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# ON SIEGEL MODULAR CUSP FORMS OF DEGREE TWO

By

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## Introduction

The purpose of this note is to study certain coincidence between Shimura's zeta functions in [11] and Andrianov's zeta functions attached to Hecke eigenforms in Maass space of Siegel modular cusp forms of degree two. In this note, we discuss a correspondence between the space of modular cusp forms of half integral weight and the space of Siegel modular cusp forms of degree two, and its application to Maass spaces, in close relation with Saito-Kurokawa's conjecture in the case of arbitrary level and arbitrary character (cf. [5], [6] and [14]).

Let $M$ be any even positive integer, $\chi$ a character modulo $M$, $\tilde{M} = \mathrm{l.c.m.}(4, M)$ and $k$ an even positive integer. In our previous paper [4], we constructed a linear mapping $\Psi_k^{M,\chi}$ of $S_{k-1/2}(\tilde{M},\chi)$ into $S_k(\Gamma_0^{(2)}(M),\bar{\chi})$ determined by the relation $\Psi_k^{M,\chi}(f)(Z) = \sum_T c_f(T)e[\mathrm{tr}(TZ)]$, where $f \in S_{k-1/2}(\tilde{M},\chi)$, $c_f(T) = \sum_{m|e(T)} \bar{\chi}(m)m^{k-1}a^{(0)}(\tilde{M}N(T)/4m^2)$ and $f \mid [W_{\tilde{M}}]_{2k-1}(z) = \sum_{n=1}^{\infty} a^{(0)}(n) \cdot e[nz]$.

In the section 2, we deduce another linear mapping $\Psi$ of $S_{k-1/2}(4N,\chi)$ into $S_k(\Gamma_0^{(2)}(2N),\chi)$ defined by

$$\Psi(f)(Z) = \sum_T \sum_{m|e(T)} \chi(m)m^{k-1}a(N(T)/m^2)e[\mathrm{tr}(TZ)],$$

where $f(z) = \sum_{n=1}^{\infty} a(n)e[nz] \in S_{k-1/2}(4N,\chi)$ and $\chi$ is a character modulo $2N$. It will be seen that $\Psi$ is more useful than $\Psi_k^{M,\chi}$ in several points and serves to generalize our result in [4]. For example, Theorem 4 in [4] is generalized in the sense that the assumption (5.1) in [4] can be dropped.

We may refer to [13] for another constructions of Siegel modular forms of degree two.

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## §1. Notations and Preliminaries

We denote by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers. For a ring $A$, we denote by $A_m^n$ the set of all $n \times m$ matrices with entries in $A$, and denote $A_1^n$ (resp. $A_n^n$) by $A^n$ (resp. $M_n(A)$). For every $z \in \mathbf{C}$, we set $e[z] = \exp(2\pi i z)$ with $i = \sqrt{-1}$ and we define $\sqrt{z} = z^{1/2}$ so that $-\pi/2 < \arg(z^{1/2}) \le \pi/2$. Further we set $z^{k/2} = (\sqrt{z})^k$ for each $k \in \mathbf{Z}$. For each positive integer $N$, set

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbf{Z},\ ad - bc = 1,\ c \equiv 0 \pmod{N} \right\}.$$

Let $\kappa$ be an odd positive integer, $N$ a positive integer divisible by 4 and $\omega$ a character modulo $N$. Let $f(z)$ be a holomorphic cusp form on the complex upper half plane which satisfies

$$f(\gamma\langle z\rangle) = \omega(d) j(\gamma, z)^{\kappa} f(z) \quad \text{for every } \gamma = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in \Gamma_0(N),$$

where $j(\gamma, z)$ is an automorphic factor defined by [11, (1.10)]. The space of such all cusp forms is denoted by $S_{\kappa/2}(N, \omega)$.

