\prod_{i=1}^{v-1} \sum_{r=1}^{p+2} \eta_{h_i+r} \right)$$

$$\leq C \sum_{j=k+1}^{N} \sum_{v=2}^{j-2} M^v C^v v^{k+1} \sum_{\{*\}, h_1 \geq 3, h_v \geq k} (h_v+3)\eta_{h_v+3} \left(\prod_{i=1}^{v-1} \sum_{r=1}^{p+2} \eta_{h_i+r} \right). \tag{5.40}$$

After some calculations, similar to those of (5.24), this leads to the inequality

$$L''_{N,3} \leq \Phi_2(\mathscr{E}_N) \sum_{j=k+1}^{N} j\eta_j \quad \text{for } |\mathscr{E}_N| < \frac{1}{(p+2)MC}, \tag{5.41}$$

where $\Phi_2(\cdot)$ is a suitable analytic function.

To continue, let us estimate $L''_{N,4}$. In this case from Assumption 5.5 we deduce that

$$\frac{\rho^{h_1}}{h_1!} \sum_{r=1}^{p+2} \sqrt{F_{h_1+r}} \leq C\rho^2, \tag{5.42}$$

because $h_1 + p + 2 = p + 4 \leq k$. Hence, we obtain

$$L''_{N,4} = C \sum_{j=k+1}^{N} j^{k-1} \sum_{v=2}^{j-2} M^v C^{v-1} v^2 \sum_{\{*\}, h_1 = 2, h_v \geq k} \left(\prod_{i=1}^{v-1} \frac{\rho^{h_i}}{h_i!} \sum_{r=1}^{p+2} \sqrt{F_{h_i+r}} \right)$$

$$\cdot \rho^{h_v+1-k} \frac{\sqrt{F_{h_v+3}}}{h_v!(h_v+3)}$$

$$= C \sum_{j=k+1}^{N} j^{k-1} \sum_{v=2}^{j-2} M^v C^{v-1} v^2 \sum_{\substack{h_2+\cdots+h_v = j-3 \\ 2 \leq h_i \leq h_v, h_v \geq k}} \left(\prod_{i=2}^{v-1} \frac{\rho^{h_i}}{h_i!} \sum_{r=1}^{p+2} \sqrt{F_{h_i+r}} \right)$$

$$\cdot \rho^{h_v+3-k} \frac{\sqrt{F_{h_v+3}}}{h_v!(h_v+3)}. \tag{5.43}$$

Then, from (5.20), (5.21), and noting that $(h_v + 3)v \geq j$, we find that

$$L''_{N,4} \leq C \sum_{j=k+1}^{N} \sum_{v=2}^{j-2} M^v C^v v^{k+1} \sum_{\substack{h_2+\cdots+h_v = j-3 \\ 2 \leq h_i \leq h_v, h_v \geq k}} (h_v+3)\eta_{h_v+3} \left(\prod_{i=2}^{v-1} \sum_{r=1}^{p+2} \eta_{h_i+r} \right). \tag{5.44}$$

After some calculations, similar to those of (5.24), we obtain that $\forall N \geq k+1$

$$L''_{N,4} \leq \Phi_3(\mathscr{E}_N) \sum_{j=k+1}^{N} j\eta_j \quad \text{for } |\mathscr{E}_N| \leq \frac{1}{(p+2)MC}, \tag{5.45}$$

with $\Phi_3(\cdot)$ a suitable analytic function.

Thus, it remains to estimate $L''_{N,5}$, where $h_1 = 1$. Since $h_1 + \cdots + h_v = j-1$ and $h_v \leq j-3$, in the terms of $L''_{N,5}$ we must have $v \geq 3$. Then

$$\begin{aligned} L''_{N,5} = C \sum_{j=k+1}^{N} j^{k-1} \sum_{v=3}^{j-2} M^v C^{v-1} v^2 \sum_{\substack{h_2+\cdots+h_v=j-3 \\ 1\leq h_i\leq h_v,\, h_v\geq k}} \left(\prod_{i=2}^{v-1} \frac{\rho^{h_i}}{h_i!} \sum_{r=1}^{p+2} \sqrt{F_{h_i+r}} \right) \\ \cdot \rho^{h_v+2-k} \frac{\sqrt{F_{h_v+3}}}{h_v!(h_v+3)} \\ \leq C \sum_{j=k+1}^{N} j^{k-1} \sum_{v=3}^{j-2} M^v C^{v-1} v^3 \sum_{\substack{h_2+\cdots+h_v=j-3 \\ 1\leq h_2\leq h_i\leq h_v,\, h_v\geq k}} \left(\prod_{i=2}^{v-1} \frac{\rho^{h_i}}{h_i!} \sum_{r=1}^{p+2} \sqrt{F_{h_i+r}} \right) \\ \cdot \rho^{h_v+2-k} \frac{\sqrt{F_{h_v+3}}}{h_v!(h_v+3)}. \end{aligned} \tag{5.46}$$

Hence, we may write $L''_{N,5} \leq L''_{N,6} + L''_{N,7}$, where $L''_{N,6}$ groups the terms in the right of (5.46) in which $h_2 = 1$ and $L''_{N,7}$ those with $h_2 \geq 2$ respectively. Operating as above, we easily obtain that

$$\begin{aligned} L''_{N,6} = C \sum_{j=k+1}^{N} j^{k-1} \sum_{v=3}^{j-2} M^v C^{v-1} v^3 \sum_{\substack{h_3+\cdots+h_v=j-3 \\ 1\leq h_i\leq h_v,\, h_v\geq k}} \left(\prod_{i=3}^{v-1} \frac{\rho^{h_i}}{h_i!} \sum_{r=1}^{p+2} \sqrt{F_{h_i+r}} \right) \\ \cdot \rho^{h_v+3-k} \frac{\sqrt{F_{h_v+3}}}{h_v!(h_v+3)} \\ \leq C \sum_{j=k+1}^{N} j^{k-1} \sum_{v=3}^{j-2} M^v C^{v-1} v^3 \sum_{\substack{h_3+\cdots+h_v=j-3 \\ 1\leq h_i\leq h_v,\, h_v\geq k}} \frac{\eta_{h_v+3}}{(h_v+3)^{k-2}} \left(\prod_{i=2}^{v-1} \sum_{r=1}^{p+2} \eta_{h_i+r} \right). \end{aligned} \tag{5.47}$$

While, applying (5.20), (5.21) as in the estimate of $L''_{N,4}$, recalling that $k \geq p+4$, we obtain

$$L''_{N,7} = C \sum_{j=k+1}^{N} j^{k-1} \sum_{v=3}^{j-2} M^v C^{v-1} v^3 \sum_{\substack{h_2+\cdots+h_v=j-3 \\ 2\le h_2\le h_i\le h_v,\, h_v\ge k}} \left(\prod_{i=2}^{v-1} \frac{\rho^{h_i}}{h_i!} \sum_{r=1}^{p+2} \sqrt{F_{h_i+r}} \right)$$

$$\cdot\, \rho^{h_v+2-k} \frac{\sqrt{F_{h_v+3}}}{h_v!(h_v+3)}$$

$$\le C \sum_{j=k+1}^{N} j^{k-1} \sum_{v=3}^{j-2} M^v C^{v-1} v^3 \sum_{\substack{h_2+\cdots+h_v=j-3 \\ 2\le h_2\le h_i\le h_v,\, h_v\ge k}} \frac{\eta_{h_v+3}}{(h_v+3)^{k-2}}$$

$$\times \left(\prod_{i=3}^{v-1} \sum_{r=1}^{p+2} \eta_{h_i+r} \right). \tag{5.48}$$

Thus, we conclude that $\forall N \ge k+1$

$$L''_{N,6}, L''_{N,7} \le \Phi_4(\mathscr{E}_N) \sum_{j=k+1}^{N} j\eta_j \quad \text{for } |\mathscr{E}_N| < \frac{1}{(p+2)MC}, \tag{5.49}$$

with $\Phi_4(\cdot)$ a suitable analytic function.

