# COMPARISON OF THE DEFINITIONS OF ABELIAN 2-CATEGORIES

By

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**Abstract.** In the efforts to define a 2-categorical analog of an abelian category, two (or three) notions of ''abelian 2-categories'' are defined in [4] and [2]. One is the *relatively exact 2-category* defined in [4], and the other(s) is the (2-)*abelian* Gpd-*category* defined by Dupont [2]. We compare these notions, using the arguments in [4] and [2]. Since they proceed independently in their own way, in different settings and terminologies, it will be worth while to collect and unify them. In this paper, by comparing their definitions and arguments, we show the relationship among these classes of 2-categories.

## 1. Introduction

Motivated by [5], we defined a general class of 2-categories '*relatively exact 2-categories*' in [4] (originally written as our master's thesis in 2006), so as to make the 2-categorical homological algebra works well in an abstract setting.

A relatively exact 2-category is a generalization of SCG (= the 2-category of symmetric categorical groups), and defined as a 2-categorical analog of an abelian category.

|  | category | 2-category |
|---|---|---|
| general theory | abelian category | relatively exact 2-category |
| example | Ab | SCG |

On the other hand, with a similar motivation, Dupont defined two classes of 2-categries '2-*abelian* Gpd-*category*' and '*abelian* Gpd-*category*' in [2]. Thus there are three different classes of 2-categories

 · (Relatively exact 2-category)
 · (2-abelian Gpd-category)
 · (abelian Gpd-category)

defined as 2-dimensional analogs of abelian categories. So it will be necessary to make explicit the relations.

We compare these notions, using the arguments in [4] and [2]. Since they proceed independently in their own way, in different settings and terminologies, it will be worth while to collect and unify them.

In this paper, by comparing their definitions and arguments, we show the relationship among three classes of 2-categories mentioned above. In Theorem 5.3, we show there are implications for these notions

$$(\text{2-Abelian Gpd}) \Rightarrow (\text{Relatively exact}) \Rightarrow (\text{Abelian Gpd}),$$

except for some minor differences (see Theorem 5.3).

## 2.  Preliminaries

Let $\mathbf{S}$ denote a 2-category (in the strict sense). We use the following notation.
$\mathbf{S}^0$, $\mathbf{S}^1$, $\mathbf{S}^2$: class of 0-cells, 1-cells, and 2-cells in $\mathbf{S}$, respectively.
$\mathbf{S}^1(A, B)$: 1-cells from $A$ to $B$, where $A, B \in \mathbf{S}^0$.
$\mathbf{S}^2(f, g)$: 2-cells from $f$ to $g$, where $f, g \in \mathbf{S}^1(A, B)$ for some $A, B \in \mathbf{S}^0$.
$\mathbf{S}(A, B)$: Hom-category between $A$ and $B$
(i.e. $\mathrm{Ob}(\mathbf{S}(A, B)) = \mathbf{S}^1(A, B)$, $\mathbf{S}(A, B)(f, g) = \mathbf{S}^2(f, g)$).

In diagrams, $\rightarrow$ represents a 1-cell, $\Rightarrow$ represents a 2-cell, $\circ$ represents a horizontal composition, and $\cdot$ represents a vertical composition. We use capital letters $A, B, \ldots$ for 0-cells, small letters $f, g, \ldots$ for 1-cells, and Greek symbols $\alpha, \beta, \ldots$ for 2-cells.

The composition of $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ is denoted by $g \circ f$, conversely to [4]. Similarly for the composition of 2-cells.

In the following arguments, any 2-cell in a 2-category is invertible. This helps us to avoid being fussy about the directions of 2-cells, and we use the word 'dual' simply to reverse the directions of 1-cells. For example, *cokernel* is the dual notion of *kernel*, and *pullback* is dual to *pushout*. As for the definitions of (co-)kernels, pullbacks, and pushouts in a 2-category, see [2] or [4]. (The definitions in [2] and [4] agree.)

### 3.  Relatively Exact 2-Category

Let SCG denote the 2-category of small symmetric categorical groups (= symmetric 2-groups). This is denoted by 2-SGp in [2]. 0-cells are symmetric categorical groups, 1-cells are symmetric monoidal functors, and 2-cells are monoidal transformations (cf. [5] or [4]).

For any symmetric monoidal functor $f : A \to B$, let $f_I$ denote the unit isomorphism $f_I : f(0_A) \overset{\cong}{\to} 0_B$, where $0_A$ and $0_B$ are respectively the unit of $A$ and $B$, with respect to $\otimes$.