Next we recall the definition of Siegel modular cusp forms of degree $n$. Let $Sp(n, \mathbf{R})$ be the real symplectic group of degree $n$, i.e.,

$$Sp(n, \mathbf{R}) = \{ M \in M_{2n}(\mathbf{R}) \mid {}^tM J_n M = J_n \}, \quad \text{where } J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

and ${}^tM$ denotes the transpose of $M$. We set $SL_2(\mathbf{R}) = Sp(1, \mathbf{R})$. Let $\mathfrak{H}_n$ be the complex Siegel upper half plane of degree $n$, i.e.,

$$\mathfrak{H}_n = \{ Z = X + iY \mid X, Y \in M_n(\mathbf{R}), {}^tZ = Z \text{ and } Y > 0 \}.$$

Define an action of $Sp(n, \mathbf{R})$ on $\mathfrak{H}_n$ by

$$Z \mapsto M\langle Z\rangle = (AZ + B)(CZ + D)^{-1} \quad \text{for all } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbf{R})$$

and for all $Z \in \mathfrak{H}_n$. Denote by $K_n$ the group of stabilizers at $iE_n \in \mathfrak{H}_n$, i.e., $K_n = \{ M \in Sp(n, \mathbf{R}) \mid M\langle iE_n\rangle = iE_n \}$. We set $Sp(n, \mathbf{Z}) = Sp(n, \mathbf{R}) \cap M_{2n}(\mathbf{Z})$. For each positive integer $N$, set

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbf{Z}) \middle| C \equiv 0 \pmod{N} \right\} \quad \text{and} \quad SL_2(\mathbf{Z}) = \Gamma_0^{(1)}(1).$$

Let $\psi$ be a character modulo $M$ and let $k$ be a positive integer. We call a holomorphic function $F$ on $\mathfrak{H}_n$ a Siegel modular cusp form of Neben-type $\psi$ and of weight $k$ with respect to $\Gamma_0^{(n)}(M)$, if the following conditions are satisfied:

(i) For every $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(M)$ and for every $Z \in \mathfrak{H}_n$, $F(\gamma\langle Z\rangle) = \bar{\psi}(\det(A))(\det(CZ+D))^k F(Z)$,

(ii) $|F(Z)|(\det(\operatorname{Im} Z))^{k/2}$ is bounded on $\mathfrak{H}_n$.

The space of such cusp forms is denoted by $S_k(\Gamma_0^{(n)}(M), \psi)$.

Let $Q$ be a non-degenerate symmetric $n \times n$ matrix. Denote by $O(Q)$ (resp. $O(Q)_0$) the real orthogonal group (resp. the connected component of the unity of $O(Q)$). We denote by $\mathscr{S}(\mathbf{R}^n)$ the space of rapidly decreasing functions on $\mathbf{R}^n$. We denote by $\gamma(*, Q)$ the representation of $SL_2(\mathbf{R})$ on $\mathscr{S}(\mathbf{R}^n)$ defined by

$$(\gamma(\sigma, Q)f)(x)$$
$$= \begin{cases} |c|^{-n/2}|\det(Q)|^{1/2} \displaystyle\int_{\mathbf{R}^n} e[(a\langle x,x\rangle - 2\langle x,y\rangle + d\langle y,y\rangle)/2c] f(y)\,dy & \text{if } c \neq 0, \\ |a|^{n/2} e[ab\langle x,x\rangle/2] f(ax) & \text{if } c = 0 \end{cases}$$

for every $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{R})$ and for every $f \in \mathscr{S}(\mathbf{R}^n)$, where $\langle x, y\rangle = {}^t xQy$. We call this representation Weil representation attached to $Q$.

Here we recall the results in [12]. Let $L$ be a lattice in $\mathbf{R}^n$ and $L^*$ be the dual lattice of $L$ with respect to $Q$. We assume that $L^* \supset L$. For an $f \in \mathscr{S}(\mathbf{R}^n)$ and an $h \in L^*/L$, we define a series by

$$\theta(f, h) = \sum_{l \in L} f(l + h).$$

Now Shintani [12] showed the following theorem.

Theorem A. *Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an element of* $SL_2(\mathbf{Z})$ *satisfying the conditions* $ab\langle x, y\rangle \equiv cd\langle y, y\rangle \equiv 0 \pmod{2}$ *for* $x, y \in L$. *Then the series* $\theta(f, h)$ *satisfies the relation*

$$\theta(\gamma(\sigma, Q)f, h) = \sum_{k \in L^*/L} c(h,k)_\sigma \theta(f,k), \quad \textit{where}$$

$$c(h,k)_\sigma = \begin{cases} \delta_{h,ak} e[ab\langle h,h\rangle] & \text{if } c = 0, \\ \sqrt{|\det(Q)|}^{-1} \left( \int_{\mathbf{R}^n/L} dx \right)^{-1} |c|^{-n/2} \sum_{\gamma \in L/cL} e[(a\langle h+\gamma, h+\gamma\rangle \\ \qquad - 2\langle k, h+\gamma\rangle + d\langle k,k\rangle)/2c] & \text{if } c \neq 0. \end{cases}$$