Summarizing up, we have proved the following:

Lemma 5.6. *Let u be a C^∞ solution of* (1.2) *in* $\mathbf{R} \times [0,T)$ *and let Assumptions* 3.1, 4.1, 5.1, 5.5 *hold. Then there exist constants* $\tilde\rho \in (0,1]$ *and* $\tilde{\mathscr{E}}, \tilde C > 0$ *s.t.*

$$\mathscr{E}'_N \le \rho' + \tilde C\rho + \sum_{j=k+1}^{N} \frac{\rho^{j-k-1}}{(j-1)!} j^k \left[\frac{j-k}{j}\rho' + \tilde C \rho \right] \sqrt{F_j}, \tag{5.50}$$

for all $N \ge k+1$, *a.e. in* (τ_1, τ_2), *provided* $0 < \rho(t) \le \tilde\rho$ *and* $0 \le \mathscr{E}_N(t) \le \tilde{\mathscr{E}}$.

Proof. It is sufficient to collect the estimates from (5.5) to (5.49). □

6. Some Consequences of Lemma 5.6

Given $u : \mathbf{R} \times [0,T) \to \mathbf{R}$, with $T > 0$, a C^∞ solution of (1.2), let us suppose that $u(\cdot,\tau_1)$, $u_t(\cdot,\tau_1)$ be uniformly analytic in the interval B_{τ_1}. Namely, we assume that:

Assumption 6.1. *There exist* $C, \Lambda_0 > 0$ *such that, for all integers* $j \ge 0$, *one has*

$$|\partial_x^j u_t(x,\tau_1)|, |\partial_x^j u(x,\tau_1)| \le C\Lambda_0^j j!, \quad \forall x \in B_{\tau_1}. \tag{6.1}$$

Then, applying Lemma 5.6, we have:

LEMMA 6.2. *Under the Assumptions* 3.1, 4.1, 5.1, 5.5, 6.1, *there exist* $\varrho \in (0, \tilde{\rho})$, $\sigma > 0$ *such that putting*

$$\rho(t) = \varrho e^{-\sigma(t-\tau_1)} \tag{6.2}$$

into Definition 5.3, *then the energies* $\mathscr{E}_N(t)$ *satisfy*:

$$\mathscr{E}_N(t) \le \mathscr{E}_N(\tau_1) \le \tilde{\mathscr{E}} \quad \textit{for } t \in [\tau_1, \tau_2), \ \forall N \ge k+1. \tag{6.3}$$

PROOF. As it is known (for instance from the arguments of proof of the classical Cauchy-Kowalewski theorem) Assumptions 4.1, 6.1 and the fact that $u(x,t)$ is a C^∞ solution of equation (1.2) imply that

$$|\partial^\alpha u(x, \tau_1)| \le C\Lambda^{|\alpha|}\alpha! \quad \text{for } x \in B_{\tau_1}, \tag{6.4}$$

for all $\alpha \in (\mathbf{Z}^+)^2$, with $C, \Lambda > 0$ suitable constants. Furthermore, from Definition 3.3, it easily follows that

$$\sqrt{F_j(\tau_1)} \le C\Lambda^j j! \quad \text{for all } j \ge 0. \tag{6.5}$$

Thus, Definition 5.3 gives

$$\mathscr{E}_N(\tau_1) \le \rho(\tau_1) + C \sum_{k=j+1}^{N} \rho(\tau_1)^{j-k}\Lambda^j j^k \le \tilde{\mathscr{E}}/2, \quad \forall N \ge k+1, \tag{6.6}$$

provided $\rho(\tau_1)$ is small, say $\rho(\tau_1) \le \tilde{\varrho}$ for a suitable $\tilde{\varrho} > 0$. Hence, we choose $\varrho = \min\{\tilde{\varrho}, \tilde{\rho}\}$ and then we define $\rho(t)$ as the solution of the Cauchy problem

$$\frac{\rho'}{k+1} + \tilde{C}\rho = 0, \quad \rho(\tau_1) = \varrho. \tag{6.7}$$

Namely, we take $\rho(t) = \varrho e^{-\sigma(t-\tau_1)}$, with $\sigma = \tilde{C}(k+1)$. Since $\rho(t) \le \tilde{\rho}$ in $[\tau_1, \infty)$, from (5.50) it immediately follows that $\mathscr{E}_N'(t) \le 0$ a.e. in (τ_1, τ_2), as long as $\mathscr{E}_N(t) \le \tilde{\mathscr{E}}$. Therefore, the initial condition (6.6) easily gives

$$\mathscr{E}_N(t) \le \mathscr{E}_N(\tau_1) \tag{6.8}$$

in the whole interval $[\tau_1, \tau_2)$, $\forall N \ge k+1$. Thus (6.3) holds. □

COROLLARY 6.3. *Under the Assumptions* 3.1, 4.1, 5.1, 5.5, 6.1, *the solution* $u(x,t)$ *is uniformly analytic in* Γ, *i.e. there exist constants* $C, \Lambda > 0$ *such that*

$$\sup_{\Gamma} |\partial^\alpha u(x,t)| \le C\Lambda^{|\alpha|}\alpha! \quad \textit{for all } \alpha \in (\mathbf{Z}^+)^2. \tag{6.9}$$

PROOF. From (6.2), (6.3) we deduce that for all $j \geq k+1$

$$\sqrt{F_j(t)} \leq \tilde{\mathscr{E}} \rho(t)^{k-j} j^{-k} j! \quad \text{in } [\tau_1, \tau_2), \tag{6.10}$$

where $\rho(t) = \varrho e^{-\sigma(t-\tau_1)}$. Hence, Assumption 5.1 and condition (5.2) imply that

$$\|\partial^j u(t)\|_\infty \leq C \sum_{r=1}^{p} \sqrt{F_{j+r}(t)} \leq C \tilde{\mathscr{E}} \rho(t)^{-j} j!, \tag{6.11}$$

for all $j \geq k+1$ and $t \in [\tau_1, \tau_2)$. Since

$$\rho(t) \geq \varrho e^{\sigma(\tau_1 - \tau_2)} \quad \text{in } [\tau_1, \tau_2], \tag{6.12}$$

we easily see that (6.9) holds. In fact, setting $j = |\alpha|$, we have $j! \leq 2^{|\alpha|} \alpha!$ for all $\alpha \in (\mathbf{Z}^+)^2$. Then, from (6.11)–(6.12), we obtain

$$|\partial^\alpha u(x,t)| \leq C \tilde{\mathscr{E}} \left(\frac{2e^{\sigma(\tau_2 - \tau_1)}}{\varrho} \right)^{|\alpha|} \alpha!, \tag{6.13}$$

for all $(x,t) \in \Gamma$. Hence, $u(x,t)$ is uniformly analytic in Γ. □

7. Proof of the Main Result

Let $u : \mathbf{R} \times [0, T) \to \mathbf{R}$, with $T > 0$, be a given C^∞ solution of equation (1.2). Besides, let us suppose that $a : \mathbf{R} \to [0, \infty)$ satisfies (1.4).

Given $x_o \in \mathbf{R}$ and $\delta > 0$, we will prove that if $u(x,0)$, $u_t(x,0)$ are uniformly analytic in $S_0 = [x_o - \delta, x_o + \delta]$, then $u(x,t)$ is uniformly analytic in the compact domains

$$D_\tau \overset{\text{def}}{=} \{(x,t) : |x - x_o| \leq \delta - \sqrt{\lambda} t, 0 \leq t \leq \tau\}, \tag{7.1}$$

for all $\tau \in \left(0, \min\left(T, \frac{\delta}{\sqrt{\lambda}}\right)\right)$. To this aim, defining

$$T_\delta \overset{\text{def}}{=} \min(T, \delta/\sqrt{\lambda}), \tag{7.2}$$

$$S_\tau \overset{\text{def}}{=} \{x : |x - x_o| \leq \delta - \sqrt{\lambda}\tau\} \quad (0 \leq \tau < \delta/\sqrt{\lambda}), \tag{7.3}$$

we first establish two preliminary lemmas.

LEMMA 7.1. *Given $\tau \in [0, T_\delta)$, let us suppose that*:

1) *$u(x,0)$, $u_t(x,0)$ are uniformly analytic in S_0, if $\tau = 0$;*

2) *$u(x,t)$ is uniformly analytic in D_τ, if $\tau > 0$.*

Then $u(x,t)$ is uniformly analytic in $D_{\tau'}$ for some $\tau' \in (\tau, T_\delta)$.