DEFINITION 3.1 (Definition 3.7 in [4]).  *A 2-category* **S** *is said to be locally SCG if the following conditions are satisfied*:

(LS1) *For every* $A, B \in \mathbf{S}^0$, *there is a given functor* $\bigotimes_{A,B} : \mathbf{S}(A,B) \times \mathbf{S}(A,B) \to \mathbf{S}(A,B)$, *and an object* $0_{A,B} \in \mathrm{Ob}(\mathbf{S}(A,B))$ *such that* $(\mathbf{S}(A,B), \bigotimes_{A,B}, 0_{A,B})$ *becomes a symmetric categorical group, and the following naturality condition is satisfied*:

$$0_{B,C} \circ 0_{A,B} = 0_{A,C} \quad (\forall A, B, C \in \mathbf{S}^0)$$

(LS2) $\mathrm{Hom} = \mathbf{S}(-,-) : \mathbf{S}^{\mathrm{op}} \times \mathbf{S} \to \mathrm{SCG}$ *is a 2-functor* (*in the strict sense*). *Moreover, for any* $A, B, C \in \mathbf{S}^0$,

(1) $$(- \circ 0_{A,B})_I = \mathrm{id}_{0_{A,C}} \in \mathbf{S}^2(0_{A,C}, 0_{A,C})$$

(2) $$(0_{B,C} \circ -)_I = \mathrm{id}_{0_{A,C}} \in \mathbf{S}^2(0_{A,C}, 0_{A,C}).$$

*are satisfied.*



(*Remark that* $(- \circ 0_{A,B})$ *and* $(0_{B,C} \circ -)$ *are symmetric monoidal functors.*)

(LS3) *There is a 0-cell* $0 \in \mathbf{S}^0$ *called zero object, which satisfies the following conditions*:

(ls3-1) *For any* $f : 0 \to A$ *in* **S**, *there exists a unique 2-cell* $\theta_f \in \mathbf{S}^2(f, 0_{0,A})$.
(ls3-2) *For any* $f : A \to 0$ *in* **S**, *there exists a unique 2-cell* $\tau_f \in \mathbf{S}^2(f, 0_{A,0})$.
(LS3+) $\mathbf{S}(0,0)$ *is the zero categorical group.*
(LS4) *For any* $A, B \in \mathbf{S}^0$, *their product and coproduct exist.*

CAUTION 3.2.  *In* [4], *zero object was also assumed to satisfy* (LS3+). *On the other hand, the definition of zero object in* [2] *only requires* (ls3-1) *and* (ls3-2). *In*

*fact, condition* (LS3+) *is not used essentially in* [4]. *So in the following, we mainly consider locally* SCG 2-*categories without condition* (LS3+).

Definition 3.3 (Definition 3.7 in [4]).   *Let* **S** *be a locally* SCG 2-*category.* **S** *is said to be relatively exact if the following conditions are satisfied*:
    (RE1) *For any* 1-*cell* $f$, *its kernel and cokernel exist.*
    (RE2) *Any* 1-*cell* $f$ *is faithful if and only if* $f = \mathrm{ker}(\mathrm{cok}(f))$.
    (RE3) *Any* 1-*cell* $f$ *is cofaithful if and only if* $f = \mathrm{cok}(\mathrm{ker}(f))$.

(For the definitions of (fully) (co-)faithfulness, see [4] or [3].)

Remark 3.4.   For any 1-cell $f : A \to B$, its kernel is defined as the triplet $(\mathrm{Ker}(f), \mathrm{ker}(f), \varepsilon_f)$

$$
\mathrm{Ker}(f) \xrightarrow[\mathrm{Ker}(f)]{} A \xrightarrow[f]{} B, \qquad \Uparrow \varepsilon_f, \quad 0
$$

universal among those $(K, k, \varepsilon)$

$$
K \xrightarrow[k]{} A \xrightarrow[f]{} B. \qquad \Uparrow \varepsilon, \quad 0
$$

For the precise definition, see [4] or [2]. Dually, the cokernel of $f$ is the universal triplet $(\mathrm{Cok}(f), \mathrm{cok}(f), \pi_f)$

$$
A \xrightarrow[f]{} B \xrightarrow[\mathrm{cok}(f)]{} \mathrm{Cok}(f). \qquad \Uparrow \pi_f, \quad 0
$$

## 4.   (2-)Abelian Gpd-Category

(2-)Abelian Gpd-categories, defined in [2], are Gpd*-categories satisfying certain conditions. By definition, a Gpd*-category is a category $\mathscr{C}$ enriched by the category Gpd* of small pointed groupoids (Proposition 70 in [2]). For any $A, B \in \mathrm{Ob}(\mathscr{C})$, the distinguished point in $\mathscr{C}(A, B)$ is denoted by $0_{A,B}$ or simply by 0.