*Moreover, assume that* $c$ *is even,* $cL^* \subset L$, $cd \neq 0$ *and* $c\langle x,x\rangle \equiv 0 \pmod 2$ *for every* $x \in L^*$. *Let* $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ *be a basis of* $L$ *over* $\mathbf{Z}$ *and set* $D = \det(\langle \lambda_i, \lambda_j\rangle)$. *Then it holds that*

$$\sqrt{i}^{-(p-q)\,\mathrm{sgn}(cd)} c(h,k)_\sigma$$

$$= \begin{cases} \delta_{h,dk} e[ab\langle h,h\rangle/2] \left(\frac{-1}{d}\right)^{-n/2} (\mathrm{sgn}\, ci)^n \left(\frac{2c}{d}\right)^n \left(\frac{D}{-d}\right) & (d < 0), \\ \delta_{h,dk} e[ab\langle h,h\rangle/2] \left(\frac{-1}{d}\right)^{n/2} \left(\frac{-2c}{d}\right)^n \left(\frac{D}{d}\right) & (d > 0), \end{cases}$$

*where* $\delta$ *is the Kronecker's delta and* $\left(\frac{*}{*}\right)$ *is the symbol given in* [11].

Now the general linear group $GL_n(\mathbf{R})$ acts on $L^2(\mathbf{R}^n)$ as follows: $(Tf)(x) = |\det(T)|^{-1/2} f(T^{-1}x)$ for all $T \in GL_n(\mathbf{R})$ and for each $f \in L^2(\mathbf{R}^n)$. We introduce two symmetric matrices $Q_0$ and $Q_1$ by

$$Q_0 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad Q_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For a positive integer $N$, set $L(N) = \{{}^t(x_1, x_2, 2Nx_3, (1/N)x_4, \sqrt{2}x_5) \mid x_i \in \mathbf{Z}\ (1 \le i \le 5)\}$ and $L'(N) = \{{}^t(x_1, x_2, x_3, (1/N)x_4, \sqrt{2}x_5) \mid x_i \in \mathbf{Z}\ (1 \le i \le 5)\}$. Let $\rho$ be the isomorphism of $Sp(2,\mathbf{R})/\{\pm E_4\}$ onto $O(Q_0)_0$ defined in [3]. Then we can show the following relation:

(1.1) $\quad \rho(g)^{-1}x \equiv (\det(A))x \pmod{L(N)}$ for every $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(2)}(2N)$ and $x \in L'(N)$.

Set $f_k(x) = \langle x, {}^t(-i, i, 1, -1, 0)\rangle_N^k \exp(-\pi N\, {}^t xx)$ for an $x \in \mathbf{R}^5$, where $\langle x, y\rangle_N = N\, {}^t x Q_0 y$. Then $f_k$ admits the transformation formulas

$$\rho(\kappa) f_k = (\det(A - Bi))^k f_k \quad \left(\kappa = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_2\right) \tag{1.2}$$

and

$$\varepsilon(k(\theta))\gamma(k(\theta), NQ_0) f_k = \exp(-i\theta)^{-(2k-1)/2} f_k, \quad \text{where } k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and $\varepsilon(k(\theta))$ is the symbol given in [12, (1.16)].

For a character $\chi$ modulo $2N$, we define a theta series $\theta_{k,\chi}^N(z, g)$ by

$$\theta_{k,\chi}^N(z, g) = v^{-(2k-1)/4} \sum_l \chi(l_3) \sum_h \rho(g)\gamma(\sigma_z, NQ_0) f_k(l + h)$$

for each $z = u + iv \in \mathfrak{H}_1$ and each $g \in Sp(2, \mathbf{R})$, where $l = {}^t(\ldots, l_3, \ldots)$ (resp. $h$) runs over $L'(N)/L(N)$ (resp. $L(N)$) and $\sigma_z = \begin{pmatrix} \sqrt{v} & u\sqrt{v}^{-1} \\ 0 & \sqrt{v}^{-1} \end{pmatrix}$. By (1.1), (1.2) and Theorem A, we deduce the following.