Proof. In both cases, by the *unique continuation principle* for analytic functions, there exist an open set Ω_τ, $S_\tau \subset \Omega_\tau \subset \mathbf{R}_x$, and $v_0, v_1 : \Omega_\tau \to \mathbf{R}$ analytic functions such that $v_0(x) = u(x,\tau)$, $v_1(x) = u_t(x,\tau)$ for all $x \in S_\tau$. Then, applying the Cauchy-Kowalewski theorem, we can solve (locally) the problem

$$v_{tt} - a(v)v_{xx} = 0, \tag{7.4}$$

$$v(x,\tau) = v_0(x), \quad v_t(x,\tau) = v_1(x), \tag{7.5}$$

obtaining a unique analytic solution $v(x,t)$ in an open neighborhood U_τ of $S_\tau \times \{\tau\}$ in $\mathbf{R}_x \times \mathbf{R}_t$. On the other hand, since $v(x,\tau) = u(x,\tau)$, $v_t(x,\tau) = u_t(x,\tau)$ for $x \in S_\tau$, by Theorems 9.1, 9.3 (b) (cf. Appendix B) we must have

$$v(x,t) = u(x,t) \quad \text{in } \{(x,t) : |x - x_o| \le \delta - \sqrt{\lambda}t, \tau \le t \le \tau'\}, \tag{7.6}$$

for some $\tau' \in (\tau, T_\delta)$. Hence, using again the *unique continuation principle*, it follows that $u(x,t)$ is uniformly analytic in $D_{\tau'}$ for some $\tau' \in (\tau, T_\delta)$. □

Lemma 7.2. *Given $\mathscr{T} \in (0, T_\delta)$, let $u(x,t)$ be uniformly analytic in D_τ for $\tau < \mathscr{T}$. Let us suppose that for all $\bar{x} \in S_{\mathscr{T}}$ there exist $\bar{\gamma} \in (0, \sqrt{\lambda}]$ and $\bar{\tau} \in [0, \mathscr{T})$ such that Assumption* 3.1 *is verified if we set*

$$\begin{gathered} \gamma_1(t) = \bar{x} - \bar{\gamma}(\mathscr{T} - t), \quad \gamma_2(t) = \bar{x} + \bar{\gamma}(\mathscr{T} - t), \\ \tau_1 = \bar{\tau}, \quad \tau_2 = \mathscr{T}, \end{gathered} \tag{7.7}$$

in (2.6)–(2.10). *Then $u(x,t)$ is uniformly analytic in $D_{\mathscr{T}}$.*

Proof. Given $\bar{x} \in S_{\mathscr{T}}$, let $\Gamma = \Gamma(\bar{x})$ be the domain defined in (2.6)–(2.10) with τ_i, γ_i $(i = 1, 2)$ as in (7.7). Assumptions 3.1, 4.1 are clearly verified. Since Γ is a triangle, by well known embedding theorems for Sobolev spaces (see [1]), there exists $C > 0$ such that for every sufficiently regular function $v(x,t)$ one has

$$\|v(\cdot,t)\|_{L^\infty} \le C\left(\sum_{j=0}^{1} \|\partial_x^j v(\cdot,t)\|_{L^2} + \sum_{|\beta| \le 2} \|\partial^\beta v\|_{L^2(\Gamma_t)}\right), \tag{7.8}$$

for all $t \in [\bar{\tau}, \mathscr{T})$. Then, for $h \in \mathbf{Z}^+$, from (2.11)–(2.12) and (3.2)–(3.4) it follows that

$$
\begin{aligned}
\|\partial^h u(\cdot,t)\|_{L^\infty} &\le C \sum_{|\alpha|=h} \left(\sum_{j=0}^{1} \|\partial_x^j \partial^\alpha u(\cdot,t)\|_{L^2} + \sum_{|\beta|\le 2} \|\partial^\beta \partial^\alpha u\|_{L^2(\Gamma_t)} \right) \\
&\le C \sum_{i=0}^{2} \left(\sum_{|\alpha|=h+i} \|\partial^\alpha u(\cdot,t)\|_{L^2} + \sum_{|\alpha|=h+i} \|\partial^\alpha u\|_{L^2(\Gamma_t)} \right) \\
&\le C \sum_{i=0}^{2} \left[\sum_{|\alpha|=h+i} \frac{\sqrt{E_\alpha(t)}}{h+i+1} + \frac{1}{h+i+1} \sum_{|\alpha|=h+i} \left(\int_{\tau_1}^{t} E_\alpha(s)\, ds \right)^{1/2} \right] \\
&\le C \sum_{i=0}^{2} \left[\frac{\sqrt{E_{h+i+1}(t)}}{h+i+1} + \frac{1}{\sqrt{h+i+1}} \left(\sum_{|\alpha|=h+i} \int_{\tau_1}^{t} E_\alpha(s)\, ds \right)^{1/2} \right] \\
&\le C \sum_{i=0}^{2} \left\{ \frac{\sqrt{E_{h+i+1}(t)}}{h+i+1} + \frac{1}{\sqrt{h+i+1}} \left[\int_{\tau_1}^{t} \left(\sum_{|\alpha|=h+i} \sqrt{E_\alpha(s)} \right)^2 ds \right]^{1/2} \right\} \\
&\le C \sum_{i=0}^{2} \left[\frac{\sqrt{E_{h+i+1}(t)}}{h+i+1} + \frac{1}{\sqrt{h+i+1}} \left(\int_{\tau_1}^{t} E_{h+i+1}(s)\, ds \right)^{1/2} \right] \\
&\le C \sum_{i=0}^{2} \frac{\sqrt{F_{h+i+1}(t)}}{\sqrt{h+i+1}}, \qquad (7.9)
\end{aligned}
$$

for all $t \in [\bar{\tau}, \mathscr{T})$. Thus, Assumption 5.1 holds with $p = 3$. Besides, taking $k = 7$ in (5.2), also Assumption 5.5 is verified, because $u \in C^\infty$. Finally, with $\tau_1 = \bar{\tau}$, Assumption 6.1 is satisfied because $\bar{\gamma} \in (0, \sqrt{\lambda}]$ and $u(x,t)$ is uniformly analytic in D_τ for $\tau \in [0, \mathscr{T})$. Hence, we can apply Corollary 6.3 which implies that $u(x,t)$ is uniformly analytic in $\Gamma = \Gamma(\bar{x})$. In particular, for all $\alpha \in (\mathbf{Z}^+)^2$ and $t \in [\bar{\tau}, \mathscr{T}]$, we have

$$
|\partial^\alpha u(\bar{x},t)| \le C \Lambda^{|\alpha|} \alpha! \quad \text{for } \bar{\tau} \le t \le \mathscr{T}, \tag{7.10}
$$

for suitable constants $C, \Lambda \ge 0$. Thanks to the *unique continuation principle* for analytic functions, (7.10) implies that for all $\varepsilon \in (0, \Lambda^{-1})$ the exists $C_\varepsilon, \Lambda_\varepsilon \ge 0$ such that

$$
|\partial^\alpha u(x,t)| \le C_\varepsilon \Lambda_\varepsilon^{|\alpha|} \alpha! \quad \text{for all } \alpha \in (\mathbf{Z}^+)^2 \tag{7.11}
$$

and for all $(x,t) \in G_\varepsilon$, where

$$
G_\varepsilon \overset{\text{def}}{=} (\{|x - \bar{x}| \le \varepsilon\} \times [\bar{\tau}, \mathscr{T})) \cap D_{\mathscr{T}}. \tag{7.12}
$$

Now, since $u(x,t) \in C^\infty$, it is clear that the inequalities (7.11) continue to hold in the closure of the G_ε; namely in $(\{|x-\bar{x}| \le \varepsilon\} \times [\bar{\tau}, \mathscr{T}]) \cap D_{\mathscr{T}}$. Finally, since $\bar{x} \in S_{\mathscr{T}}$ is arbitrary and $S_{\mathscr{T}}$ is compact, we conclude that $u(x,t)$ is uniformly analytic in $D_{\mathscr{T}}$. □

REMARK 7.3. Let consider the statement of Lemma 7.2. Given $\bar{x} \in S_{\mathscr{T}}$, if we further suppose that

$$a(u(\bar{x}, \mathscr{T})) = 0, \tag{7.13}$$

then i) of Assumption 3.1 is automatically satisfied provided $\bar{\tau} \in [\mathscr{T} - \epsilon, \mathscr{T})$ with $\epsilon > 0$ sufficiently small. Indeed, given any $\bar{\gamma} > 0$, by (7.13) we have $a(u(x,t)) \le \bar{\gamma}^2$ in a neighborhood of $(\bar{x}, \mathscr{T})$. Thus, in order to apply Lemma 7.2, we need only to show that there exits $\bar{\gamma} \in (0, \sqrt{\Lambda}]$, $\bar{\tau} \in [0, \mathscr{T})$ such that ii) of Assumption 3.1 holds.

