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SIMILARITIES INVOLVING UNBOUNDED NORMAL OPERATORS

By

Mohammed Hichem Mortad

Abstract. We prove and disprove some generalizations of a result about some similarities involving normal operators due to M. R. Embry in 1970. Some interesting consequences are also given.

1. Introduction

Mary R. Embry wrote many interesting research papers on operator theory (mainly bounded operators) in late sixties and the seventies of the last century.

One particular paper which is of interest to us is [6]. The main theorem in that paper is

Theorem 1. *If H and K are two commuting normal operators and $AH = KA$, where $0 \notin W(A)$ (where $W(A)$ is to be defined below), then $H = K$.*

The paper [6] has been since cited several times and it has had many applications. Some of them may be found in some of the author's recent papers (see e.g. [11] or [13]). It also permits to solve problems by bypassing the Fuglede-Putnam theorem (as done in [1], [9], [19] and [22]).

The aim of this paper is to try to give a follow-up to Embry's paper. The outline of the paper is as follows. First of all, we give a different version of Theorem 1 by imposing a self-adjointness condition on A and dropping the commutativity condition on H and K. Then, we generalize Theorem 1 to unbounded H and K and keeping A bounded. The theorem remains valid in this case and it allows us to obtain important consequences. Finally, we show that

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Theorem 1, unfortunately, fails to be true if all the operators involved are unbounded. This will be illustrated by an explicit example.

Different results and notions are needed in order to achieve this aim. We recall the most important ones. Any other notion will be assumed to be known by the reader. The general references on operator theory are many. We cite [3, 4, 8] among others.

We first recall the Fuglede-Putnam theorem [7, 16]. It states that if A, N and M are bounded operators such that M and N are normal, then

$$AN = MA \Rightarrow AN^* = M^*A.$$

If N and M are unbounded normal operators, then "$=$" is replaced by "$\subset$". In the same context and also in [16] it is known that if $AN \subset MA$, then $AP_{B_R}(N) = P_{B_S}(M)A$ where $P_{B_R}(N)$ and $P_{B_S}(M)$ are the spectral projections of N and M respectively and where

$$B_R = \{z \in \mathbf{C} : |z| \leq R\} \quad \text{and} \quad B_S = \{z \in \mathbf{C} : |z| \leq S\}$$

are two closed balls in $\mathbf{C}$ where R and S are two positive numbers (this result and these notations will be used below).

Also note that bounded operators are assumed to be defined on the whole Hilbert space while the unbounded ones are assumed to be densely defined. The numerical range of an operator A, defined on a Hilbert space $\mathscr{H}$, is denoted by $W(A)$ and is defined as

$$W(A) = \{\langle Af, f\rangle : f \in \mathscr{H}, \|f\| = 1\}$$

and if A is unbounded, then $\mathscr{H}$ is replaced by $D(A)$.

The question of commutativity for two unbounded operators is not an easy matter. See e.g. [17] for some very informative discussion on the subject and also for the famous Nelson's example which shows that the relation $AB = BA$, on some common dense domain, does not necessarily mean that A and B commute.

We adopt the following definition of commutativity of two unbounded normal operators: Two normal operators are said to commute if their associated spectral projections do.

As introduced by Devinatz-Nussbaum in [5], we say that the unbounded operators N, H and K have the property P if they are normal and if $N = HK = KH$. Devinatz-Nussbaum proved (in the same paper) the following result

Theorem 2. *If N, H and K have the property P, then H and K commute.*

Finally, we need the following lemma, due to M. R. Embry [6], which we state in the version we need.

Lemma 1. *Let A and E be two bounded operators. If $0 \notin W(A)$ such that $AE = -EA$ where A is self-adjoint, then $E = 0$.*

2. Main Results

2.1. Positive Results. We first note (as alluded to in the introduction) that by imposing a stronger condition on A we can drop the commutativity hypothesis on H and K.