Lemma 1. *Under the above notations, $\theta_{k,\chi}^N(z, g)$ satisfies the following transformation formulas*:

(i) $\theta_{k,\chi}^N(\sigma\langle z\rangle, g) = \left(\frac{N}{d}\right)\chi(d) j(\sigma, z)^{2k-1} \theta_{k,\chi}^N(z, g)$ *for every* $\sigma = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in \Gamma_0(4N)$,

(ii) $\theta_{k,\chi}^N(z, \gamma g \kappa) = \bar{\chi}(\det(A')) \det(A - Bi)^k \theta_{k,\chi}^N(z, g)$ *for every* $\gamma = \begin{pmatrix} A' & * \\ * & * \end{pmatrix} \in \Gamma_0^{(2)}(2N)$ *and for every* $\kappa = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_2$.

## § 2. Construction of Siegel Modular Cusp Forms

In this section we shall construct Siegel modular cusp forms of degree two from modular cusp forms of half integral weight.

Now, for each function $f \in S_{k-1/2}(4N, \chi)$, we define a function $\Psi(f)$ on $\mathfrak{H}_2$ by

$$\Psi(f)(Z) = J(g, iE_2)^k \int_{D_0(4N)} v^{(2k-1)/2} f \mid [W_{4N}]_{2k-1}(z) \overline{\theta_{k,\bar{\chi}}^N(z, g)} v^{-2}\, du dv$$

with $Z = g\langle iE_2\rangle$, where $D_0(4N)$ is the fundamental domain for $\Gamma_0(4N)$, $J(g, iE_2) = \det(Ci + D)$ $\left(g = \begin{pmatrix} * & * \\ C & D \end{pmatrix}\right)$ and $f \mid [W_{4N}]_{2k-1}(z) = f(-1/4Nz) \cdot (4N^{1/4}(-iz)^{1/2})^{-(2k-1)}$. Applying Lemma 1 and the arguments in [9] and [10], we deduce that $\Psi(f)$ is a Siegel modular cusp form of Neben-type $\chi$ and of weight $k$ with respect to $\Gamma_0^{(2)}(2N)$. To calculate coefficients of the Fourier expansion of $\Psi(f)$ at infinity, we define three theta series. For a non-negative integer $\varepsilon$, we define theta series $\theta_{1,\varepsilon}^{N}(z, Y_1)$, $\theta_{1,\varepsilon}^{*}(z, Y_1)$ and $\theta_{2,\varepsilon}^{N,\bar{\chi}}(z, y)$ by

$$\theta_{1,\varepsilon}^{N}(z, Y_1) = (Nv)^{(-\varepsilon+2)/2} \sum_{l} H_\varepsilon(\sqrt{2\pi Nv}(y_1, -y_3, -\sqrt{2}y_2)l) e[(Nu\,{}^t l Q_0 l + Niv\,{}^t l R(Y_1) l)/2],$$

$$\theta_{1,\varepsilon}^{*}(z, Y_1) = v^{(-\varepsilon+2)/2} \sum_{l'} H_\varepsilon(\sqrt{2\pi v}(y_1, -y_3, -\sqrt{2}y_2)l') e[(u\,{}^t l' Q_1 l' + iv\,{}^t l' R(Y_1) l')/2]$$

and

$$\theta_{2,\varepsilon}^{N,\bar{\chi}}(z, y) = (Nv)^{(1-\varepsilon)/2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \bar{\chi}(m) \exp(-2\pi mnui - \pi v(m^2(\sqrt{N}y)^2 + n^2(\sqrt{N}y)^{-2})) \times H_\varepsilon(\sqrt{2\pi v}(-(\sqrt{N})^{-1} n y^{-1} + \sqrt{N} m y))$$

for each $z = u + iv \in \mathfrak{H}_1$, $y > 0$ and $Y_1 = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix}$ ($Y_1 > 0$ and $\det(Y_1) = 1$), where $H_\varepsilon(x) = (-1)^\varepsilon \exp(x^2/2) \frac{d^\varepsilon}{dx^\varepsilon}(\exp(-x^2/2))$, $L_1^* = \{y \in \mathbf{R}^3 \mid {}^t y Q_1 x \in \mathbf{Z}$ for all $x \in L_1\}$, $L_1 = \{{}^t(x_1, x_2, \sqrt{2}x_3) \mid (x_1, x_2, x_3) \in \mathbf{Z}_3^1\}$,

$$Q_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad R(Y_1) = \begin{pmatrix} y_1^2 & -y_2^2 & -\sqrt{2}y_1y_2 \\ -y_2^2 & y_3^2 & \sqrt{2}y_2y_3 \\ -\sqrt{2}y_1y_2 & \sqrt{2}y_2y_3 & 1 + 2y_2^2 \end{pmatrix}$$

and the summation $\sum_l$ (resp. $\sum_{l'}$) is taken over all $l \in L_1$ (resp. $l' \in 2L_1^*$).