7.1. Conclusion of the Proof of Theorem 1.1

Assuming $u(x,0)$, $u_t(x,0)$ uniformly analytic in S_0, from now on we define:

$$\mathscr{T} \stackrel{\text{def}}{=} \sup\{\tau \in (0, T_\delta) \,|\, u(x,t) \text{ is uniformly analytic in } D_\tau\}. \tag{7.14}$$

Our aim is to prove that $\mathscr{T} = T_\delta$.

By Lemma 7.1 we have $0 < \mathscr{T} \le T_\delta$. To see that $\mathscr{T} = T_\delta$, we argue by contradiction. Namely, assuming $\mathscr{T} < T_\delta$, we prove that $u(x,t)$ is uniformly analytic in $D_{\mathscr{T}}$. Therefore, applying Lemma 7.1 once again, $u(x,t)$ is uniformly analytic in D_τ for some $\tau \in (\mathscr{T}, T_\delta)$.

In view of Lemma 7.2, it is enough to show that for any

$$\bar{x} \in S_{\mathscr{T}} \tag{7.15}$$

there exist $\bar{\gamma} \in (0, \sqrt{\lambda}]$ and $\bar{\tau} \in [0, \mathscr{T})$ such that, setting $\tau_1 = \bar{\tau}$, $\tau_2 = \mathscr{T}$ and defining $\gamma_1(t)$, $\gamma_2(t)$ as in (7.7), the conditions i), ii) of Assumption 3.1 are verified.

To do this, we distinguish different cases:

(1) Case $a(u(\bar{x}, \mathscr{T})) > 0$. We set $\bar{\gamma} = \sqrt{\lambda}$ and then we take $\bar{\tau} \in [0, \mathscr{T})$ such that

$$\inf_{(x,t) \in \Gamma} a(u(x,t)) > 0, \tag{7.16}$$

where Γ is the triangle

$$\Gamma = \{(x,t) : |x - \bar{x}| \le \sqrt{\lambda}(\mathscr{T} - t), \bar{\tau} \le t \le \mathscr{T}\}. \tag{7.17}$$

Then the conditions i), ii) of Assumption 3.1 are clearly verified. See Remark 3.2.

(2) Case $a(u(\bar{x}, \mathscr{T})) = 0$. By Remark 7.3, we have only to find $\bar{\gamma} \in (0, \sqrt{\lambda}]$ and $\bar{\tau} \in [0, \mathscr{T})$ such that ii) of Assumption 3.1 holds. For simplicity, we suppose

$$u(\bar{x}, \mathscr{T}) = 0, \quad a(0) = 0 \tag{7.18}$$

(the general case, i.e. $u(\bar{x}, \mathscr{T}) = z$ with $a(z) = 0$, is only formally more complicated; see Remark 7.4 below). Then, since $a : \mathbf{R} \to [0, \infty)$ satisfies (1.4) we may assume

$$a(s) = a_0(s)s^{2l} \quad \text{with } a_0(s) \geq \eta \quad \text{for } |s| < \varepsilon, \tag{7.19}$$

where $a_0 : \mathbf{R} \to [0, \infty)$ is analytic, $l \geq 1$ is an integer; $\varepsilon, \eta > 0$ are suitable constants.

To continue let us fix $\chi \in C_0^\infty(\mathbf{R})$ such that:

$$\chi(x) = 1 \quad \text{in } |x - x_o| \leq \delta + 1; \quad \chi(x) = 0 \quad \text{for } |x - x_o| \geq \delta + 2. \tag{7.20}$$

Then, for $\tau_a \in [0, \mathscr{T})$, we consider the Cauchy problem

$$v_{tt} - a(v)v_{xx} = 0, \quad (x, t) \in \mathbf{R} \times [\tau_a, \infty), \tag{7.21}$$

$$v(x, \tau_a) = \chi(x)u(x, \tau_a), \quad v_t(x, \tau_a) = \chi(x)u_t(x, \tau_a). \tag{7.22}$$

By Corollary 9.2 and Theorem 9.3 (a) (cf. Appendix B), we may select $\tau_a \in [0 \vee (\mathscr{T} - 1), \mathscr{T})$ such that (7.21), (7.22) has a unique local C^∞ solution

$$v : \mathbf{R} \times [\tau_a, \tilde{\mathscr{T}}) \to \mathbf{R}, \tag{7.23}$$

with $\tilde{\mathscr{T}} \in (\mathscr{T}, T)$, and there exist C^∞ functions $g, h : \mathbf{R} \times [\tau_a, \tilde{\mathscr{T}}) \to \mathbf{R}$ such that

$$v(x, t) = g(x, t)\chi(x)u(x, \tau_a) + h(x, t)\chi(x)u_t(x, \tau_a) \quad \text{in } \mathbf{R} \times [\tau_a, \tilde{\mathscr{T}}), \tag{7.24}$$

$$g(x, \tau_a) = 1, \quad g_t(x, \tau_a) = 0, \quad h(x, \tau_a) = 0, \quad h_t(x, \tau_a) = 1, \quad x \in \mathbf{R}, \tag{7.25}$$

$$|g_t(x, t)|, |h_t(x, t) - 1| \leq 1/4 \quad \text{in } \mathbf{R} \times [\tau_a, \tilde{\mathscr{T}}). \tag{7.26}$$

In particular, we have

$$v(x, t) = g(x, t)u(x, \tau_a) + h(x, t)u_t(x, \tau_a) \quad \text{in } S_0 \times [\tau_a, \tilde{\mathscr{T}}), \tag{7.27}$$

and, by Theorem 9.3 (b),

$$v(x, t) = u(x, t) \quad \text{in } \{|x - x_o| \leq \delta - \sqrt{\lambda}(t - \tau_a), \tau_a \leq t < \tilde{\mathscr{T}}\}. \tag{7.28}$$

Then, we consider the following subcases:

$$\text{(2a)} \ u(\bar{x}, \tau_a) \neq 0, \quad \text{(2b)} \ u(\bar{x}, \tau_a) = 0. \tag{7.29}$$

(2a) We may suppose $u(\bar{x},\tau_a)>0$ (if $u(\bar{x},\tau_a)<0$ the argument is similar). Then $u(\bar{x},\mathscr{T})=0$ and (7.24)–(7.28) imply

$$u_t(\bar{x},\tau_a) = -\frac{g(\bar{x},\mathscr{T})}{h(\bar{x},\mathscr{T})}u(\bar{x},\tau_a) \le -\frac{1-(\mathscr{T}-\tau_a)/4}{5(\mathscr{T}-\tau_a)/4}u(\bar{x},\tau_a)$$

$$\le -\frac{3}{5(\mathscr{T}-\tau_a)}u(\bar{x},\tau_a), \tag{7.30}$$

because $\tau_a \in [0\vee(\mathscr{T}-1),\mathscr{T})$. It follows that

$$\partial_t u(\bar{x},\mathscr{T}) = g_t(\bar{x},\mathscr{T})u(\bar{x},\tau_a) + h_t(\bar{x},\mathscr{T})u_t(\bar{x},\tau_a)$$

$$\le \left(\frac{1}{4}-\frac{9}{20}\right)u(\bar{x},\tau_a) < 0. \tag{7.31}$$

Since $\partial_t u(\bar{x},\mathscr{T})<0$, there exists

$$Q \overset{\text{def}}{=} \{|x-\bar{x}|\le\sigma_1\}\times\{|t-\mathscr{T}|\le\sigma_2\}, \tag{7.32}$$