In [2], it is remarked that any Gpd*-category $\mathscr{C}$ is equivalent to a *strictly described* one, and thus $\mathscr{C}$ is assumed to be strictly described, namely, it satisfies the following:

(SD1) For any sequence of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

in $\mathscr{C}$,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

is satisfied.

(SD2) For any $f : A \to B$ in $\mathscr{C}$,

$$f \circ \mathrm{id}_A = f$$

$$\mathrm{id}_B \circ f = f$$

are satisfied.

(SD3) For any $f : A \to B$ and any pair of objects $A'$, $B'$ in $\mathscr{C}$,

$$f \circ 0_{A', A} = 0_{A', B}$$

$$0_{B, B'} \circ f = 0_{A, B'}$$

are satisfied.

(SD4) For any $A \underset{g}{\overset{f}{\underset{\alpha \Downarrow}{\rightleftarrows}}} B$ and any pair of objects $A'$, $B'$ in $\mathscr{C}$,

$$\alpha \circ 0_{A', A} = \mathrm{id}_{0_{A', B}}$$

$$0_{B, B'} \circ \alpha = \mathrm{id}_{0_{A, B'}}$$

are satisfied.

REMARK 4.1.  A Gpd$^*$-category $\mathscr{C}$ is regarded as a 2-category in the following, and we use 2-categorical terminologies, e.g. '0-cell' for an object, '1-cell' for an arrow.

DEFINITION 4.2 (Definition 165 in [2]).  *An abelian Gpd-category is a* Gpd$^*$-*category $\mathscr{C}$ with a zero object, finite (co-)products and (co-)kernels, satisfying the following conditions*:

(AG1) *Every* 0-*monomorphic* 1-*cell* $f$ *satisfies* $f = \ker(\mathrm{cok}(f))$.

(AG2) *Every* 0-*epimorphic* 1-*cell* $f$ *satisfies* $f = \mathrm{cok}(\ker(f))$.

(AG3) *Fully* 0-*faithful* 1-*cells and* 0-*monomorphic* 1-*cells are stable under pushout*.

(AG4) *Fully* 0-*cofaithful* 1-*cells and* 0-*epimorphic* 1-*cells are stable under pullback*.

DEFINITION 4.3 (Definition 179 and 183 in [2]).   *A 2-abelian Gpd-category is a* $\mathrm{Gpd}^*$*-category* $\mathscr{C}$ *with a zero object, finite (co-)products and (co-)kernels, satisfying the following conditions*:

(2AG1) *If* $f$ *is a* 0*-faithful* 1*-cell, then* $f = \ker(\mathrm{cok}(f))$.

(2AG2) *If* $f$ *is a* 0*-cofaithful* 1*-cell, then* $f = \mathrm{cok}(\ker(f))$.

(2AG3) *Any fully* 0*-faithful* 1*-cell is canonically the root of its copip.*

(2AG4) *Any fully* 0*-cofaithful* 1*-cell is canonically the coroot of its pip.*

For the definitions of (co-)roots and (co-)pips, see [2]. We do not require them explicitly in the following arguments. We introduce the rest of the notions appearing in the above definitions. The definition of 0-monomorphic 1-cells is the following. 0-epimorphicity is defined dually.

DEFINITION 4.4 (Definition 118 in [2]).   *A* 1*-cell* $f : A \to B$ *is* 0*-monomorphic if, for any* 1*-cell* $a : X \to A$ *and any* 2*-cell* $\beta : f \circ a \Rightarrow 0$ *compatible with* $\pi_f$ (*of Remark* 3.4), *there exists a unique* $\alpha : a \Rightarrow 0$ *such that* $f \circ \alpha = \beta$.

The definitions of (fully) 0-faithful 1-cells are the following. (Fully) 0-cofaithful 1-cells are defined dually.

DEFINITION 4.5 (Definition 78, 80 in [2]).   *Let* $\mathscr{C}$ *be a* $\mathrm{Gpd}^*$*-category, and* $f : A \to B$ *be a* 1*-cell in* $\mathscr{C}$.