Now, a direct calculation yields

$$(\gamma(\sigma_z, NQ_0) f_k)(\rho(g)^{-1} l) = v^{(2k+5)/4} \langle l, \Phi'(g)^{-1t}(-i, i, 1, -1, 0)\rangle_N^k e[(u\langle l, l\rangle_N + iNv\,{}^t l \Phi'(g^2) l)/2],$$

where $g = \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y}^{-1} \end{pmatrix} \in Sp(2, \mathbf{R})$ $(Y = yY_1)$ and $\Phi'(g)$ is the one in [3, p. 67]. The theta series $\theta^N_{k,\bar{\chi}}(z, g)$ can be transformed into the form

$$N^{k-3/2}\sqrt{2\pi}^{-k}\sum_{\varepsilon=0}^{k} {}_kC_\varepsilon(-i)^\varepsilon \theta^N_{1,\varepsilon}(z, Y_1)\theta^{N,\bar{\chi}}_{2,k-\varepsilon}(z, y), \tag{2.1}$$

where $Y = yY_1$ $(\det(Y_1) = 1)$ and $\varepsilon$ runs over all even integers such that $0 \le \varepsilon \le k$. The same arguments as in [8, p. 152] show that

$$\begin{aligned} &\theta^{N,\bar{\chi}}_{2,\varepsilon}(z, y) \\ &\quad = \sqrt{2\pi}^{\varepsilon} i^\varepsilon N y^{\varepsilon+1}(\mathrm{Im}\, v)^{-\varepsilon} \sum_{m=-\infty}^{\infty} \bar{\chi}(m) \sum_{n=-\infty}^{\infty} (m\bar{z} + n)^\varepsilon \exp(-\pi N y^2 |mz + n|^2/v), \end{aligned}$$

which shows

$$\begin{aligned} &\theta^{N,\bar{\chi}}_{2,\varepsilon}(-1/4Nz, y) \\ &\quad = \chi(-1)(\sqrt{2\pi}i)^\varepsilon N y^{\varepsilon+1} z^\varepsilon v^{-\varepsilon} \sum_{n=1}^{\infty} \bar{\chi}(n) n^\varepsilon \sum_{\gamma} \bar{\chi}(d) J\overline{(\gamma, z)}^{\varepsilon} k(\gamma\langle z\rangle, n, y), \end{aligned} \tag{2.2}$$

where $k(z, n, y) = \exp(-\pi n^2 y^2/4v)$ and $\gamma = \begin{pmatrix} * & * \\ * & d \end{pmatrix}$ runs over $\Gamma_\infty \backslash \Gamma_0(4N)$ with $\Gamma_\infty = \left\{ \pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbf{Z} \right\}$. The following formula has been proved in [3, Lemma 3.1]:

$$\theta^N_{1,\varepsilon}(-1/4Nz, Y_1) = (4z)^{(2\varepsilon-1)/2}\sqrt{i/2}2^{2-\varepsilon}\theta^*_{1,\varepsilon}(z, Y_1). \tag{2.3}$$

By virtue of the definition of $\Psi(f)$ and (2.1), we see that

$$\begin{aligned} \Psi(f)(iY) &= c'y^{-k}\sum_{\varepsilon} {}_kC_\varepsilon\overline{(-i)}^{\varepsilon} \int_{D_0(4N)} v^{(2k-1)/2} f \mid [W_{4N}]_{2k-1}(z) \\ &\qquad \times \overline{\theta^N_{1,\varepsilon}(z, Y_1)}\,\overline{\theta^{N,\bar{\chi}}_{2,k-\varepsilon}(z, y)} v^{-2}\, du dv \\ &= c'y^{-k}\sum_{\varepsilon} {}_kC_\varepsilon i^\varepsilon \int_{D_0(4N)} (v/4N|z|^2)^{(2k-1)/2} f(z)((4N)^{1/4}(-iz)^{1/2})^{2k-1} \\ &\qquad \times \overline{\theta^N_{1,\varepsilon}(-1/4Nz, Y_1)}\,\overline{\theta^{N,\bar{\chi}}_{2,k-\varepsilon}(-1/4Nz, y)} v^{-2}\, du dv. \end{aligned}$$