with $\sigma_1\in(0,\delta)$ and $\sigma_2\in(0,(\mathscr{T}-\tau_a)\wedge(\tilde{\mathscr{T}}-\mathscr{T}))$, such that

$$\partial_t u(x,t)<0 \quad \text{in } Q \tag{7.33}$$

and, by the implicit function theorem,

$$\{(x,t): u(x,t)=0\}\cap Q \tag{7.34}$$

is the graph of a C^∞ function, say $f:[\bar{x}-\sigma_1,\bar{x}+\sigma_1]\to[\mathscr{T}-\sigma_2,\mathscr{T}+\sigma_2]$, such that

$$f(\bar{x})=\mathscr{T}. \tag{7.35}$$

Now, we take $\bar{\gamma}\in(0,\sqrt{\lambda}]$ such that

$$\bar{\gamma}\le \min_{x\in[\bar{x}-\sigma_1,\bar{x}+\sigma_1]}\frac{1}{1+|f'(x)|}, \tag{7.36}$$

and then we define $\gamma_1(t)$, $\gamma_2(t)$ as in (7.7). In this way

$$\{|x-\bar{x}|\le\bar{\gamma}(\mathscr{T}-t), t\le\mathscr{T}\}\cap Q \subseteq \{(x,t): t\le f(x), x\in[\bar{x}-\sigma_1,\bar{x}+\sigma_1]\}. \tag{7.37}$$

Finally, since $u(\bar{x},\mathscr{T})=a(u(\bar{x},\mathscr{T}))=0$, we may select $\bar{\tau}\in[\mathscr{T}-\sigma_2,\mathscr{T})$ such that

$$\Gamma=\{(x,t): |x-\bar{x}|\le\bar{\gamma}(\mathscr{T}-t), \bar{\tau}\le t\le\mathscr{T}\}\subset Q \tag{7.38}$$

and

$$\max_{(x,t)\in\Gamma} a(u(x,t)) \le \bar{\gamma}^2, \tag{7.39}$$

$$u(x,t) \in [-\varepsilon, \varepsilon] \quad \text{for } (x,t) \in \Gamma, \tag{7.40}$$

where $\varepsilon > 0$ is the constant of (7.19). Then condition i) of Assumption 3.1 is verified. While, from (7.33)–(7.37), it immediately follows that

$$\partial_t u^{2l} \le 0 \quad \text{in } \Gamma. \tag{7.41}$$

Hence, by (7.19) and Remark 3.2, condition ii) of Assumption 3.1 holds.

(2b) Since $u(\bar{x}, \tau_a) = u(\bar{x}, \mathscr{T}) = 0$, by (7.26) we must have

$$u_t(\bar{x}, \tau_a) = 0. \tag{7.42}$$

Hence, we further distinguish two cases:

$$\begin{aligned} &(2\mathrm{b}_1)\ u(\cdot, \tau_a) \equiv 0 \quad \text{or} \quad u_t(\cdot, \tau_a) \equiv 0 \quad \text{near } \bar{x}, \\ &(2\mathrm{b}_2)\ u(\cdot, \tau_a) \not\equiv 0 \quad \text{and} \quad u_t(\cdot, \tau_a) \not\equiv 0 \quad \text{near } \bar{x}. \end{aligned} \tag{7.43}$$

(2b$_1$) Conditions i), ii) of Assumption 3.1 are easily verified with $\bar{\gamma} = \sqrt{\lambda}$ and $\bar{\tau} \in [\tau_a, \mathscr{T})$ sufficiently close to $\mathscr{T}$. In fact, if $u_t(\cdot, \tau_a) \equiv 0$, (7.26) and (7.27) give

$$|\partial_t u(x,t)| = |g_t(x,t)|\,|u(x,\tau_a)| \le \frac{1}{4}|u(x,\tau_a)|, \tag{7.44}$$

$$|u(x,t)| = |g(x,t)|\,|u(x,\tau_a)| \ge \frac{3}{4}|u(x,\tau_a)|, \tag{7.45}$$

for $t \in [\tau_a, \tilde{\mathscr{T}})$ and $|x - \bar{x}|$ small enough. Thus $|\partial_t u(x,t)| \le \frac{1}{3}|u(x,t)|$ where (7.44), (7.45) hold. Conversely, if $u(\cdot, \tau_a) \equiv 0$ in a neighborhood of $\bar{x}$, then we have

$$|u(x,t)| \ge \frac{t - \tau_a}{2}|u_t(x,\tau_a)|, \tag{7.46}$$

for $t \in [\tau_a, \tilde{\mathscr{T}})$ and $|x - \bar{x}|$ small enough. While

$$|\partial_t u(x,t)| = |h_t(x,t)|\,|u_t(x,\tau_a)| \le \frac{5}{4}|u_t(x,\tau_a)|. \tag{7.47}$$

Hence, we have

$$|\partial_t u(x,t)| \le \frac{5}{\mathscr{T} - \tau_a}|u(x,t)|, \tag{7.48}$$

provided $t \in \left[\frac{\tau_a + \mathscr{T}}{2}, \mathscr{T}\right]$ and $|x - \bar{x}|$ is sufficiently small.

In conclusion, in both cases we obtain that

$$\partial_t u^{2l} \le C u^{2l} \quad \text{in } \Gamma, \tag{7.49}$$

provided we take $\bar{\tau} \in [\tau_a, \mathscr{T})$ close enough to $\mathscr{T}$. By Remark 3.2 it follows that condition ii) of Assumption 3.1 holds.

($2b_2$) $u(\cdot, \tau_a) \not\equiv 0$ and $u_t(\cdot, \tau_a) \not\equiv 0$ near $\bar{x}$. Since $u(\cdot, \tau_a)$, $u_t(\cdot, \tau_a)$ are analytic in a neighborhood of $\bar{x}$, it follows that

$$u(x, \tau_a) = w_1(x)(x - \bar{x})^m, \quad u_t(x, \tau_a) = w_2(x)(x - \bar{x})^n, \tag{7.50}$$

with $m, n \ge 1$ integers and $w_1(x)$, $w_2(x)$ analytic and such that

$$w_1(\bar{x}) \ne 0, \quad w_2(\bar{x}) \ne 0. \tag{7.51}$$

Hence,

$$u(x, t) = g(x, t) w_1(x)(x - \bar{x})^m + h(t, x) w_2(x)(x - \bar{x})^n, \tag{7.52}$$

provided $|x - \bar{x}|$ is sufficiently small and $t \in [\tau_a, \tilde{\mathscr{T}})$.

Then we have two possibilities:

$$m \ne n, \quad m = n. \tag{7.53}$$

Case $m \ne n$. If $m \ne n$, we set $\bar{\gamma} = \sqrt{\lambda}$, so that i) of Assumption 3.1 holds. Condition ii) is easily verified provide we take $\bar{\tau} \in [\tau_a, \mathscr{T})$ sufficiently close to $\mathscr{T}$. In fact, if $|x - \bar{x}| \le \sqrt{\lambda}(\mathscr{T} - t)$ and t is sufficiently close to $\mathscr{T}$, there exists $C > 0$ such that

$$|u(x, t)| \ge C|x - \bar{x}|^m \quad \text{if } m < n, \tag{7.54}$$

$$|u(x, t)| \ge C|x - \bar{x}|^n \quad \text{if } m > n. \tag{7.55}$$

Following almost the same argument of ($2b_1$), we obtain that

$$\partial_t u^{2l} \le C u^{2l} \quad \text{in } \Gamma, \tag{7.56}$$

for a suitable constant $C \ge 0$, provided $\bar{\tau}$ is close enough to $\mathscr{T}$. Then we may conclude recalling Remark 3.2.

Case $m = n$. Finally, if $m = n$, we have

$$u(x, t) = (x - \bar{x})^m [g(x, t) w_1(x) + h(t, x) w_2(x)]. \tag{7.57}$$

Then, if

$$g(\bar{x}, \mathcal{T})w_1(\bar{x}) + h(\bar{x}, \mathcal{T})w_2(\bar{x}) \neq 0, \tag{7.58}$$

we can operate as in the case (1) above, where $a(u(\bar{x}, \mathcal{T})) \neq 0$. Otherwise, if

$$g(\bar{x}, \mathcal{T})w_1(\bar{x}) + h(\bar{x}, \mathcal{T})w_2(\bar{x}) = 0, \tag{7.59}$$

noting that $w_1(\bar{x}) \neq 0$, we can follow almost the same proof of the case (2a) (with $a(u(\bar{x}, \mathcal{T})) = 0$, $u(\bar{x}, \tau_a) \neq 0$) which was discussed earlier.