Proposition 1. *Let A be a bounded self-adjoint operator such that $0 \notin W(A)$. If H and K are bounded normal operators such that $AH = KA$, then $H = K$.*

Proof. We have $AH = KA$. Since H and K are normal, then the Fuglede-Putnam theorem gives us $AH^* = K^*A$. Taking the adjoint of the previous equation and by the self-adjointness of A we obtain $HA = AK$. Thus,

$$A(H - K) = -(H - K)A.$$

Since A is self-adjoint and $0 \notin W(A)$, then Lemma 1 gives us the desired result, i.e. $H = K$. □

Corollary 1. *Let A be a bounded self-adjoint operator such that $0 \notin W(A)$. If H is a bounded normal operator such that $AH = H^*A$, then H is self-adjoint.*

The previous corollary also appeared in [6]. It can also be used to give an alternative way of answering the following question: When is the normal product of two self-adjoint operators self-adjoint? This was answered in [19, 1, 9, 10] (the first two references are for bounded operators and the last two are for unbounded ones) and the condition of establishing this was that one of the operators, say K, must satisfy the asymmetric condition $\sigma(K) \cap \sigma(-K) \subseteq \{0\}$ (in [19] it was assumed that K was positive but this is a consequence of the condition just mentioned). We show that the result remains valid if $0 \notin W(K)$ with a rather simple proof.

Corollary 2. *Assume that H and K are two bounded self-adjoint operators such that $0 \notin W(K)$. If HK is normal, then it is self-adjoint.*

Proof. Set $N = HK$. Then the proof follows from the previous corollary and the following observation

$$KN = KHK = N^*K. \qquad \square$$

We will get back to a similar question for unbounded H and K below the next coming theorem.

Now we give the generalization to unbounded H and K.

Theorem 3. *Assume N, H and K are unbounded operators having the property P. Also assume that $D(H) \subset D(K)$. Assume further that A is a bounded operator for which $0 \notin W(A)$ and such that $AH \subset KA$. Then $H = K$.*

Proof. Let $P_{B_R}(H)$ and $P_{B_S}(K)$ (as introduced in the introduction) be the spectral projections of H and K by respectively. Then $HP_{B_R}(H)$ and $KP_{B_S}(K)$ are two bounded normal operators. The property P (more precisely Theorem 2) then guarantees that they are commuting operators.

Now since $AH \subset KA$ and since ran $P_{B_R}(H) \subset D(H)$ (by the spectral theorem), we immediately see that

$$AHP_{B_R}(H) = KAP_{B_R}(H).$$

Whence (by the remark below the Fuglede-Putnam theorem in the introduction)

$$AHP_{B_R}(H) = KAP_{B_R}(H) = KP_{B_S}(K)A.$$

Therefore we are in a bounded setting and Theorem 1 then applies and implies that

$$P_{B_R}(H)Hf = KP_{B_S}(K)f \quad \text{for all } f \in D(H).$$

Sending both R and S to infinity (in the strong operator topology) gives us

$$Hf = Kf \quad \text{for all } f \in D(H) \ (\subset D(K)).$$

Whence $H \subset K$. Now since normal operators are maximally normal (see [20]), $H = K$. The proof is complete. $\square$

An interesting application of the previous theorem is the following.

Corollary 3. *Assume that A is a bounded operator such that $0 \notin W(A)$. If H is an unbounded normal operator such that $AH \subset H^*A$, then H is self-adjoint.*

Proof. Obvious since $HH^* = H^*H$ as H is normal and also since HH^* is self-adjoint (see [20]). □

Remark. The condition $AH \subset H^*A$ actually implies (after taking the adjoint) that $A^*H \subset H^*A^*$. This is a bit stronger than quasi-similarity (as we have $0 \notin W(A)$). But it does imply the self-adjointness of H (cf. Proposition 4.2 and Remark 4.3 in [15]. See also [21]).

Remark. The necessity of $0 \notin W(A)$ was justified in [6] by a counterexample. Now we give an example which shows that Property P cannot be completely eliminated.

Take $A = I$ (the identity operator on the whole Hilbert space). Now take any non-closed symmetric operator H and hence it is neither self-adjoint nor normal (e.g., take H such that $Hf(x) = -if'(x)$ on $D(H) = C_0^\infty(\mathbf{R})$). Then Property P is not fulfilled, $AH \subset H^*A$, $0 \notin W(A)$ but H is not self-adjoint.