This combined with (2.2) and (2.3) shows that

$$\Psi(f)(iY) = c'' \sum_{\varepsilon} {}_k C_\varepsilon 2^{\varepsilon+1} \sqrt{2\pi}^{-\varepsilon} y^{1-\varepsilon} \sum_{n=1}^{\infty} \chi(n) n^{k-\varepsilon} \int_{D_0(4N)} v^{\varepsilon-1/2}$$

$$\times \sum_{\gamma} \chi(d) J(\gamma, z)^{k-\varepsilon} k(\gamma\langle z\rangle, n, y) \overline{\theta_{1,\varepsilon}^*(z, Y_1)} f(z) v^{-2} \, du dv,$$

where $\sum_\varepsilon$ is taken over all non-negative even integers $\varepsilon$ such that $0 \leq \varepsilon \leq k$ and $\sum_\gamma$ is taken over all $\gamma = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_0(4N)$. By [3, Lemma 3.1], we obtain

$$\int_{D_0(4N)} v^{\varepsilon-1/2} \sum_{\gamma} \chi(d) J(\gamma, z)^{k-\varepsilon} k(\gamma\langle z\rangle, n, y) \overline{\theta_{1,\varepsilon}^*(z, Y_1)} f(z) v^{-2} \, du dv$$

$$= \int_0^\infty \int_0^1 v^{\varepsilon-1/2} k(z, n, y) \overline{\theta_{1,\varepsilon}^*(z, Y_1)} f(z) v^{-2} \, du dv$$

$$= \int_0^\infty v^{\varepsilon/2-3/2} \sum_{l'} a({}^t l' Q_1 l'/2) H_\varepsilon(\sqrt{2\pi v}(y_1, -y_3, -\sqrt{2} y_2) l')$$

$$\times \exp(-\pi v({}^t l' Q_1 l' + {}^t l' R(Y_1) l')) \exp(-\pi n^2 y^2/4v),$$

where $l'$ runs over $2L_1^*$.

Here we recall the following formula (cf. [2, p. 173]):

$$\int_0^\infty v^{(\varepsilon-3)/2} \exp(-\alpha v - \beta v^{-1}) H_\varepsilon(\sqrt{2\alpha v}) \, dv = \beta^{(\varepsilon-1)/2} \sqrt{\pi} \sqrt{2}^{\varepsilon} \exp(-2\sqrt{\alpha\beta})$$

for each $\alpha, \beta > 0$, which shows

$$\Psi(f)(iY) = c''' \sum_{n=1}^{\infty} \chi(n) n^{k-1} \sum_{T} a(4 \det(T)/n^2) \exp(-2\pi n \operatorname{tr}(TY)),$$

where $T$ runs over all $T = \begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix}$ under the condition $T > 0$, $(n_1, n_2, n_3) \in \mathbf{Z}_3^1$, $n|\text{g.c.d.}(n_1, n_2, n_3)$, $f(z) = \sum_{n=1}^{\infty} a(n) e[nz]$ and $c'$, $c''$ and $c'''$ are non zero constants not depending upon $f$ and $T$.

Set $P_2 = \left\{ T = \begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix} \middle| T > 0 \text{ and } (n_1, n_2, n_3) \in \mathbf{Z}_3^1 \right\}$. For every $T = \begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix} \in P_2$, we set $e(T) = \text{g.c.d.}(n_1, n_2, n_3)$ and $N(T) = 4 \det(T)$.

Let $f(z) = \sum_{n=1}^{\infty} a(n)e[nz]$ be an element in $S_{k-1/2}(4N, \chi)$. Then we deduce the following.