Remark 7.4. If, in case (2), $a(u(\bar{x}, \mathcal{T})) = 0$ with $u(\bar{x}, \mathcal{T}) = z \neq 0$, we use Theorem 9.1 which gives a local representation of the solution $u(x, t)$ near the point $(\bar{x}, \mathcal{T})$. Indeed, since $a(z) = 0$, we have

$$a(s) = a_1(s)(s - z)^{2l}, \quad \text{with } a_1(s) \geq \eta \quad \text{for } |s - z| < \varepsilon, \tag{7.60}$$

where $a_1 : \mathbf{R} \to [0, \infty)$ is analytic, $l \geq 1$ is an integer; $\varepsilon, \eta > 0$ are suitable constants. Then, using (7.60) and the local representation (9.4)–(9.6), the rest of the proof proceeds in a similar fashion.

8. Appendix A

8.1. Square Root of Absolutely Continuous Functions

Let $I \subset \mathbf{R}$ be an open interval and let $f : I \to [0, \infty)$ be absolutely continuous. The following holds:

Lemma 8.1. *If* $\frac{f'}{\sqrt{f}}$ *is integrable in the open set* $\{x \in I : f > 0\}$, *then* $\sqrt{f}$ *is absolutely continuous in* I *and* $(\sqrt{f})' = 0$ *a.e. in the set* $\{x \in I : f = 0\}$.

Proof. It is clear that $\sqrt{f}$ is absolutely continuous in every close interval $J \subset \{f > 0\}$. Moreover, $(\sqrt{f})' = \frac{f'}{2\sqrt{f}}$ a.e. in J. Setting

$$g = \begin{cases} \frac{f'}{2\sqrt{f}} & \text{if } f > 0, \\ 0 & \text{if } f = 0, \end{cases} \tag{8.1}$$

it easily follows that $g \in L^1(I)$ and that $|\sqrt{f(t)} - \sqrt{f(s)}| \leq \int_s^t |g|\, d\tau$ for all $s, t \in I$, $s \leq t$. This means that $\sqrt{f}$ is absolutely continuous in I and that $(\sqrt{f})'$ exists a.e. In particular, we must have $(\sqrt{f})' = 0$ a.e. in the close set $\{f = 0\}$. □

8.2. Leibniz' Formula for Composite Functions

Let $m \geq 1$ and let $g : \mathbf{R}^n_y \to \mathbf{R}$, $f : \mathbf{R} \to \mathbf{R}$ be C^m functions. Then, for every multi-index $\alpha \in (\mathbf{Z}^+)^n$, with $1 \leq |\alpha| \leq m$, the following identity holds:

$$\frac{\partial^\alpha}{\partial y^\alpha} f(g(y)) = \sum_{v=1}^{|\alpha|} \frac{f^{(v)}(g(y))}{v!} \sum_{\substack{\beta_1+\cdots+\beta_v=\alpha \\ |\beta_i|>0}} \frac{\alpha!}{\beta_1!\cdots\beta_v!} \partial_y^{\beta_1} g(y) \cdots \partial_y^{\beta_v} g(y), \tag{8.2}$$

where $\beta_i \in (\mathbf{Z}^+)^n$.

PROOF. To verify (8.2) we may suppose f of class C^{m+1}. By Taylor's formula with integral remainder, given $\alpha \in (\mathbf{Z}^+)^n$, $1 \leq |\alpha| \leq m$, and $s, \tilde{s} \in \mathbf{R}$, we have:

$$f(s) = f(\tilde{s}) + \sum_{v=1}^{|\alpha|} \frac{f^{(v)}(\tilde{s})}{v!}(s-\tilde{s})^v + \frac{1}{|\alpha|!}\int_{\tilde{s}}^{s} f^{(|\alpha|+1)}(z)(s-z)^{|\alpha|}\,dz. \tag{8.3}$$

The remainder $R(s,\tilde{s}) = \frac{1}{|\alpha|!}\int_{\tilde{s}}^{s} f^{(|\alpha|+1)}(z)(s-z)^{|\alpha|}\,dz$ satisfies

$$\frac{\partial^k}{\partial s^k} R(s,\tilde{s})|_{s=\tilde{s}} = 0 \quad \text{for } 0 \leq k \leq |\alpha|. \tag{8.4}$$

Therefore, putting $s = g(y)$, $\tilde{s} = g(\tilde{y})$ into (8.3) and deriving with respect to y, the usual Leibniz' formula gives the identity

$$\frac{\partial^\alpha}{\partial y^\alpha} f(g(y)) = \sum_{v=1}^{|\alpha|} \frac{f^{(v)}(g(\tilde{y}))}{v!} \sum_{\beta_1+\cdots+\beta_v=\alpha} \frac{\alpha!}{\beta_1!\cdots\beta_v!} \left(\prod_{i=1}^{v} \partial_y^{\beta_i}(g(y)-g(\tilde{y})) \right)$$

$$+ \partial_y^\alpha R(g(y), g(\tilde{y})). \tag{8.5}$$

Finally, setting $y = \tilde{y}$ in (8.5), formula (8.2) easily follows. Indeed, (8.4) gives

$$\partial_y^\beta R(g(y), g(\tilde{y}))|_{y=\tilde{y}} = 0 \quad \text{for } |\beta| \leq |\alpha|. \tag{8.6}$$

□

8.3. Some Other Identities

1) Let $n, v \geq 1$ be positive integers and let

$$\phi : ((\mathbf{Z}^+)^n)^v \to \mathbf{R}$$

(i.e. $\phi = \phi(\beta_1, \ldots, \beta_v)$ with $\beta_i \in (\mathbf{Z}^+)^n$) be a given function.

Then, for all $k \geq \nu$, we easily have the following identity:

$$\sum_{|\alpha|=k} \sum_{\substack{\beta_1+\cdots+\beta_\nu=\alpha \\ |\beta_i|\geq 1}} \phi(\beta_1,\ldots,\beta_\nu) = \sum_{\substack{|h|=k \\ h_i\geq 1}} \sum_{|\beta_1|=h_1} \cdots \sum_{|\beta_\nu|=h_\nu} \phi(\beta_1,\ldots,\beta_\nu). \tag{8.7}$$

where $h \in (\mathbf{Z}^+)^\nu$ is the multi-index $h = (h_1,\ldots,h_\nu)$.

Moreover, if we suppose $\phi(\beta_1,\ldots,\beta_\nu) \geq 0$ and symmetric with respect to the variables $\beta_1,\ldots,\beta_\nu \in (\mathbf{Z}^+)^n$, we also have

$$\sum_{\substack{|h|=k \\ h_i\geq 1}} \sum_{|\beta_1|=h_1} \cdots \sum_{|\beta_\nu|=h_\nu} \phi(\beta_1,\ldots,\beta_\nu) \leq \nu \sum_{\substack{|h|=k \\ 1\leq h_i\leq h_\nu}} \sum_{|\beta_1|=h_1} \cdots \sum_{|\beta_\nu|=h_\nu} \phi(\beta_1,\ldots,\beta_\nu). \tag{8.8}$$

2) Given $\eta_i \in \mathbf{R}$ for $1 \leq i \leq N$, let us consider the sum

$$\mathscr{E} = \sum_{i=1}^{N} \eta_i. \tag{8.9}$$

Then, for all integer $\nu \geq 1$, one has

$$\mathscr{E}^\nu = \sum_{j=\nu}^{\nu N} \sum_{\substack{h_1+\cdots+h_\nu=j \\ 1\leq h_i\leq N}} \eta_{h_1}\cdots\eta_{h_\nu}. \tag{8.10}$$

Besides, if we suppose $\eta_i \geq 0$ for $1 \leq i \leq N$, then given any integer p, $0 \leq p \leq N-1$, the following inequality holds:

$$\sum_{j=\nu}^{\nu(N-p)} \sum_{\substack{h_1+\cdots+h_\nu=j \\ 1\leq h_i\leq N-p}} \prod_{i=1}^{\nu} \left(\sum_{r=0}^{p} \eta_{h_i+r} \right) \leq (p+1)^\nu \mathscr{E}^\nu. \tag{8.11}$$

9. Appendix B: Well-Posedness in C^∞ and Local Representation

We recall here the results of [23, Th. 1.1], [24, Th. 1.1, 2.3] of well-posedness in C^∞ and local representation of solutions of the Cauchy problem:

$$v_{tt} - a(v)v_{xx} = 0, \quad (x,t) \in \mathbf{R} \times [0,\infty), \tag{9.1}$$

$$v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x), \tag{9.2}$$

where $a : \mathbf{R} \to [0,\infty)$ is a bounded analytic function.