Now we give an analog of Corollary 2 for unbounded H (this is also akin to a result obtained in [9]). We have

Corollary 4. *Assume that H and K are two self-adjoint operators such that H is unbounded and $0 \notin W(K)$. Assume further that K is bounded. If HK is normal, then it is self-adjoint.*

Proof. Set $N = HK$ where H and K are self-adjoint, H unbounded and $0 \notin W(K)$. We have

$$KN = KHK \subset (HK)^*K = N^*K.$$

Since N is normal, then Theorem 3 applies and gives us $N = N^*$, i.e. N is self-adjoint. □

We also have the following result.

Corollary 5. *Let H be an unbounded normal operator. Let A be a bounded operator such that $0 \notin W(A)$. If $AH \subset -HA$, then $H = 0$.*

Proof. Apply Theorem 3. □

2.2. A Counterexample. The case where all operators are unbounded fails to be true in general even if A is assumed to be self-adjoint and even if $\subset$ is replaced by $=$ in the assumption $AH \subset KA$. We have

Theorem 4. *Let A, H and K be unbounded operators. Assume that N, H and K have the property P. Also assume that A is self-adjoint. Then $AH = KA$ and $0 \notin W(A)$* ***do not*** *necessarily imply that $H = K$.*

Proof. We give a counterexample. Consider the following operators A and H defined by

$$Af(x) = (1 + |x|)f(x) \quad \text{and} \quad Hf(x) = -i(1 + |x|)f'(x)$$

on their respective domains

$$D(A) = \{f \in L^2(\mathbf{R}) : (1 + |x|)f \in L^2(\mathbf{R})\}$$

and

$$D(H) = \{f \in L^2(\mathbf{R}) : (1 + |x|)f' \in L^2(\mathbf{R})\}.$$

In order to find H^*, the adjoint of H, some technical work is required. One has to do it first for $f \in C_0^\infty(\mathbf{R}^*)$ (which is a core for H), the space of smooth functions with compact support away from the origin. Then one has to mimic the arguments used in [9] for slightly different operators. One finds the following

$$H^*f(x) = \mp f(x) - i(1 + |x|)f'(x) \quad \text{on}$$

$$D(H^*) = \{f \in L^2(\mathbf{R}) : (1 + |x|)f' \in L^2(\mathbf{R})\}.$$

Now simple calculations yield

$$AH^*f(x) = HAf(x) = \mp i(1 + |x|)f(x) - i(1 + |x|)^2 f'(x)$$

for every f in

$$D(AH^*) = D(HA) = \{f \in L^2(\mathbf{R}) : (1 + |x|)f, (1 + |x|)^2 f' \in L^2(\mathbf{R})\}.$$

This shows that $AH^* = HA$. Now since H is normal (see [10]), then so is H^* and besides, $HH^* = H^*H$ (and HH^* is self-adjoint) and hence Property P is verified. Obviously A is self-adjoint on $D(A)$ and $0 \notin W(A)$.

As one can see, all these assumptions are **not** sufficient to make $H = H^*$. □

A Question

The following question is asked (this will have important consequences in another interesting problem for exponentials of operators if it has a positive answer): Let A be a bounded self-adjoint (*non-unitary*) and invertible operator.

Let N be a *non-normal* and invertible operator. Is $AN = N^*A$, then when will we have N self-adjoint? (for normal N, this is known in [2] and for normal M in lieu of N^*, see [12] or [18]).

Conclusion

Although the proof of Theorem 1, as it appeared in [6], is neither hard nor very technical, one can ask whether it is possible to give another proof of that theorem, i.e. whether one can prove Theorem 1 using algebraic techniques only and by bypassing the spectral theorem. If one can do that, then one may prove similar theorems for non-normal operators for which there is no spectral theorem and hence obtain more results.

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Département de Mathématiques
Université d'Oran (Es-Senia)
B.P. 1524, El Menouar
Oran 31000, Algeria
Mailing address:
Dr Mohammed Hichem Mortad
BP 7085 Seddikia
Oran 31013, Algeria
E-mail address: mhmortad@gmail.com
mortad@univ-oran.dz