(i) $f$ *is* 0*-faithful if for any* $X \underset{0}{\overset{0}{\underset{\alpha\Downarrow}{\rightrightarrows}}} A$ *in* $\mathscr{C}$,

$$f \circ \alpha = \mathrm{id}_0 \;\Rightarrow\; \alpha = \mathrm{id}_0$$

*is satisfied.*

(ii) $f$ *is fully* 0*-cofaithful if for any* 1*-cell* $a : X \to A$ *and any* 2*-cell* $X \underset{0}{\overset{f\circ\alpha}{\underset{\alpha\Downarrow}{\rightrightarrows}}} B$, *there exists a unique* 2*-cell* $\alpha : a \Rightarrow 0$ *such that* $\beta = f \circ \alpha$.

FACT 4.6.   *In* [2], *it is shown that any* 2*-abelian* Gpd*-category* $\mathscr{C}$ *admits a weak enrichment by* SCG, *i.e.,* $\mathscr{C}$ *is preadditive, in the terminology of* [2].

For the general definition of a preadditive Gpd-category, see [2]. We only consider the case where $\mathscr{C}$ is strictly described. (In this case, the natural transformations appearing in Definition 218 in [2] are identities.)

DEFINITION 4.7. *A strictly described* Gpd-*category* $\mathscr{C}$ *is preadditive if it satisfies the following*:

(o) *For any pair of* 0-*cells* $A$, $B$ *in* $\mathscr{C}$, Hom-*category* $\mathscr{C}(A, B)$ *is equipped with a structure of a symmetric categorical group* $(\mathscr{C}(A, B), \otimes, 0)$.

(a1) *For any* 1-*cell* $A \xrightarrow{f} B$ *and any* 0-*cell* $C$ *in* $\mathscr{C}$, *the composition by* $f$

$$- \circ f : \mathscr{C}(B, C) \to \mathscr{C}(A, C)$$

*is symmetric monoidal*.

(a2) *The dual of* (a1).

(b1) *For any* $A \xrightarrow{f} B \xrightarrow{g} C$ *and any* 0-*cell* $D$ *in* $\mathscr{C}$, *we have*

$$
\begin{array}{ccc}
\mathscr{C}(C, D) & \xrightarrow{\ -\circ g\ } & \mathscr{C}(B, D) \\
& \circlearrowleft & \\
{\scriptstyle -\circ(g\circ f)}\searrow & & \swarrow{\scriptstyle -\circ f} \\
& \mathscr{C}(A, D) &
\end{array}
$$

*as monoidal functors*.

(b2) *The dual of* (b1).

(c) *For any* $A \xrightarrow{f} B$ *and* $C \xrightarrow{g} D$, *we have*

$$
\begin{array}{ccc}
\mathscr{C}(B, C) & \xrightarrow{\ -\circ f\ } & \mathscr{C}(A, C) \\
{\scriptstyle g\circ -}\downarrow & \circlearrowleft & \downarrow{\scriptstyle g\circ -} \\
\mathscr{C}(B, D) & \xrightarrow[\ -\circ f\ ]{} & \mathscr{C}(A, D)
\end{array}
$$

*as monoidal functors*.

(d) *For any pair of* 0-*cells* $A$ *and* $B$ *in* $\mathscr{C}$, *we have*

$$
\begin{array}{c}
\mathrm{id}_B\circ - \\
\circlearrowleft \\
\mathscr{C}(A, B) \xrightarrow{\ \mathrm{id}\ } \mathscr{C}(A, B) \\
\circlearrowleft \\
-\circ\mathrm{id}_A
\end{array}
$$

(e1) *For any* 0-*cell* $X$ *and any* $\ell, k : X \to A$,

$$(- \circ f)_{\ell, k} : (\ell \otimes k) \circ f \Rightarrow (\ell \circ f) \otimes (k \circ f) \quad (\forall f : A \to B)$$

*is natural in* $f$.

$$X \underset{k}{\overset{\ell}{\rightrightarrows}} A \xrightarrow{f} B$$

(e2) *The dual of* (e1).

(f1) *For any* $X$,

$$(- \circ f)_I : 0_{X,A} \circ f \Rightarrow 0_{X,B} \quad (\forall f : A \to B)$$

*is natural in* $f$.

(f2) *The dual of* (f1).

*Here, since* $(- \circ f)$ *is monoidal,* $(- \circ f)_{\ell,k}$ *denotes the structure isomorphism*

$$(- \circ f)_{\ell,k} : (\ell \otimes k) \circ f \Rightarrow (\ell \circ f) \otimes (k \circ f)$$

*natural in* $\ell, k : X \to A$.