Theorem 9.1 ([24, Th. 1.1]). *Let* $a : \mathbf{R} \to [0, \infty)$ *satisfy* (1.4) *and let* $\phi, \psi \in C_0^\infty$, *then* (9.1), (9.2) *has a unique solution* $v \in C^\infty(\mathbf{R} \times [0, T_{\phi\psi}))$, *with* $T_{\phi\psi} > 0$, $T_{\phi\psi} \to \infty$ *as* $\phi, \psi \to 0$ *in* C_0^∞. *Given any* $z \in [\min \phi, \max \phi]$ *s.t.* $a(z) = 0$, *there exists an open neighborhood* $\Omega_z \subset \mathbf{R}$ *of* $\{x : \phi(x) = z\}$ *and* C^∞ *functions*

$$g, h : \Omega_z \times [0, T_{\phi\psi}) \to \mathbf{R} \tag{9.3}$$

such that:

$$v(x, t) = z + g(x, t)[\phi(x) - z] + h(x, t)\psi(x) \quad \text{in } \Omega_z \times [0, T_{\phi\psi}), \tag{9.4}$$

$$g(x, 0) = 1, \quad \partial_t g_i(x, 0) = 0, \quad h(x, 0) = 0, \quad \partial_t h(x, 0) = 1, \quad x \in \Omega_z, \tag{9.5}$$

$$|\partial_t g(x, t)|, |\partial_t h(x, t) - 1| \le 1/4 \quad \text{in } \Omega_z \times [0, T_{\phi\psi}). \tag{9.6}$$

Corollary 9.2. *Under the assumptions of Theorem* 9.1, *if we suppose that*

$$a(0) = 0 \quad \text{and} \quad a(s) > 0 \quad \text{for } s \in [\min \phi, \max \phi] \backslash \{0\}, \tag{9.7}$$

then there exist C^∞ *functions* $g, h : \mathbf{R} \times [0, T_{\phi\psi}) \to \mathbf{R}$ *such that*:

$$v(x, t) = g(x, t)\phi(x) + h(x, t)\psi(x) \quad \text{in } \mathbf{R} \times [0, T_{\phi\psi}), \tag{9.8}$$

$$g(x, 0) = 1, \quad g_t(x, 0) = 0, \quad h(x, 0) = 0, \quad h_t(x, 0) = 1, \quad x \in \mathbf{R}, \tag{9.9}$$

$$|g_t(x, t)|, |h_t(x, t) - 1| \le 1/4 \quad \text{in } \mathbf{R} \times [0, T_{\phi\psi}). \tag{9.10}$$

Moreover, by direct inspection of the proofs of [24] it is easily seen that:

Theorem 9.3. *Let* $a : \mathbf{R} \to [0, \infty)$ *satisfy* (1.4). *Then the following facts hold*:
(a) *If* $\phi_k \to \phi$ *and* $\psi_k \to \psi$ *in* C_0^∞ *as* $k \to +\infty$, *then*

$$\liminf_{k \to +\infty} T_{\phi_k \psi_k} > 0. \tag{9.11}$$

(b) *Given* $x_o \in \mathbf{R}$, $\delta > 0$, *let* U *be an open neighborhood of* $[x_o - \delta, x_o + \delta] \times \{0\}$ *in* $\mathbf{R} \times [0, \infty)$. *Let* $v_1, v_2 : U \to \mathbf{R}$ *be* C^∞ *solutions of* (9.1) *in* U *such that*

$$v_1(x, 0) = v_2(x, 0), \quad \partial_t v_1(x, 0) = \partial_t v_2(x, 0) \quad \text{for } |x - x_o| \le \delta. \tag{9.12}$$

Then there exists $\varepsilon > 0$ *such that*

$$v_1(x, t) = v_2(x, t) \quad \text{in } D_\varepsilon, \tag{9.13}$$

with $D_\varepsilon = \{(x, t) : |x - x_o| \le \delta - \sqrt{\lambda} t, 0 \le t \le \varepsilon\} \subset U$.

Sketch of the proof of (b). The statement is obvious in the *strictly hyperbolic* case, i.e. if

$$a(v_i(x,0)) > 0 \quad \text{for } |x - x_0| \le \delta, \quad i = 1,2. \tag{9.14}$$

Indeed, the function

$$w = v_1 - v_2 \tag{9.15}$$

is a C^∞ solution in U of the linear, homogeneous equation

$$w_{tt} - a(v_1)w_{xx} + b(v_1, v_2)w = 0, \tag{9.16}$$

where the coefficient $b(v_1, v_2)$ is the C^∞ function

$$b(v_1, v_2) = -v_{2xx}\int_0^1 a'(v_1 + s(v_2 - v_1))\,ds. \tag{9.17}$$

Now, by (9.14), $\inf_{D_\varepsilon} a(v_1) > 0$ provided $\varepsilon > 0$ is small enough. Thus, (9.16) is strictly hyperbolic in D_ε for $\varepsilon > 0$ small. Then, since $w(x,0) = \partial_t w(x,0) = 0$ for $|x - x_0| \le \delta$ and $a(v_1) \le \lambda$, by standard arguments it easily follows that $w \equiv 0$ in D_ε.

When (9.14) does not hold, by the same arguments of [24, Proposition 2.2], we can restrict ourselves to the following particular situation:

$$a(s) = \tilde{a}(s)s^{2l}, \tag{9.18}$$

with $l \ge 1$ integer, $\tilde{a} : \mathbf{R} \to [0, \infty)$ analytic s.t. $\tilde{a}(v_i) \ge \eta > 0$ in U, for $i = 1,2$. Then, taking $\chi \in C_0^\infty(\mathbf{R})$ s.t.

$$\chi(x) = 1 \quad \text{for } |x - x_o| \le \delta, \quad \operatorname{supp}\{\chi\} \times \{0\} \subset U, \tag{9.19}$$

we consider the Cauchy problem

$$v_{tt} - \tilde{a}(v)v^{2l}v_{xx} = 0, \quad (x,t) \in \mathbf{R} \times [0, \infty), \tag{9.20}$$

$$v(x,0) = \chi(x)v_1(x,0) \stackrel{\text{def}}{=} \phi(x), \quad v_t(x,0) = \chi(x)\partial_t v_1(x,0) \stackrel{\text{def}}{=} \psi(x). \tag{9.21}$$

Since $\phi, \psi \in C_0^\infty$, by Corollary 9.2 problem (9.20)–(9.21) has a unique C^∞ local solution v in $\mathbf{R} \times [0, T_{\phi\psi})$, with $T_{\phi\psi} > 0$. As a above, the difference

$$u = v - v_1 \tag{9.22}$$

is a C^∞ solution, in a neighborhood $\tilde{U}$ of $[x_o - \delta, x_o + \delta] \times \{0\}$ in $\mathbf{R} \times [0, \infty)$, of the linear homogeneous equation

$$u_{tt} - \tilde{a}(v)v^{2l}u_{xx} + b(v, v_1)u = 0, \tag{9.23}$$

with $b(v, v_1)$ defined as in (9.17). Now, using the representation (9.8)–(9.10), we can apply to u a suitable variant of the energy estimates of [24, Sections 5, 6]. More precisely, denoting with

$$\gamma : \Omega_{\phi\psi} \to \mathbf{R}, \quad \Omega_{\phi\psi} = \{x : |\phi| + |\psi| > 0\}, \tag{9.24}$$

the *separating curve* introduced in [24, Section 4], we set

$$\tilde{\gamma}(x) \stackrel{\text{def}}{=} \min\left\{\gamma(x), \frac{\delta - |x - x_o|}{\sqrt{\lambda}}\right\}, \quad x \in \Omega_{\phi\psi}. \tag{9.25}$$