Theorem 1. *Suppose that $\chi$ is a Dirichlet character modulo $2N$ and $k(>5)$ is even. Then $\Psi(f)(Z)$ is a Siegel modular cusp form in $S_k(\Gamma_0^{(2)}(2N), \chi)$ and the Fourier expansion of $\Psi(f)$ at infinity has the form*

$$\Psi(f)(Z) = c \sum_{T} \sum_{n|e(T)} \chi(n) n^{k-1} a(N(T)/n^2) e[\mathrm{tr}(TZ)],$$

*where $\sum_T$ is the sum taken over $P_2$, $\sum_{n|e(T)}$ is the summation taken over all positive integers $n$ with $n \mid e(T)$ and $c$ is a non zero constant not depending upon $f$ and $T$.*

## §3. Hecke Operators

In this section we shall investigate relations between our mapping $\Psi$ and Hecke operators. We denote by $T_{k-1/2,\chi}^{4N}(p^2)$ (resp. $\tilde{T}_{k,\chi}^{2N}(n)$) Hecke operators defined on $S_{k-1/2}(4N, \chi)$ (resp. $S_k(\Gamma_0^{(2)}(2N), \chi)$) (cf. [1], [7] and [11]). Assume that $F$ is an eigen-function of Hecke operators $\tilde{T}_{k,\chi}^{2N}(n)$ for all positive integers $n$. Then the zeta function $Z_F(s)$ in the sense of Andrianov has the form

$$\begin{aligned} Z_F(s) = \prod_{p|2N} (1 - \lambda(p)p^{-s})^{-1} \prod_{p \nmid 2N} (1 - \lambda(p)p^{-s} + (\lambda(p)^2 - \lambda(p^2) \\ - p^{2k-4}\chi(p)^2)p^{-2s} - p^{2k-3}\chi(p)^2\lambda(p)p^{-3s} + p^{4k-6}\chi(p)^4 p^{-4s})^{-1}, \end{aligned}$$

where $\tilde{T}_{k,\chi}^{2N}(n)F = \lambda(n)F$ for all positive integers $n$.

Now, a relation between our mapping $\Psi$ and the above Hecke operators can be stated as follows.

Theorem 2. *Suppose that $f$ is a cusp form of $S_{k-1/2}(4N, \chi)$ such that $T_{k-1/2,\chi}^{4N}(p^2)f = \omega(p)f$ for all primes $p$. Then $\Psi(f)$ is an eigen-function of Hecke operators $\tilde{T}_{k,\chi}^{2N}(n)$ for every integer $n(>0)$. Moreover, the zeta function $Z_{\Psi(f)}(s)$ associated with $\Psi(f)$ is equal to*

$$L(s-k+1, \chi) L(s-k+2, \chi) \prod_p (1 - \omega(p)p^{-s} + \chi(p)^2 p^{2k-3-2s})^{-1},$$

*where $L(s, \chi)$ is the zeta function associated with $\chi$.*

The following theorem has been shown by Andrianov [1, Prop. 2.1.2] (cf. also [7]).

Theorem B. *Let* $F(Z) = \sum_T c(T)e[\mathrm{tr}(TZ)]$ *be a cusp form of* $S_k(\Gamma_0^{(2)}(M), \psi)$ *and let* $p$ *be a prime. Then* $\tilde{T}^M_{k,\psi}(p^n)F(Z) = \sum_T c(p^n : T)e[\mathrm{tr}(TZ)]$ *and*

$$c(p^n : T) = \begin{cases} c(p^nT) & \text{if } p \mid M, \\ \sum p^{(k-2)\beta+(2k-3)\gamma}\psi(p)^{\beta+2\gamma}\Delta^-(p^\gamma)\Pi(p^\beta)\Delta^+(p^\alpha)c(T) & \text{if } p \nmid M, \end{cases}$$

*where the summation* $\sum$ *is taken over all* $(\alpha, \beta, \gamma) \in \mathbf{Z}_3^1$ *with* $\alpha, \beta, \gamma \geq 0$, $\alpha + \beta + \gamma = n$, $\Delta^-(*)$, $\Pi(*)$, $\Delta^+(*)$ *are the symbols given in* [7] *and* $T$ *runs over* $P_2$.

For a $T \in P_2$, we denote by $d(T)$ the discriminant of the imaginary quadratic field $Q(\sqrt{-N(T)})$. We set $-N(T) = d(T)f^2$ with a positive integer $f$. For each prime $p$, we have

$$SL_2(\mathbf{Z})\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}SL_2(\mathbf{Z}) = \bigcup_{i=1}^{p+1} SL_2(\mathbf{Z})\sigma_i \quad \text{(a disjoint union).}$$

The following lemma was shown in [3, Lemma 4.1].