*Similarly,* $(- \circ f)_I$ *denotes the unit isomorphism.*

## 5.  Comparison

LEMMA 5.1.  *If* $\mathscr{C}$ *is a strictly described preadditive* Gpd-*category, then*

$$\mathrm{Hom} = \mathscr{C}(-,-) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{SCG}$$

*is a 2-functor.*

PROOF.  For the definition of a 2-functor, see Definition 7.2.1 in [1]. It can be easily shown that, to show the lemma, it suffices to show the following conditions:

(i) For any 1-cells $A' \xrightarrow{f} A$ and $B \xrightarrow{g} B'$,

$$g \circ - \circ f : \mathscr{C}(A,B) \to \mathscr{C}(A',B')$$

is a symmetric monoidal functor.

(ii) For any $A' \underset{f'}{\overset{f}{\Rightarrow}} A$ and $B' \underset{g'}{\overset{g}{\Rightarrow}} B$,

$$\beta \circ - \circ \alpha : g \circ - \circ f \Rightarrow g' \circ - \circ f'$$

is a monoidal transformation.

(iii) For any $A'' \xrightarrow{f'} A' \xrightarrow{f} A$ and $B \xrightarrow{g} B' \xrightarrow{g'} B''$, we have

$$\begin{array}{ccc}
\mathscr{C}(A,B) & \xrightarrow{g\circ-\circ f} & \mathscr{C}(A',B') \\
{\scriptstyle(g'\circ g)\circ-\circ(f'\circ f)}\searrow & \circlearrowleft & \swarrow{\scriptstyle g'\circ-\circ f'} \\
& \mathscr{C}(A'',B'') &
\end{array}$$

as monoidal functors.

(iv) For any pair of 0-cells $A$ and $B$ in $\mathscr{C}$, we have

$$\mathscr{C}(A,B) \underset{\text{id}}{\overset{\text{ido}-\text{oid}}{\circlearrowleft}} \mathscr{C}(A,B)$$

as monoidal functors.

(iv) follows from (d). (i) follows from (a1), (a2) (and (c)). (iii) follows from (b1), (b2) (and (c)). (ii) follows from (e1), (e2), (f1), (f2). $\qquad\square$

LEMMA 5.2.  *Let $f : A \to B$ be any 1-cell in a relatively exact 2-category. Then the following are satisfied.*
  (i) *$f$ is faithful if and only if it is 0-faithful, if and only if it is 0-monomorphic.*
  (ii) *$f$ is fully faithful if and only if it is fully 0-faithful.*

PROOF.  (i) By Corollary 3.24 in [4], $f$ is faithful if and only if it is 0-faithful. As remarked after Definition 118 in [2], any 0-monomorphic 1-cell is faithful. Conversely, if $f$ is faithful, then $f$ satisfies $f = \ker(\mathrm{cok}(f))$, and becomes 0-monomorphic by Lemma 3.19 in [4].
  (ii) This is nothing other than Lemma 3.22 (2) in [4]. $\qquad\square$

THEOREM 5.3.  *There are implications among the conditions on 2-categories*

$$(\text{2-Abelian Gpd}) \Rightarrow (\text{Relatively exact}) \Rightarrow (\text{Abelian Gpd}).$$

*More precisely, we have*:
  (i) *Any strictly described 2-abelian Gpd-category is a relatively exact 2-category without condition (LS3+).*
  (ii) *Any relatively exact 2-category without condition (LS3+) is an abelian Gpd-category not necessarily strictly described.*

PROOF.  First remark that each of these 2-categories is a Gpd$^*$-category with a zero object, finite (co-)products and (co-)kernels.
  (i) Let $\mathscr{C}$ be a 2-abelian Gpd-category. $\mathscr{C}$ satisfies (LS1), as a particular case of (SD3). By Lemma 5.1, $\mathrm{Hom} = \mathscr{C}(-,-) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{SCG}$ is a 2-functor. Moreover, (1) and (2) in (LS2) follow from (SD4). Thus $\mathscr{C}$ satisfies (LS2). By Proposition 180 in [2], any 1-cell in $\mathscr{C}$ is 0-faithful if and only if it is faithful. Thus (RE2) follows from (2AG1). Dually, (RE3) follows from (2AG2).

(ii) Let **S** be a relatively exact 2-category. By the duality, it suffices to show (AG1) and (AG3). By Lemma 5.2, we have equivalences of the notions

$$\text{faithful} = 0\text{-faithful} = 0\text{-monomorphic}$$

$$\text{fully faithful} = \text{fully } 0\text{-faithful}$$

for 1-cells in **S**. Thus (AG1) follows from (RE2), and (AG3) follows from the duals of Proposition 3.32 and Proposition 5.12 in [4].

Remark also that (SD1) and (SD2) are satisfied, but (SD3) and (SD4) are not satisfied in general. So **S** is not necessarily strictly described as a Gpd$^*$-category.

□

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