Then we consider the energies:

$$\tilde{E}(\tau) \stackrel{\text{def}}{=} \int_{\tilde{\gamma}(x) > \tau} e^{\theta t}(u_t^2 + a|\nabla u|^2 + u^2)|_{t=\tau}\, dx, \tag{9.26}$$

where $\{\tilde{\gamma}(x) > \tau\} = \{x \in \Omega_{\phi\psi} : \tilde{\gamma}(x) > \tau\}$, $\theta \in \mathbf{R}$ is a suitable constant;

$$\tilde{F}(\tau) \stackrel{\text{def}}{=} e^{-\beta t} \iint_{\tilde{G}_\tau} e^{\theta t} u^2\, dxdt, \tag{9.27}$$

where $\beta \in \mathbf{R}$ is a constant, $\tilde{G}_\tau$ is the open set

$$\tilde{G}_\tau = \left\{(x,t) \in \Omega_{\phi\psi} \times (0, \infty) : \gamma(x) < t < \min\left(\tau, \frac{\delta - |x - x_o|}{\sqrt{\lambda}}\right)\right\}. \tag{9.28}$$

Since

$$u(x, 0) \equiv 0, \quad u_t(x, 0) \equiv 0 \quad \text{for } |x - x_o| \le \delta, \tag{9.29}$$

operating as in the estimates of [24, Lemmas 5.1, 5.2, Prop. 6.1] (see also [11, Lemmas 2, 3]), we deduce that there exists $\varepsilon_1 > 0$ s.t. $u \equiv 0$ in the set

$$\{(x,t) : x \in \Omega_{\phi\psi}, |x - x_o| \le \delta - \sqrt{\lambda}t, 0 \le t \le \varepsilon_1\} \subset \tilde{U}. \tag{9.30}$$

On the other hand, since $v(x,t) = 0$ for $x \notin \Omega_{\phi\psi}$, $t \in [0, T_{\phi\psi})$, from (9.23), (9.29) we have $u(x,t) = 0$ for $x \notin \Omega_{\phi\psi}$, $|x - x_o| \le \delta$ and $0 \le t \le \varepsilon_1$. Hence, we obtain that $u \equiv 0$ in D_{ε_1}. Finally, considering the difference

$$u^* = v - v_2, \tag{9.31}$$

one can easily see that the situation is exactly the same. Therefore, $u^* \equiv 0$ in D_{ε_2} for some $\varepsilon_2 > 0$. In conclusion, $v_1 \equiv v_2$ in D_ε with $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. □

References

[1] R. A. Adams, Sobolev Spaces. Academic Press, San Diego, 1975.

[2] S. Alinhac, G. Métivier, Propagation de l'analyticité des solutons de systèmes hyperboliques non-linéaires. Invent. Math. **75** (1984), 189–203.

[3] J. M. Bony, P. Schapira, Existence et prolongement des solutions holomorphes des équations aux dérivées partielles. Invent. Math. **17** (1972), 95–105.

[4] J. M. Bony, P. Schapira, Prolongement et existence des solutions des systèmes hyperboliques non stricts à coefficients analytiques. Partial differential equations (Proc. Sympos. Pure Math., Vol. 23, Univ. California, Berkeley, 1971), pp. 85–95. Amer. Math. Soc., Providence, R.I., 1973.

[5] M. Cicognani, Propagation of the analytic regularity in nonlinear hyperbolic equations with constant multiplicity. Ann. Univ. Ferrara—Sez. VII—Sc. Mat. Suppl. Vol. **41** (1995), 167–174.

[6] M. Cicognani, L. Zanghirati, Propagation of the analytic and Gevrey regularity for a class of semi-linaer weakly hyperbolic equations. Rend. Sem. Mat. Univ. Padova **94** (1995), 99–111.

[7] M. Cicognani, L. Zanghirati, Analytic regularity for solutions of nonlinear weakly hyperbolic equations. Boll. Un. Mat. It. (7) **11**-B (1997), 643–679.

[8] M. Cicognani, L. Zanghirati, Quasilinear weakly hyperbolic equations with Levi's condiions. Rend. Sem. Mat. Univ. Politec. Torino **55** (1997), 123–144.

[9] F. Colombini, S. Spagnolo, An example of a weakly hypebolic Cauchy problem not well-posed in C^∞. Acta Math. **148** (1982), 243–253.

[10] F. Colombini, E. Jannelli, S. Spagnolo, Non-uniqueness in hyperbolic Cauchy problems. Ann. of Math. **126** (1987), 495–524.

[11] P. D'Ancona, Well posedness in C^∞ for a weakly hyperbolic second order equation. Rend. Sem. Mat. Univ. Padova **91** (1994), 65–83.

[12] P. D'Ancona, S. Spagnolo, On the life span of the analytic solutions to quasilinear weakly hyperbolic equations. Indiana Univ. Math. J. **40** (1991), 71–98.

[13] P. D'Ancona, S. Spagnolo, Small Analytic solutions to nonlinear weakly hyperbolic systems. Ann. Sc. Norm. Sup. Pisa **22** (1995), 469–491.

[14] P. D'Ancona, S. Spagnolo, Quasi-symmetrization of hyperbolic systems and propagation of the analytic regularity. Boll. Un. Mat. Ital. (8) **1**-B (1998), 169–185.

[15] G. Gleaser, Racine carrée d'une funcion différentiable. Ann. Inst. Fourier **13** (1963), 203–210.

[16] L. Hörmander, A counterexample of Gevrey class to the uniqueness of the Cauchy problem. Math. Res. Letters **7** (2000), 615–624.

[17] J. Hounie, J. R. dos Santos Filho, Well posed Cauchy problems for complex nonlinear equations must be semilinear. Math. Ann. **294** (1992), 439–447.

[18] E. Jannelli, Analytic solutions of non linear hyperbolic systems. Boll. Un. Mat. Ital. **5**-B (1986), 487–501.

[19] K. Kajitani, K. Yamaguti, Propagation of analyticity of the solutions of the Cauchy problem for nonlinear symmetrizable systems. Ann. Sc. Norm. Sup. Pisa **28** (1999), 471–487.

[20] P. D. Lax, Nonlinear hyperbolic equations. Comm. Pure Appl. Math. **6** (1953), 231–258.

[21] R. Manfrin, Some results of analytic and Gevrey regularity for semilinear weakly hyperbolic equations of Oleink type. Rend. Sem. Mat. Univ. Padova **94** (1995), 165–213.

[22] R. Manfrin, Analytic regularity for a class of semilinear weakly hyperbolic equattions of second order. NoDEA **3** (1996), 371–394.

[23] R. Manfrin, Local solvability in C^∞ for quasi-linear weakly hyperbolic equations of second order. Comm. in P.D.E. **21** (1996), 1487–1519.

[24] R. Manfrin, Well posedness in the C^∞ class for $u_{tt} = a(u)\Delta u$. Nonlinear Analysis TMA **36** (1999), 177–212.

[25] G. Métivier, Counterexamples to Hölmgren's uniqueness for analytic non linear Cauchy problems. Invent. Math. **112** (1993), 217–222.

[26] S. Mizohata, Analyticity of solutions of hyperbolic systems with analytic coefficients. Comm. Pure Appl. Math. **14** (1961), 547–559.

[27] S. Mizohata, On hyperbolicity in the domain of real analytic functions and Gevrey classes. Sém. Vaillant, 1980–81.

[28] T. Nishitani, On the Lax-Mizohata theorem in the analytic and Gevrey classes. J. Math. Kyoto Univ. **18** (1978), 547–559.

[29] S. Spagnolo, Analytic regularity of the solutions of a semi-linear weakly hyperbolic equation. Ricerche di Matematica, Suppl. Vol. **36** (1987), 193–202.

[30] S. Spagnolo, Some results of analytic regularity for the semi-linear weakly hyperbolic equations of the second order. Rend. Sem. Mat. Univ. Politec. Torino, Fascicolo Speciale (1988), 203–229.

Renato Manfrin
Università IUAV di Venezia
Tolentini S. Croce 191, 30135 Venezia, Italy
E-mail address: manfrin@iuav.it