Lemma B. *Suppose that* $T \in P_2$ *satisfies* $e(T) = 1$. *Then the folloing statements hold*:

(1) *Among* $(p+1)$ *matrices* $\{\sigma_i T {}^t\sigma_i\}_{i=1}^{p+1}$, *there are* $p - \left(\frac{d(T)}{p}\right)$ *matrices with* $e(\sigma_i T {}^t\sigma_i) = 1$ *and* $1 + \left(\frac{d(T)}{p}\right)$ *matrices with* $e(\sigma_i T {}^t\sigma_i) = p$, *if* $f$ *is prime to* $p$.

(2) *Among* $(p+1)$ *matrices* $\{\sigma_i T {}^t\sigma_i\}_{i=1}^{p+1}$, *there are* $p$ *matrices with* $e(\sigma_i T {}^t\sigma_i) = 1$ *and one matrix with* $e(\sigma_i T {}^t\sigma_i) = p^2$, *if* $f$ *is divisible by* $p$.

Proof of Theorem 2. It is sufficient to show the following relations:

$$\tilde{T}^{2N}_{k,\chi}(p)\Psi(f) = (\omega(p) + \chi(p)(p^{k-1} + p^{k-2}))\Psi(f) \quad \text{for all primes } p, \tag{3.1}$$

and

$$\tilde{T}^{2N}_{k,\chi}(p^2)\Psi(f) = (\omega(p)^2 + \chi(p)(p^{k-1} + p^{k-2})\omega(p) + \chi(p)^2 p^{2k-2})\Psi(f) \tag{3.2}$$

$$\text{for all primes } p \text{ such that } p \nmid 2N.$$

These can be proved in the same fashion as in [4] by virtue of Theorem B and Lemma B. Therefore we may omit the details.

Now we recall the definition of Maass forms (cf. [4], [5]). We call an element $F(Z) = \sum_{T \in P_2} c(T)e[\mathrm{tr}(TZ)]$ of $S_k(\Gamma_0^{(2)}(M), \psi)$ a Maass form if the following condition is satisfied:

$$c\left(\begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix}\right) = \sum_d \psi(d) d^{k-1} c\left(\begin{pmatrix} 1 & n_2/2d \\ n_2/2d & n_1 n_3/d^2 \end{pmatrix}\right),$$

where $\sum_T$ (resp. $\sum_d$) is the sum taken over all $T \in P_2$ (resp. positive integers with $d \mid (n_1, n_2, n_3)$). Let $N$ be a positive integer such that $(2, N) = 1$. Let $f(z) = \sum_{n=1}^{\infty} a(n)e[nz]$ $(a(1) \neq 0)$ (resp. $F(Z) = \sum_{T \in P_2} c(T)e[\mathrm{tr}(TZ)]$ $\left(c\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \neq 0\right)$) be a cusp form of $S_{k-1/2}(4N, \chi)$ (resp. a Maass form of $S_k(\Gamma_0^{(2)}(2N), \chi)$) such that $T_{k-1/2,\chi}^{4N}(p^2)f = \omega(p)f$ (resp. $\tilde{T}_{k,\chi}^{2N}(n)F = \lambda(n)F$) for all primes $p$ (resp. positive integers $n$). We denote by $\tilde{\mathfrak{S}}_{k-1/2}(4N, \chi)$ (resp. $\mathcal{M}_k(\Gamma_0^{(2)}(2N), \chi)$) the vector space spanned by all such elements $f$ (resp. $F$). We deduce the following theorem.

THEOREM 3. *Under the above notations, $\Psi$ induces a linear isomorphic mapping between $\tilde{\mathfrak{S}}_{k-1/2}(4N, \chi)$ and $\mathcal{M}_k(\Gamma_0^{(2)}(2N), \chi)$. Furthermore, if $f \in \tilde{\mathfrak{S}}_{k-1/2}(4N, \chi)$ satisfies $T_{k-1/2,\chi}^{4N}(p^2)f = \omega(p)f$ for all primes $p$, then $\Psi(f)$ is an eigen-function of Hecke operators $\tilde{T}_{k,\chi}^{2N}(n)$ for all positive integers $n$ and the zeta function $Z_{\Psi(f)}(s)$ attached to $\Psi(f)$ is equal to*

$$L(s-k+1, \chi)L(s-k+2, \chi)\prod_p (1 - \omega(p)p^{-s} + \chi(p)^2 p^{2k-3-2s})^{-1}.$$

Since the above theorem can be verified in the same manners as those of [4], we may omit the details of the proof.

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